Functiones et Approximatio
XXXIX.1 (2008), 61–70

A REMARK ON THE MÖBIUS FUNCTION

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Dedicated to
Professor Władysław Narkiewicz
for his 70th birthday

Abstract: It is proved that for every positive B there exist real numbers $0 = a_0 < a_1 < \ldots < a_N = 1$ and $\max_{1\le j\le N}(a_{j-1}/a_j) \le \theta < 1$ such that

$$\limsup_{x\to\infty} \frac{1}{\sqrt{x}} \sum_{j=1}^{N} \sum_{\theta a_j x < n \le a_j x} \mu(n) \ge B$$

and

$$\liminf_{x\to\infty} \frac{1}{\sqrt{x}} \sum_{j=1}^{N} \sum_{\theta a_j x < n \le a_j x} \mu(n) \le -B,$$

where $\mu(n)$ denotes the Möbius function.
Keywords: Möbius function, Mertens conjecture, omega estimates

1. Introduction and statement of the Theorem

Let $\mu(n)$ denote the Möbius function, and let us write

$$M(x) = \sum_{n\le x} \mu(n),$$

$$m^- = \liminf_{x\to\infty} \frac{1}{\sqrt{x}} M(x) \quad \text{and} \quad m^+ = \limsup_{x\to\infty} \frac{1}{\sqrt{x}} M(x).$$

The most important unproved conjecture concerning these quantities predicts that

$$m^- = -\infty \quad \text{and} \quad m^+ = \infty. \tag{1.1}$$

In particular it is expected that

$$\limsup_{x\to\infty} \frac{1}{\sqrt{x}} |M(x)| = \infty. \tag{1.2}$$

Mathematics Subject Classification: 11N25, 11N37, 11M26
Partially supported by the KBN grant N N201 1482 33.

The best result in this direction is due to Odlyzko and te Riele [5] who showed that

$$m^- \le -1.009 \quad \text{and} \quad m^+ \ge 1.06$$

disproving in this way the famous Mertens conjecture

$$|M(x)| < \sqrt{x} \quad \text{for} \quad x > 1$$

(see also [6]). Another type of approximation to the above conjectures was discussed in [3]. It was proved there that for every real $a \neq 0$ we have as $x \to \infty$

$$\left|\sum_{n\le x} \mu(n)\right| + \left|\sum_{n\le x} \mu(n)\cos\left(\frac{ax}{n}\right)\right| = \Omega\left(x^{1/2}\log\log\log x\right), \tag{1.3}$$

so that at least one of the sums on the left is very large infinitely often. Observe that if we could pass to the limit as $a \to 0$, then (1.3) would imply

$$M(x) = \Omega(x^{1/2}\log\log\log x),$$

a result much stronger than (1.2).

In this paper we prove the following result.

Theorem 1.1. *For every positive B there exist real numbers*

$$0 = a_0 < a_1 < \ldots < a_N = 1 \tag{1.4}$$

and a real number θ satisfying

$$\max_{1\le j\le N}(a_{j-1}/a_j) \le \theta < 1 \tag{1.5}$$

such that

$$\limsup_{x\to\infty} \frac{1}{\sqrt{x}} \sum_{j=1}^{N} \sum_{\theta a_j x < n \le a_j x} \mu(n) \ge B$$

and

$$\liminf_{x\to\infty} \frac{1}{\sqrt{x}} \sum_{j=1}^{N} \sum_{\theta a_j x < n \le a_j x} \mu(n) \le -B.$$

It is an interesting problem to estimate N in terms of B. Our method of proof gives $N \ll B^2(\log B)^C$ for certain positive C. Sufficiently sharp estimates of this type would have important consequences. For instance $N = o(B)$ easily implies (1.2). Indeed, suppose in contrary, that $M(x) \ll \sqrt{x}$. Then

$$\frac{1}{\sqrt{x}} \sum_{j=1}^{N} \sum_{\theta a_j x < n \le a_j x} \mu(n) = \frac{1}{\sqrt{x}} \sum_{j=1}^{N} (M(a_j x) - M(\theta a_j x)) \ll N.$$

Passing to the limit as $x \to \infty$ over a suitably chosen values of x, and applying Theorem 1.1 we obtain $B \ll N$. If $N = o(B)$ this leads to contradiction, and hence (1.2) holds.

Acknowledgement. The author thanks Alberto Perelli who read the first version of this paper and made a number of valuable remarks.

