Functiones et Approximatio
XXXVII.1 (2007), 213–231

THE INTEGER POINTS IN A PLANE CURVE

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Dedicated to Professor Jean-Marc Deshouillers
on the occasion of his 60th birthday

Abstract: Bombieri and Pila gave sharp estimates for the number of integer points (m,n) on a given arc of a curve $y=F(x)$, enlarged by some size parameter M, for algebraic curves and for transcendental analytic curves. The transcendental case involves the maximum number of intersections of the given arc by algebraic curves of bounded degree. We obtain an analogous result for functions $F(x)$ of some class C^k that satisfy certain differential inequalities that control the intersection number. We allow enlargement by different size parameters M and N in the x- and y-directions, and we also estimate integer points close to the curve, with

$$\left|n-NF\left(\frac{m}{M}\right)\right| \leqslant \delta,$$

for δ sufficiently small in terms of M and N.

As an appendix we obtain a determinant mean value theorem which is a quantitative version of a linear independence theorem of Pólya.
Keywords: determinant mean value theorem, interpolation, polynomial, Grobner basis.

1. Introduction

Let $f(x)$ be a real function of some class C^k (k continuous derivatives) on an interval I of length M. We would like to count, or to estimate from above, the number R of pairs of integers m, n with

$$|n-f(m)| \leqslant \delta,$$

where δ is a small positive number, or zero. Often $f(x)=NF(x/M)$, where the function $F(x)$ is independent of the size parameters M and N. For δ large we expect an asymptotic formula

$$R = 2\delta M + O(E), \tag{1.1}$$

2000 Mathematics Subject Classification: primary 11P21, secondary 11J54.

where the expression E does not depend on δ. For δ small, R may be zero, and we can expect only an upper bound

$$R = O(E), \tag{1.2}$$

where E may depend on δ, but E will not tend to zero as $\delta \to 0$, because the case $F(x) = \sqrt{x}$ gives many integer points for infinitely many pairs of values of M and N. The classical results have $E = (MN)^{1/3}$ in (1.1) (Van der Corput [15]) for class C^2, and $E = M^{2/3}$ in (1.2) when $M = N$, $\delta = 0$, $f(x)$ of class C^2, even to a precise constant in the upper bound (Jarník [10]). The function $f(x)$ must satisfy non-triviality conditions: certain expressions in the derivatives of $f(x)$ are non-zero or bounded away from zero. Large δ results of the type (1.1) are discussed in [4]. Swinnerton-Dyer [14] obtained (1.2) with $E = (MN)^{3/10+\epsilon}$ (for any $\epsilon 0$) in the case $M = N$, $\delta = 0$; the result can be extended to certain cases with $M \neq N$, δ small (see [13], [4], [5]).

Bombieri and Pila [1] considered (1.2) with the function $F(x)$ fixed, $\delta = 0$, $M = N$, as $N \to \infty$. For $F(x)$ an algebraic function for which the curve $y = F(x)$ has degree d, they obtained $E = M^{1/d+\epsilon}$ in (1.2). Secondly, for $F(x)$ a transcendental analytic function, and d any positive integer, they obtained

$$E = B(d)C(F)M^{\frac{8}{3(d+3)}}, \tag{1.3}$$

where $C(F)$ is constructed from upper bounds for the derivatives of $F(x)$, whilst $B(d)$ is the Bombieri-Pila intersection number, the maximum number of intersections of the curve $y = F(x)$ with any algebraic curve of degree at most d. The intersection number $B(d)$ is finite, but $B(d) \geqslant d(d+3)/2$, with equality for the case $F(x) = x^{\alpha}$, α irrational. Otherwise $B(d)$ is an ineffective constant, and we cannot deduce a result uniform in some family of curves. However, since d can be arbitrarily large in (1.3), we have (1.2) with $E = A(F)M^{\epsilon}$, where the constant $A(F)$ depends ineffectively on ϵ and on the function $F(x)$. Thirdly, for $F(x)$ of class C^k with finite k, they obtained

$$E = C(F)M^{\frac{1}{2}+\frac{8}{3(d+3)}+\epsilon}, \tag{1.4}$$

provided that $k \geqslant d(d+3)/2$. The cost of removing $B(d)$ from (1.3) is the large factor $M^{1/2+\epsilon}$.

