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THREE-SPACE-PROBLEMS AND SMALL BALL PROPERTIES FOR FRÉCHET SPACES

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Dedicated to the memory of Susanne Dierolf

Abstract: We unify several known three-space-properties in the context of Fréchet spaces by using the small ball properties for Fréchet spaces introduced by Frerick and the author [8].

Keywords: Fréchet spaces, Three-space-properties, small ball properties.

1. Introduction

A natural problem for topological vector spaces is the so-called *three-space-problem*, that is, given a topological vector space X and a closed subspace $L \subset X$ such that L and the quotient space X/L satisfy certain property P, does X also satisfy P? The classical article of Dierolf and Roelcke [7] is certainly the best reference for positive (and also negative) solutions to this problem for a large variety of properties.

Our purpose is, within the framework of Fréchet spaces, to unify several known positive solutions to the three-space-problem. To do this, we will reformulate the problem in terms of small ball properties for Fréchet spaces as defined in [8].

Definition 1 ([8]). *Let (M, d) be a metric space and let $\mathcal{A}$ be a family of subsets of M (shortly, $\mathcal{A} \subset 2^M$) stable under finite unions. $D \subset M$ has the $\mathcal{A}$-*small ball property *($\mathcal{A}$-*sbp*) if, for each decreasing zero sequence $(\varepsilon_n)_n$, there is $(A_n)_n \in \mathcal{A}^{\mathbb{N}}$ such that*

$$D \subset \bigcup_{n\in\mathbb{N}} B(A_n, \varepsilon_n),$$

where $B(A, \delta) := \{x \in M \ ; \ d(x, A) < \delta\}$.

The above definition was inspired by the small ball property introduced by Behrends and Kadets [2]. We also refer to [1] for quantitative results on Behrends small balls property. Our main motivation was to study sbp for Fréchet spaces, and the following result was the key to do it.

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Proposition 2 ([8]). *Let $T : (M_1, d_1) \to (M_2, d_2)$ be a uniformly continuous map between metric spaces where (M_1, d_1) is in addition complete and let $\mathcal{A} \subset 2^{M_2}$ stable under finite unions. If $T(M_1)$ has the $\mathcal{A}$-sbp then there exists an $r > 0$ such that for all $\varepsilon > 0$ there are $x \in M_1$ and $A \in \mathcal{A}$ with $T(B(x, r)) \subset B(A, \varepsilon)$.*

From now on, if E is a Fréchet space, we denote by $\mathcal{U}_0(E)$ (respectively, $\mathcal{B}(E)$) the family of all absolutely convex and closed 0-neighbourhoods (respectively, bounded sets). If $U \in \mathcal{U}_0(E)$ and $B \in \mathcal{B}(E)$, the corresponding local Banach spaces are denoted by E_U and E_B.

We will need the following slight generalization of Lemma 1 in [8], whose proof is not included since it follows the same lines as in [8].

Lemma 3. *Let $T : E \to F$ be an operator between Fréchet spaces, and let $\mathcal{A} \subset 2^F$ stable under finite unions such that*

(i) *for all $A \in \mathcal{A}$ and $\lambda \geqslant 0$, we have $\lambda A \in \mathcal{A}$,*
(ii) *for all $A \in \mathcal{A}$, $U \in \mathcal{U}_0(F)$, and $x \in E$, there is $B \in \mathcal{A}$ satisfying $T(x) + A \subset B + U$.*

Then $T(E)$ has the $\mathcal{A}$-sbp if and only if there exists $V \in \mathcal{U}_0(E)$ such that for every $U \in \mathcal{U}_0(F)$, there is $A \in \mathcal{A}$ with

$$T(V) \subset A + U.$$

From now on, any family $\mathcal{A} \subset 2^E$ of subsets of a Fréchet space E for which we are considering the sbp, will be supposed to satisfy

(i) $\mathcal{A} \subset \mathcal{B}(E)$, and it is stable under finite unions and finite sums.
(ii) For all $A \in \mathcal{A}$ and $\lambda \geqslant 0$, we have $\lambda A \in \mathcal{A}$.
(iii) $\overline{\bigcup_{A \in \mathcal{A}} A} = E$.

Observe, for $\mathcal{A} \subset 2^F$, that the first and the last conditions imply condition (ii) of Lemma 3. Therefore this Lemma is valid under our assumptions.

Let E be a Fréchet space. Examples for families $\mathcal{A}$ satisfying the above assumptions are the system $\mathcal{E}(E)$ of all finite subsets of E, the system $\mathcal{B}(E)$ of all bounded subsets of E, and the system $\Sigma(E)$ of all sets which are contained in some absolutely convex and $\sigma(E, E')$-compact subset of E. Also, if a bounded absolutely convex set $A_0 \subset E$ is such that $\overline{\operatorname{span}(A_0)} = E$, then the class generated by A_0, defined as $\mathcal{A}(A_0) = \{\lambda A_0 \ ; \ \lambda \geqslant 0\}$ satisfies our assumptions.