2. Lemmas

Lemma 2.1. *Suppose the Riemann Hypothesis is true. Then for almost all* $y \in [x, 2x]$, $x \geq 2$, *there is a prime* $p \in [y, y + f(y)\log^2 y]$, *where* $f(y)$ *is any positive function tending to infinity when* $y \to \infty$.

This is a classical result proved by Selberg [7]. Let us remark that 'almost all' in the formulation of the lemma means that the Lebesgue measure of exceptions is $o(x)$ as $x \to \infty$.

Following [2] let us denote by $\mathfrak{A}$ the set of all functions defined on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ by the formula

$$F(z) = \sum_{n=1}^{\infty} a_n e^{iw_n z}, \tag{2.1}$$

and satisfying the following conditions:

1. $0 \leq w_1 < w_2 < \ldots$ are real numbers;
2. $a_n \in \mathbb{C}$, $n = 1, 2, 3, \ldots$;
3. There exists a non-negative integer D such that

$$\sum_{n=2}^{\infty} |a_n| w_n^{-D} < \infty \;;$$

4. There exists $L_0 = L_0(F) \geq 0$ such that the limit

$$P(x) = \lim_{y \to 0^+} \Re F(x + iy)$$

exists for every real $x \geq L_0$ and represents a locally bounded function of $x \in [L_0, \infty)$.
5. For every bounded interval $I \subset (L_0, \infty)$ we have

$$\Re F(x + iy) \ll_I 1$$

for $x \in I$ and $y > 0$.

Note that in [2] condition 5 was erroneously omitted. With this notation we have the following result, which is the basis for the proof of Theorem 1.1.

Lemma 2.2. (See [2], Corollary 2.) *Let* $F \in \mathfrak{A}$. *Then*

$$\liminf_{x \to \infty} P(x) = \inf\nolimits_{z \in \mathbb{H}} \Re F(z)$$

and

$$\limsup_{x \to \infty} P(x) = \sup_{z \in \mathbb{H}} \Re F(z).$$

In order to construct an $F(z)$ suitable for our purposes, we consider subsidiary functions $m(z)$ and $\mathcal{M}(z)$ defined as follows. Let $\zeta(s) = \zeta(\sigma + it)$ denote as usual the Riemann zeta function. The function $m(z)$ is defined for z from the upper half plane by the following formula

$$m(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{1}{\zeta(s)} e^{sz}\, ds. \tag{2.2}$$

The path of integration consists of the half-line $s = -1/2 + it$, $\infty > t \geq 0$, the line segment $[-1/2, 3/2]$, and the half-line $s = 3/2 + it$, $0 \leq t < \infty$. Since $1/\zeta(s)$ is bounded on $\mathcal{C}$, the integral converges absolutely and uniformly for $z \in \mathbb{H}$, and hence represents a holomorphic function on this half-plane. Moreover, for $z \in \mathbb{H}$ we put

$$\mathcal{M}(z) = \int_{z+i\infty}^{z} m(w)\, dw,$$

where the integration is taken along the vertical half-line $w = z + it$, $\infty > t \geq \mathrm{Im}(z)$.

In the case when all non-trivial zeros are simple and $|\zeta'(\rho)| \gg e^{-\varepsilon|\gamma|}$ for every positive ε, we have for $z \in \mathbb{H}$

$$m(z) = \sum_{\gamma > 0} \frac{1}{\zeta'(\rho)} e^{\rho z}$$

and

$$\mathcal{M}(z) = \sum_{\gamma > 0} \frac{1}{\rho\zeta'(\rho)} e^{\rho z}.$$

Basic analytic properties of $m(z)$ were established in [1] and [3]. In particular, it was proved that $m(z)$ admits meromorphic continuation to the whole complex plane with simple poles at logarithms of positive squarefree integers and corresponding residues

$$\mathrm{Res}_{z=\log n} = -\frac{\mu(n)}{2\pi i} \qquad (n \geq 1). \tag{2.3}$$

Moreover, $m(z)$ satisfies the following functional equation

$$m(z) + \overline{m(\overline{z})} = A(z), \tag{2.4}$$

where $A(z)$ is an entire function defined as follows

$$A(z) = -2 \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos\left(\frac{2\pi}{n} e^{-z}\right). \tag{2.5}$$

For real x, we write

$$\mathcal{M}_{\Re}(x) = \lim_{y \to 0^+} \Re\mathcal{M}(x + iy).$$

The limit exists for all x, and we have

$$\mathcal{M}_{\Re}(x) = \frac{1}{2}(\mathcal{M}_{\Re}(x-0) + \mathcal{M}_{\Re}(x+0)). \tag{2.6}$$

Discontinuities occur only at $x = \log n$, where $\mu(n) \neq 0$. We have also the following result, which is implicitly contained in [1]. However, for the sake of completeness, we shall give a detailed proof.