In this paper we study the Bombieri-Pila method in its first non-trivial case $k = 5$, $d = 2$. We allow $M \neq N$, and δ non-zero, but small. Subject to four non-vanishing conditions, which imply $B(2) = 5$, we obtain (1.2) with

$$E = C(F)(MN)^{4/15},$$

in agreement with (1.3). Our factor $C(F)$ also involves lower bounds for certain combinations of derivatives. It is natural to expect lower bound conditions, because the curve can also be written as $x = f^{-1}(y)$, but an upper bound for the derivative of $f^{-1}(y)$ is a lower bound for $f'(x)$. If the non-vanishing conditions are relaxed to allow finitely many zeros, then we must have $\delta = 0$, $M = N$, and

$$E = A(F)M^{8/15}$$

with a constant $A(F)$ depending non-uniformly on the function $F(x)$.

We also obtain results of the strength (1.3) for $d \geqslant 3$, $k = d(d+3)/2$, but the $k - d$ non-vanishing conditions have been left in determinant form, and not simplified or factorised.

Theorem 1. *Let $f(x)$ be a real function, five times continuously differentiable on an interval I of length $M \geqslant 1$, for which $f''(x)$, $f^{(3)}(x)$,*

$$G(f,x) = 4f^{(3)}(x)^2 - 3f''(x)f^{(4)}(x), \tag{1.5}$$

$$H(f,x) = 40f^{(3)}(x)^3 - 45f''(x)f^{(3)}(x)f^{(4)}(x) + 9f''(x)^2 f^{(5)}(x) \tag{1.6}$$

do not vanish, and whose derivatives satisfy the inequalities

$$\frac{|f^{(r)}(x)|}{r!} \leqslant \frac{C^{r+1}N}{M^r} \tag{1.7}$$

for $r = 1, \ldots, 5$,

$$\frac{|f^{(r)}(x)|}{r!} \geqslant \frac{N}{C^{r+1}M^r} \tag{1.8}$$

for $r = 2, 3$, and

$$\frac{|G(f,x)|}{3!^2} \geqslant \frac{4N^2}{C^8 M^6}, \tag{1.9}$$

for some parameters $C \geqslant 1$, $N \geqslant 1$. Let S be a set of R integer points (m_i, n_i) with m_i on I, $m_1 < m_2 < \ldots < m_R$ with $n_i = f(m_i)$. Then

$$R \leqslant 5\left(C^{79}M^4N^4\right)^{1/15} + 5. \tag{1.10}$$

Corollary. *Let $F(x)$ be a real function, five times continuously differentiable on an interval J of length 1, for which $F''(x)$ is non-zero, and $F^{(3)}(x)$, $G(F,x)$ and $H(F,x)$ each have finitely many zeros, and no two of $F^{(3)}(x)$, $G(F,x)$ and $H(F,x)$ have a common zero. Let $f(x) = MF(x/M)$, $I = MJ$, and let the set S be as in Theorem 1. Then*

$$R \leqslant A(F)M^{8/15}, \tag{1.11}$$

where the constant $A(F)$ depends on the function $F(x)$.

The condition (1.7) is the scaling law for $f(x) = NF(x/M)$, where $F(x)$ and its derivatives are bounded. The non-vanishing of $f''(x)$ implies $B(1) = 2$. The differential equation $H(f,x) = 0$ is satisfied by all algebraic curves of degree at most two. Pila [11] showed that the non-vanishing of $H(f,x)$ implies $B(2) = 5$. The expression $H(f,x)$ arises as a determinant, unchanged under rotations of the coordinate axes. The proof of Theorem 1 requires lower bounds for a nested sequence of minor determinants. We break symmetry by considering the principal minors; the conditions become (1.8) for $r = 3$ and (1.9). To obtain a result for δ non-zero, we require a lower bound for the determinant $H(f,x)$ as well.