An easy consequence of Lemma 3 are the following characterizations of small ball properties, which were implicit in [8].

Lemma 4. *Let E be a Fréchet space and $\mathcal{A}$ a family of bounded sets in E satisfying our assumptions. The following characterizations hold:*

(a) *For every operator T from E into a Banach space X the set $T(E)$ has the $T(\mathcal{A})$-sbp if, and only if,*

$$\forall U \in \mathcal{U}_0(E) \ \ \exists V \in \mathcal{U}_0(E) \ \ \forall \varepsilon > 0 \ \ \exists A \in \mathcal{A} \ : \ V \subset A + \varepsilon U.$$

(b) *For every operator T from any Banach space X into E the set $T(X)$ has the $\mathcal{A}$-sbp if, and only if,*

$$\forall B \in \mathcal{B}(E) \quad \forall U \in \mathcal{U}_0(E) \quad \exists A \in \mathcal{A} \; : \; B \subset A + U.$$

With this at hand, we were able to characterize several properties for Fréchet spaces in terms of small ball properties. We recall them:

A Fréchet space E is *quasinormable* (respectively, *Schwartz*) if, for every $U \in \mathcal{U}_0(E)$, there is $V \in \mathcal{U}_0(E)$ such that, for all $\varepsilon > 0$, there is $A \in \mathcal{B}(E)$ (respectively, $A \in \mathcal{E}(E)$) such that

$$V \subset A + \varepsilon U.$$

Reflexivity can be also equivalently formulated in terms of sets, as in (b) of Lemma 4, so that, E is *Montel* (respectively, *reflexive*) if, for every $B \in \mathcal{B}(E)$ and $U \in \mathcal{U}_0(E)$, there is $A \in \mathcal{E}(E)$ (respectively, $A \in \Sigma(E)$) such that,

$$B \subset A + U.$$

The following result was obtained in [8], which follows from the (implicit) Lemma 4.

Corollary 5 ([8]). *Let E be a Fréchet space. Then*

1. *E is a Schwartz space if, and only if, for all continuous linear maps from E into a Banach space X the set $T(E)$ has the $T(\mathcal{E}(E))$-sbp.*
2. *E is quasinormable if, and only if, for all continuous linear maps from E into a Banach space X the set $T(E)$ has the $T(\mathcal{B}(E))$-sbp.*
3. *E is Montel if, and only if, for every operator T from any Banach space X into E the set $T(X)$ has the $\mathcal{E}(E)$-sbp.*
4. *E is reflexive if, and only if, for every operator T from any Banach space X into E the set $T(X)$ has the $\Sigma(E)$-sbp.*

We are also interested in the *density condition* (DC) which, for a Fréchet space E, can be formulated as (see [3]): There exists a bounded set $A_0 \in \mathcal{B}(E)$ such that, for each $B \in \mathcal{B}(E)$ and for every $U \in \mathcal{U}_0(E)$, there is $\lambda > 0$ such that

$$B \subset \lambda A_0 + U.$$

Taking into account Lemma 4, the (DC) can be characterized in terms of small ball properties.

Corollary 6. *Let E be a Fréchet space. E has the (DC) if, and only if, there exists a bounded absolutely convex subset $A_0 \subset E$ such that, for every operator T from any Banach space X into E the set $T(X)$ has the $\mathcal{A}(A_0)$-sbp.*

2. Three-space-problems

We intend to provide known positive solutions to three-space-problems for Fréchet spaces under a unified formulation of small ball properties. Some concepts about lifting of sets are also needed. Given a Fréchet space E and a closed subspace $F \subset E$, we say that the quotient map $q : E \to E/F$ *lifts bounded sets* if, for each $B \in \mathcal{B}(E/F)$, there is $C \in \mathcal{B}(E)$ such that $B \subset q(C)$. We recall that, by a result of Bonet and Dierolf [4], it suffices that q *lifts bounded sets with closure*, i.e., for each $B \in \mathcal{B}(E/F)$, there is $C \in \mathcal{B}(E)$ such that $B \subset \overline{q(C)}$.

Given $\mathcal{A} \subset \mathcal{B}(E/F)$ and $\mathcal{A}' \subset \mathcal{B}(E)$, we will say that q has the $(\mathcal{A}', \mathcal{A})$*-lifting property with closure* if, for each $A \in \mathcal{A}$, there is $A' \in \mathcal{A}'$ such that $A \subset \overline{q(A')}$.