Lemma 2.3. *For real x we have*

$$\mathcal{M}_{\Re}(x) = \frac{1}{2}M_0(e^x) + 1 + H(x), \tag{2.7}$$

where

$$M_0(x) = \frac{1}{2}(M(x-0) + M(x+0)),$$

and H is an entire function which for $z \in \mathbb{C}$ is defined as follows

$$H(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n)!\zeta(2n+1)}(2\pi e^{-z})^{2n}.$$

In particular for $x > 0$ we have

$$\mathcal{M}_{\Re}(x) = \frac{1}{2}M_0(e^x) + 1 + O(e^{-2x}). \tag{2.8}$$

Let us remark that in this paper we do not need as precise formulae as provided by Lemma 2.3. We formulate them in the full generality for the sake of a possible future references.

Proof of Lemma 2.3. Because of (2.6) we can assume without the loss of generality that $x \neq \log n$, $\mu(n) \neq 0$. Let $a < \min(0, x)$, and let us denote by $l(a, x)$ a smooth curve $\tau : [0,1] \to \mathbb{C}$ such that $\tau(0) = a$, $\tau(1) = x$, and $\operatorname{Im}(\tau(t)) > 0$ for $0 < t < 1$. Then using (2.3) and (2.4) we obtain

$$\begin{aligned}
\mathcal{M}(x) &= \mathcal{M}(a) + \int_{l(a,x)} m(z)\,dz \\
&= \mathcal{M}(a) + M(e^x) + \int_{\overline{l(a,x)}} m(z)\,dz \\
&= \mathcal{M}(a) + M(e^x) - \overline{\int_{l(a,x)} m(z)\,dz} - \int_a^x A(t)\,dt \\
&= -\overline{\mathcal{M}(x)} + M(e^x) + 2\mathcal{M}_{\Re}(a) + 2H(x) - 2H(a).
\end{aligned}$$

Hence

$$2\mathcal{M}_{\Re}(x) = M(e^x) + c_a + 2H(x),$$

where

$$c_a = 2\mathcal{M}_{\Re}(a) - 2H(a),$$

and all what remains to be proved is that $c_a = 2$. To this end let us consider the integral

$$I_a = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{e^{as}}{s\zeta(s)}\,ds,$$

where $\mathcal{C} = \mathcal{C}(0,\delta)$ denotes the circle with center 0 and radius $\delta > 0$. Obviously,

$$I_a = \frac{1}{\zeta(0)} = -2.$$

On the other hand we have

$$I_a = \lim_{\eta\to 0^+}\left\{\frac{1}{2\pi i}\int_{\mathcal{C}_-}\frac{e^{(a+i\eta)s}}{s\zeta(s)}\,ds + \frac{1}{2\pi i}\int_{\mathcal{C}_+}\frac{e^{(a-i\eta)s}}{s\zeta(s)}\,ds,\right\}$$

where

$$\mathcal{C}_- = \{\delta e^{i\varphi} : 0 \le \varphi \le \pi\} \quad \text{and} \quad \mathcal{C}_+ = \{\delta e^{i\varphi} : -\pi \le \varphi \le 0\}.$$

Let k be a real number greater than 1, and let $\mathcal{L}_k$ be the contour consisting of the vertical half-line $[-k+i\infty, -k+i]$, the polygon line with vertices $-k+i$, $-1+i$, -1, $-\delta$, the half-circle $-\mathcal{C}_+$, the line segment $[\delta, k]$ and the vertical half-line $[k, k+i\infty]$. For $\eta > 0$ and sufficiently small positive δ we have

$$\frac{1}{2\pi i}\int_{\mathcal{L}_k}\frac{e^{(a+i\eta)s}}{s\zeta(s)}\,ds = \mathcal{M}(a+i\eta).$$