Theorem 2. *Let $f(x)$ be a real function, five times continuously differentiable on an interval I of length $M \geqslant 1$, whose derivatives satisfy the inequalities* (1.7) *for* $r = 1, \ldots, 5$, (1.8) *for* $r = 2, 3$, (1.9) *and*

$$\frac{|H(f,x)|}{3!^3} \geqslant \frac{20N^3}{C^{12}M^9}, \tag{1.12}$$

for some parameters $C \geqslant 1$, $N \geqslant 1$. Let $\delta < \frac{1}{2}$, and let S be a set of R integer points (m_i, n_i) with m_i on I, $m_1 < m_2 < \ldots < m_R$, and

$$|n_i - f(m_i)| \leqslant \delta.$$

Then there is a constant A (not depending on C, M, N, I or the function $f(x)$) for which

$$R \leqslant A \left(1 + \frac{\delta C^6 M^5}{N}\right)^{11/75} \left(C^{79} M^4 N^4\right)^{1/15}. \tag{1.13}$$

The coefficient of δ in (1.13) is very large, so the bound becomes useless even for $\delta = 1/M$. Large values of δ permit the existence of major arcs, sets of six or more consecutive integer points close to the curve which lie on a conic (an algebraic curve of degree two). When δ is large, a good estimate requires four steps.

A. Spacing on the minor arcs: six consecutive integer points that lie on a conic cannot be close together.
B. A bound for the length of major arcs.
C. The arithmetic structure of major arcs.
D. A repulsion lemma: on each side of a major arc, there is a gap containing no integer points.

The papers [7] and [8] carry out this programme in an analogous problem of rational points close to a curve. The lemmas of this paper form part of step A, but it is not clear whether carrying out the whole programme would greatly extend the range of δ in which Theorem 2 is better than the results of [5], based on the case $d = 1$, $k = 3$ with the Swinnerton-Dyer refinement. A Swinnerton-Dyer refinement would be possible in this paper if we could show that the determinant in Lemma 1 is usually large.

To state the non-vanishing conditions in the general case $d \geqslant 1$, we list the monomials $x^i y^j$ with $0 \leqslant i \leqslant d$, $0 \leqslant j \leqslant d - i$ lexicographically by "$x^i y^j$ precedes $x^g y^h$ if $j < h$ or if $j = h$, $i < g$". The number of monomials is $(d+1)(d+2)/2 = k+1$. Let z_{j-1} denote the j-th monomial, and let $z_{j-1}^{(i)}$ denote its i-th derivative as a function of x. Let $E(x,y)$ be the determinant

$$E(x,y) = \det \left(z_{d+j}^{(d+i)}\right)_{(k-d)\times(k-d)}. \tag{1.14}$$

Each entry in the determinant $E(x,y)$ is a polynomial in x, y, and the derivatives $d^i y / dx^i$. For $r = 1, \ldots, k-d$ we write $E_r(x,y)$ for the principal minor determinant in which i, j run from 1 to r.

Theorem 3. *Let d be a positive integer, and let $k = d(d+3)/2$. Let $F(x)$ be a real function, k times continuously differentiable on an interval J of length 1. Let $M \geqslant 1$ and $N \geqslant 1$ be parameters, and let $f(x) = NF(x/M)$. Let S be a set of R ordered pairs of integers (m, n) with $m/M \in J$, and*

$$|n - f(m)| \leqslant \delta. \tag{1.15}$$

(i). *Suppose that $\delta = 0$, so $n = f(m)$ in* (1.15). *Suppose that the functional determinant $E(x, F(x))$ does not vanish on the interior of J, and that the principal minors $E_r(x, F(x))$ are bounded by*

$$0 < a_r \leqslant |E_r(x, F(x))| \leqslant b_r \tag{1.16}$$

for $r = 1, \dots, k-d-1$, and

$$0 < |E_{k-d}(x, F(x))| \leqslant b_{k-d}. \tag{1.17}$$

Then

$$R \leqslant C(F)(MN)^{\frac{4}{3(d+3)}},$$

where $C(F)$ is a constant constructed from d and the numbers $a_1, \dots a_{k-d-1}$, $b_1, \dots, b_{k-d}$.

(ii). *Suppose that the functional determinant $E(x, F(x))$ and all its principal minors $E_r(x, F(x))$ satisfy* (1.16) *for $r = 1, \dots, k-d$ (this includes $E(x, F(x)) = E_{k-d}(x, F(x))$). Then*

$$R \leqslant C(F)\left(1 + \frac{\delta M^k}{N}\right)^{\frac{3d+5}{3(d+3)k}} (MN)^{\frac{4}{3(d+3)}},$$

where $C(F)$ is a constant constructed from d and the numbers $a_1, \dots, a_{k-d}, b_1, \dots,$ b_{k-d}, and $c_1, \dots, c_k$, where $c_r = \max|F^{(r)}(x)|$.