Theorem 7. *Let E be a Fréchet space, $F \subset E$ a closed subspace, and let $\mathcal{A}_1 \subset \mathcal{B}(F)$, $\mathcal{A}_2 \subset \mathcal{B}(E)$, $\mathcal{A}_3 \subset \mathcal{B}(E/F)$ satisfy our assumptions. If $\mathcal{A}_1 \subset \mathcal{A}_2$, the quotient map $q : E \to E/F$ has the $(\mathcal{A}_2, \mathcal{A}_3)$-lifting property with closure, and the following two properties are satisfied*

- *$T(F)$ satisfies the $T(\mathcal{A}_1)$-sbp for every operator T from F into any Banach space X,*
- *$T(E/F)$ satisfies the $T(\mathcal{A}_3)$-sbp for every operator T from E/F into any Banach space X,*

then $T(E)$ satisfies the $T(\mathcal{A}_2)$-sbp for every operator T from E into any Banach space X.

Proof. Let $U \in \mathcal{U}_0(E)$. By assumption on F, there is $V \in \mathcal{U}(E)$ ($V \subset U$) such that, for every $\lambda, \varepsilon > 0$, there exists $C \in \mathcal{A}_1$ such that

$$\lambda V \cap F \subset C + \varepsilon U.$$

Also, by applying directly the $T(\mathcal{A}_3)$-sbp of $T(E/F)$ with respect to the canonical operator from E/F into the local Banach space associated to $q(V)$, and since $q : E \to E/F$ has the $(\mathcal{A}_2, \mathcal{A}_3)$-lifting property with closure, we find a double sequence $(B_{n,j})_{n,j}$ with $B_{n,j} \in \mathcal{A}_2$ for each $n, j \in \mathbb{N}$, such that

$$E = \bigcup_{n \in \mathbb{N}} \left(B_{n,j} + \frac{1}{2nj} V + F \right), \qquad \forall j \in \mathbb{N}.$$

Fix a double sequence $(\lambda_{n,j})_{n,j}$ of positive numbers such that $B_{n,j} \subset \lambda_{n,j} V$ for all $n, j \in \mathbb{N}$. There is a double sequence $(C_{n,j})_{n,j}$ of elements in $\mathcal{A}_1$ satisfying

$$(2 + \lambda_{n,j}) V \cap F \subset C_{n,j} + \frac{1}{2nj} U,$$

for every $n, j \in \mathbb{N}$. Therefore,

$$V \cap \left(B_{n,j} + \frac{1}{2nj} V + F \right) \subset B_{n,j} + \frac{1}{2nj} V + (2 + \lambda_{n,j}) V \cap F \subset B_{n,j} + C_{n,j} + \frac{1}{nj} U,$$

for each $n, j \in \mathbb{N}$. That is,

$$V \subset \bigcup_{n\in\mathbb{N}} \left(B_{n,j} + C_{n,j} + \frac{1}{nj} U \right), \qquad \forall j \in \mathbb{N}.$$

Let $A_n \in \mathcal{A}_n$ such that $j(B_{k,j} + C_{k,j}) \subset A_n$ for every $k, j \leqslant n$, and for each $n \in \mathbb{N}$. We easily obtain

$$jV \subset \bigcup_{n\in\mathbb{N}} \left(A_n + \frac{1}{n} U \right), \qquad \forall j \in \mathbb{N},$$

and, since V is absorbing, we reach

$$E = \bigcup_{n\in\mathbb{N}} \left(A_n + \frac{1}{n} U \right).$$

This yields the conclusion since, as we observed in [8], a subset $D \subset M$ of a metric space M satisfies the $\mathcal{A}$-small ball property as soon as, for each $\varepsilon > 0$, there are a sequence $(A_n)_n$ in $\mathcal{A}$ and a sequence of positive scalars $(\delta_n)_n$ converging to 0 with $\delta_n < \varepsilon$, $n \in \mathbb{N}$, such that

$$D \subset \bigcup_{n\in\mathbb{N}} B(A_n, \delta_n),$$

because $\mathcal{A}$ is stable under finite unions. ■

As a consequence, we easily obtain that quasinormability and Schwartz properties are three-space properties. We just need to observe that, if E/F is Schwartz, then bounded sets are relatively compact, thus they are lifted. On the other hand, if F is quasinormable, a classical result of De Wilde [6] tells us that q lifts bounded sets.