It is easy to show that the integrals along vertical half-lines tend to 0 as $k \to \infty$. Therefore

$$\mathcal{M}(a+i\eta) = \frac{1}{2\pi i}\left(\int_{\mathcal{L}_+} - \int_{\mathcal{C}_+} + \int_{\delta}^{\infty}\right)\frac{e^{(a+i\eta)s}}{s\zeta(s)}\,ds,$$

where $\mathcal{L}_+$ the the infinite polygon line with vertices $-\infty + i$, $-1+i$, -1 and $-\delta$. In the similar way, but working on the lower half-plane we obtain

$$\overline{\mathcal{M}(a+i\eta)} = \frac{1}{2\pi i}\left(-\int_{\mathcal{L}_-} - \int_{\mathcal{C}_-} - \int_{\delta}^{\infty}\right)\frac{e^{(a-i\eta)s}}{s\zeta(s)}\,ds,$$

where $\mathcal{L}_- = \overline{\mathcal{L}_-}$. Adding the above two formulae and passing to the limit as $\eta \to 0$ and then counting residues, we obtain

$$\begin{aligned} 2\mathcal{M}_{\Re}(a) &= \frac{1}{2\pi i}\left(-\int_{\mathcal{L}_-\cup(-\mathcal{L}_+)} - \int_{\mathcal{C}}\right)\frac{e^{as}}{s\zeta(s)}\,ds \\ &= 2 + \sum_{n=1}^{\infty}\frac{e^{-2na}}{2n\zeta'(-2n)} \\ &= 2 + 2H(a). \end{aligned}$$

Consequently $c_a = 2$, and the result follows. ∎

Lemma 2.4. *Suppose $m^- > -\infty$ or $m^+ < \infty$. Then the Riemann Hypothesis is true, all non-trivial zeros of the Riemann zeta function are simple, and moreover*

$$\frac{1}{\zeta'(\rho)} \ll |\gamma|. \tag{2.9}$$

This is well known and classical (see for instance [4], Section 15.1).

Lemma 2.5. *Suppose $m^- > -\infty$ or $m^+ < \infty$. Then the function*

$$F(z) = e^{-z/2}\mathcal{M}(z)$$

belongs to the class $\mathfrak{A}$. More generally, for arbitrary real numbers $b_1, \ldots, b_k, c_1, \ldots,$ c_k the function

$$G(z) = \sum_{n=1}^{k} b_k F(z + c_k)$$

belongs to $\mathfrak{A}$.

Proof. Using Lemma 2.4 we can assume Riemann Hypothesis and simplicity of zeros. For $z \in \mathbb{C}$ we have

$$F(z) = \sum_{\gamma>0} \frac{1}{\rho\zeta'(\rho)} e^{i\gamma z},$$

and hence it is of the form (2.1). Other conditions in the definition of $\mathfrak{A}$ easily follow from described earlier properties of $\mathcal{M}(z)$. According to (2.9), condition 3 is satisfied with $D = 3$. Finally, $G \in \mathfrak{A}$ since $\mathfrak{A}$ is a real vector space which is invariant under the shifts of arguments by real numbers. ∎

3. Proof of the Theorem

We can assume $m^- > -\infty$ or $m^+ < \infty$ since otherwise Theorem 1.1 follows with $N = 1$ and $\theta = 0$. Consequently, using Lemma 2.4, we can assume Riemann Hypothesis and simplicity of zeros.

Let X be sufficiently large and write $L = [\log X]$. By Lemma 2.1 almost all intervals $[x, x + L^3]$, where $X \le x \le 2X$, contain primes. Applying the same lemma for $X/2$ in place of X we see that also almost all of them contain even P_2 almost primes, i.e. numbers of the form $2p$, where p is a prime, $X/2 \le p \le X$. It follows also that almost all intervals $[x, x+L^3]$, where $X \le x \le 2X$, contain both a prime and an even almost prime. Applying the pigeon hole principle we infer that there exists a subinterval $I \subset [X, 2X]$ of length XL^{-4} containing at least $\frac{1}{2}XL^{-7}$ disjoint subintervals of the form $[x, x + L^3]$ containing both a prime and an even almost prime. Applying the pigeon hole principle once more we infer that there are $\gg XL^{-10}$ disjoint subintervals $[x, x + L^3] \subset I$ containing a prime p and an even almost prime $2q$, with a fixed absolute value of the difference $|p-2q| = h$ for certain

$h \leq L^3$. Consequently, it is easy to see that there exists $X \leq Y \leq 2X - XL^{-4}$ and a sequence of integers

$$Y \leq n_1 < n_2 < \ldots < n_N \leq Y + XL^{-4}$$

satisfying the following properties:

$$n_j - n_{j-1} \geq L^3 \quad \text{for every} \quad j = 1, \ldots, N; \tag{3.1}$$

$$N \gg XL^{-10}; \tag{3.2}$$

$$\mu(n_j) = \mu(n_{j'}) \quad \text{for} \quad 1 \leq j, j' \leq N; \tag{3.3}$$

$$\mu(n_j)\mu(n_j - h) = -1 \quad \text{for} \quad j = 1, \ldots, N, \tag{3.4}$$

where h is fixed and $\leq L^3$, and we put $n_0 = 0$.