The determinants in Theorem 3 are complicated, but in the case of a monomial function $y = Ax^s$, they are of the form $AP(s)x^{L(s)}$, where $P(s)$ is a polynomial with rational coefficients and $L(s)$ is a linear function with integer coefficients; this calculation is done in section 4 of [7]. The only way for the determinant to vanish is for $P(s)$ to vanish, so that the determinant has to vanish identically in x. This is the condition for x and y to satisfy an algebraic equation of degree d (with certain coefficients zero if the first determinant that vanishes identically is one of the minors). For given degree d there is a finite set of rational values of s that must be excluded. For $d = 2$ the values $s = -1$, 0, 1, 2, 1/2 correspond to algebraic curves of degree at most two, hyperbola, straight line, or parabola. For a monomial function every value of s for which some minor determinant vanishes identically also makes the big determinant $E(x, F(x))$ vanish identically.

There is some overlap between the aims and methods of this paper and those of Pila [11]. Pila finds a determinant condition for $B(d)$ to take its minimum value, calling such a curve 'd-averse'. Pila takes the viewpoint of projective geometry, which is non-local. In [11] Pila uses an indirect argument to replace the constant $C(F)$ by a power of the parameter M.

Our approach is motivated by the affine question of integer points lying within a distance δ of a given curve. The Bombieri-Iwaniec method for estimating Van der Corput exponential sums (see [4]) involves two spacing problems for the coefficient vectors of Taylor polynomials. The First Spacing Problem concerns the number of integer points close to a 4-manifold in real affine 8-space, one of the real components of the intersection of four algebraic varieties. The Second Spacing Problem concerns integer points close to one of a family of 'resonance curves' (see [9]), which are sections of a 7-manifold in real affine 10-space. These curves are not algebraic except in special cases.

Integer points close to a surface are usually treated by considering the surface as a family of curves. The constant $C(F)$ varies, and we require an uniform upper bound for $C(F)$ over the family. Approximations require local conditions. Sometimes non-local projective arguments can be used as well, as in [3].

As an appendix we prove a mean value theorem for the Wronskian determinant of derivatives of n functions at n points, which is used in [7] and [8], as well as in this paper and in subsequent investigations. Pólya states the corresponding non-vanishing result as Theorem V of [12] , but we require explicit estimates.

This paper forms part of INTAS research project 03-51-5070, Analytic and Combinatoric Methods in Number Theory and Geometry. I would like to thank Professor Sargos and the Université Henri Poincaré for the invitation to Nancy, where much of this paper was written.

2. Inequalities for determinants

We start by expressing certain functional determinants in terms of Vandermonde determinants

$$V(x_1,\ldots,x_n)=\det\left(x_i^{j-1}\right)_{n\times n}=\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}(x_j-x_i).$$

The underlying method is explained in the Appendix. Let $\Delta(x_1,\ldots,x_6,y_1,\ldots,y_6)$ be the 6×6 determinant whose i-th row is

$$(1,x_i,x_i^2,y_i,x_iy_i,y_i^2).$$

Lemma 1. *The determinant $\Delta(x_1,\ldots,x_6,y_1,\ldots,y_6)$ is zero if and only if the six points $(x_1,y_1),\ldots,(x_6,y_6)$ lie on a conic section.*

Proof. We write the equation of a conic as

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The determinant is the eliminant of the coefficients a, b, c, f, g, h. ■

We want to estimate the determinant

$$F(x_1, \ldots, x_6) = \Delta(x_1, \ldots, x_6; f(x_1), \ldots, f(x_6)).$$

This is a special case of the functional determinant

$$D(f_1(x), \ldots, f_n(x); x_1, \ldots, x_n) = \det\left(f_j(x_i)\right)_{n\times n}$$

considered in the Appendix.

To simplify the statement of Lemma 2, we write $G(x)$ and $H(x)$ for the expressions $G(f,x)$ and $H(f,x)$ in (1.5) and (1.6).