Corollary 8 ([7]). *Let E be a Fréchet space and $F \subset E$ a closed subspace.*

(a) *If F and E/F are Schwartz, then E is Schwartz.*
(b) *If F and E/F are quasinormable, then E is quasinormable.*

Before proceeding with the next result, we would like to recall that every Fréchet space E satisfies the *strict Mackey condition*, that is, for any bounded set $B \subset E$, there is a bounded absolutely convex and closed subset $C \subset E$ with $B \subset C$ such that the spaces E_C and E induce the same topology on B. This allows us to observe that, given $\mathcal{A} \subset \mathcal{B}(E)$ satisfying our assumptions, $T(X)$ satisfies the $\mathcal{A}$-sbp for every operator T from any Banach space X into E if and only if

$$\forall B \in \mathcal{B}(E) \;\; \exists C \in \mathcal{B}(E) \;\; \forall (\varepsilon_n)_n \subset]0, +\infty[\;\; \exists (A_n)_n \subset \mathcal{A} : \quad \operatorname{span}(B) \subset \bigcup_{n\in\mathbb{N}} (A_n + \varepsilon_n C). \tag{2.1}$$

Theorem 9. *Let E be a Fréchet space, $F \subset E$ a closed subspace, and let $\mathcal{A}_1 \subset \mathcal{B}(F)$, $\mathcal{A}_2 \subset \mathcal{B}(E)$, $\mathcal{A}_3 \subset \mathcal{B}(E/F)$ satisfy our assumptions. If $\mathcal{A}_1 \subset \mathcal{A}_2$, the quotient map $q : E \to E/F$ lifts bounded sets, q has the $(\mathcal{A}_2, \mathcal{A}_3)$-lifting property with closure, and the following two properties are satisfied*

- *$T(X)$ satisfies the $\mathcal{A}_1$-sbp for every operator T from any Banach space X into F,*
- *$T(X)$ satisfies the $\mathcal{A}_3$-sbp for every operator T from any Banach space X into E/F,*

then $T(X)$ satisfies the $\mathcal{A}_2$-sbp for every operator T from any Banach space X into E.

Proof. Fix a basis $(U_n)_n$ of absolutely convex 0-neighbourhoods in E, and let $B \subset E$ be a bounded and absolutely convex set. Since $q(E_B)$ satisfies the $\mathcal{A}_3$-sbp, by the lifting properties of $q : E \to E/F$ and taking into account equation (2.1), there is a double sequence $(B_{n,j})_{n,j} \subset \mathcal{A}_2$ and an absolutely convex bounded set $C \subset E$, $B \subset C$, absorbing the sequence $(B_{n,j})_{n,j}$, such that

$$\operatorname{span}(B) \subset \bigcup_{n\in\mathbb{N}} \left(B_{n,j} + \frac{1}{2j}(C \cap U_n) + F \right), \qquad \forall j \in \mathbb{N}.$$

Fix a double sequence $(\lambda_{n,j})_{n,j} \subset]0, +\infty[$ of such that $B_{n,j} \subset \lambda_{n,j} C$ for all $n, j \in \mathbb{N}$. By hypothesis on F, there is a double sequence $(C_{n,j})_{n,j} \subset \mathcal{A}_1$ satisfying

$$(2 + \lambda_{n,j})C \cap F \subset C_{n,j} + \frac{1}{2j} U_n,$$

for every $n, j \in \mathbb{N}$. Therefore,

$$B \cap \left(B_{n,j} + \frac{1}{2j}(C \cap U_n) + F \right) \subset B_{n,j} + \frac{1}{2j} U_n + (2+\lambda_{n,j})C \cap F \subset B_{n,j} + C_{n,j} + \frac{1}{j} U_n,$$

for each $n, j \in \mathbb{N}$. That is,

$$B \subset \bigcup_{n\in\mathbb{N}} \left(B_{n,j} + C_{n,j} + \frac{1}{j} U_n \right), \qquad \forall j \in \mathbb{N}.$$

Let $A_n \in \mathcal{A}_2$ such that $j(B_{k,j} + C_{k,j}) \subset A_n$ for every $k, j \leqslant n$, and for each $n \in \mathbb{N}$. We easily obtain

$$jB \subset \bigcup_{n\in\mathbb{N}} (A_n + U_n), \qquad \forall j \in \mathbb{N},$$

to conclude that

$$\operatorname{span}(B) \subset \bigcup_{n\in\mathbb{N}} (A_n + U_n).$$

■

The previous Theorem allows us to establish the final (known) result.

Corollary 10. *Let E be a Fréchet space and $F \subset E$ a closed subspace.*

(a) *If F and E/F are Montel, then E is Montel (see [7]).*
(b) *If F and E/F are reflexive, then E is reflexive (see [7]).*
(c) *If $q : E \to E/F$ lifts bounded sets, F has the (DC) and E/F has the (DC), then E has the (DC) (see [5]).*

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