Let $\omega = 1/(2Y)$ and define $F(z)$ for z from the upper half-plane by the following formula

$$F(z) = e^{-z/2} \sum_{j=1}^{N} \left(\mathcal{M}(z + \log \frac{n_j}{n_N}) - \mathcal{M}(z + \log \frac{n_j}{n_N} - \omega) \right).$$

According to Lemma 2.5, $F(z)$ belongs to the class $\mathfrak{A}$. We put

$$a_j = \frac{n_j}{n_N}, \qquad (j = 0, \ldots, N) \quad \text{and} \quad \theta = e^{-\omega}$$

(recall that $n_0 = 0$). These numbers obviously satisfy (1.4), and for sufficiently large X we have using (3.1)

$$\begin{aligned} \frac{a_{j-1}}{a_j} &= \frac{n_{j-1}}{n_j} \leq \frac{n_j - L^3}{n_j} \\ &< 1 - \frac{L^3}{2Y} < e^{-1/(2Y)} = \theta < 1, \end{aligned}$$

and consequently (1.5) holds as well. By (2.8), for real $x \to \infty$, we have

$$\begin{aligned} 2\Re F(x) &= e^{-x/2} \sum_{j=1}^{N} \left(M_0(e^x \frac{n_j}{n_N}) - M_0(e^{x-\omega} \frac{n_j}{n_N}) + O(e^{-2x}) \right) \\ &= e^{-x/2} \sum_{j=1}^{N} \sum_{\theta a_j e^x < n \leq a_j e^x} \mu(n) + o(1). \end{aligned} \tag{3.5}$$

Hence, using Lemma 2.2, we see that the assertion of Theorem 1.1 will follow if we find two real numbers x_1 and x_2 both being regular points of $F(z)$ such that $|\Re(F(x_j))| \geq B/2$, $j = 1, 2$, and $\Re(F(x_1))\Re(F(x_2)) < 0$.

Let us put $x_1 = \log n_N + \omega/2$. Then for every $j = 1, 2, \ldots, N$ we have

$$a_j e^{x_1} = n_j e^{\omega/2} = n_j \left(1 + \frac{1}{4Y} + O\left(\frac{1}{X^2}\right) \right) = n_j + \frac{n_i}{4Y} + O(\frac{1}{X})$$

and similarly

$$\theta a_j e^{x_1} = n_j e^{-\omega/2} = n_j - \frac{n_i}{4Y} + O(\frac{1}{X}).$$

Since for large X

$$0 < \frac{n_j}{4Y} + O\left(\frac{1}{X}\right) < 1,$$

we have

$$n_j - 1 < \theta a_j e^{x_1} < n_j < a_j e^{x_1} < n_j + 1. \tag{3.6}$$

Moreover, let us put $x_2 = x_1 - h/Y$. Then

$$\begin{aligned} a_j e^{x_2} = a_j e^{x_1} e^{-h/Y} &= n_j \left(1 + \frac{1}{4Y} + O\left(\frac{1}{X^2}\right)\right)\left(1 - \frac{h}{Y} + O\left(\frac{L^6}{X^2}\right)\right) \\ &= n_j - h + \frac{n_j}{4Y} + O\left(\frac{1}{L}\right), \end{aligned}$$

and similarly

$$\theta a_j e^{x_2} = n_j - h - \frac{n_j}{4Y} + O\left(\frac{1}{L}\right).$$

Consequently, for large X we have

$$n_j - h - 1 < \theta a_j e^{x_2} < n_j - h < a_j e^{x_2} < n_j - h + 1. \tag{3.7}$$

Applying (3.5), (3.6), (3.3) and (3.2) we obtain

$$\begin{aligned} \mu(n_1)\Re F(x_1) &= \mu(n_1) e^{-x_1/2} \sum_{j=1}^{N} \mu(n_j) + o(1) \\ &= e^{-x_1/2} N + o(1) \gg X^{1/2} L^{-10}. \end{aligned}$$

Similarly, but using (3.7) in place of (3.6) we prove

$$\mu(n_1 - h)\Re F(x_2) \gg X^{1/2} L^{-10}.$$

Hence

$$|\Re F(x_j)| \geq B/2$$

for $j = 1, 2$ if X is large enough. Moreover, because of (3.4), we have

$$\Re F(x_1)\Re F(x_2) < 0,$$

and Theorem 1.1 follows.

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Received: 28 March 2007