Lemma 2. *Suppose that $f(x)$ is a real function five times continuously differentiable on an interval I, with $f^{(3)}(x)$ and $G(x)$ non-zero. Let $a_1 < a_2 < \ldots < a_6$ be points of I. Then there are points a_{ij} in $a_1 < a_{ij} < a_6$ for which*

$$\begin{aligned}&\frac{F(a_1, \ldots, a_6)}{V(a_1, \ldots, a_6)} \\ &\quad = \frac{2f''(a_{11})f^{(3)}(a_{11})f^{(3)}(a_{13})f^{(3)}(a_{23})f^{(3)}(a_{33})G(a_{12})G(a_{22})H(a_{11})}{3!4!5!f^{(3)}(a_{12})^2 f^{(3)}(a_{22})^2 G(a_{11})^2}.\end{aligned} \tag{2.1}$$

Proof. We write $g(x) = xf(x)$, $h(x) = f(x)^2$, so that

$$F(x_1, \ldots, x_6) = D(1, x, x^2, f(x), g(x), h(x); x_1, \ldots, x_6).$$

Repeated use of Lemma A2 of the Appendix gives

$$\begin{aligned}\frac{F(a_1, \ldots, a_6)}{V(a_1, \ldots, a_6)} &= \frac{2D(1, x, f', g', h'; b_1, \ldots, b_5)}{5!V(b_1, \ldots, b_5)} \\ &= \frac{2D(1, f'', g'', h''; c_1, \ldots, c_4)}{4!5!V(c_1, \ldots, c_4)} \\ &= \frac{2D(f^{(3)}, g^{(3)}, h^{(3)}; d_1, d_2, d_3)}{3!4!5!V(d_1, d_2, d_3)}\end{aligned}$$

for some $b_1, \ldots, b_5$, $c_1, \ldots, c_4$ and d_1, d_2, d_3 in the open interval (a_1, a_6). We apply Lemma A3 of the Appendix with $n = 3$. The subsidiary determinants are

$$\begin{aligned}E_1(x) &= f^{(3)}(x), \\ E_2(x) &= \begin{vmatrix} f^{(3)}(x) & g^{(3)}(x) \\ f^{(4)}(x) & g^{(4)}(x) \end{vmatrix} = \begin{vmatrix} f^{(3)}(x) & 3f''(x) \\ f^{(4)}(x) & 4f^{(3)}(x) \end{vmatrix} = G(x), \\ E_3(x) &= \begin{vmatrix} f^{(3)}(x) & g^{(3)}(x) & h^{(3)}(x) \\ f^{(4)}(x) & g^{(4)}(x) & h^{(4)}(x) \\ f^{(5)}(x) & g^{(5)}(x) & h^{(5)}(x) \end{vmatrix} = \begin{vmatrix} f^{(3)}(x) & 3f''(x) & 0 \\ f^{(4)}(x) & 4f^{(3)}(x) & 6f''(x)^2 \\ f^{(5)}(x) & 5f^{(4)}(x) & 20f''(x)f^{(3)}(x) \end{vmatrix} \\ &= 2f''(x)H(x).\end{aligned}$$

Theorem A1 of the Appendix gives

$$\begin{aligned}&\frac{D(f^{(3)}, g^{(3)}, h^{(3)}; d_1, d_2, d_3)}{V(d_1, d_2, d_3)}\\ &\quad= \frac{1}{2}\prod_{r=1}^{3}\prod_{i=1}^{r}\frac{E_{n-r-1}(a_{ir})E_{n-r+1}(a_{ir})}{E_{n-r}(a_{ir})^2}\\ &\quad= \frac{E_1(a_{11})E_1(a_{13})E_1(a_{23})E_1(a_{33})E_2(a_{12})E_2(a_{22})E_3(a_{11})}{2E_1(a_{12})^2E_1(a_{22})^2E_2(a_{11})^2}.\end{aligned}$$

We deduce the result of the Lemma. ■

Proof of Theorem 1. Under the hypotheses of Theorem 1, each factor on the right of (2.1) of Lemma 2 is non-zero. Hence the determinant $F(x_1, \ldots, x_6)$ is non-zero. By Lemma 1 no six distinct points on the curve $y = f(x)$ lie on a conic, and the curve is 2-averse, with $B(2) = 5$. Substituting the bounds (1.7), (1.8) and (1.9) of Theorem 1, we have

$$\begin{aligned}&\frac{|F(x_1, \ldots, x_6)|}{V(x_1, \ldots, x_6)}\\ &\leqslant \frac{2}{3!4!5!}\frac{2C^3N}{M^2}\left(\frac{3!C^4N}{M^3}\right)^4\left(\frac{3!N}{C^4M^3}\right)^{-4}\left(\frac{2!3!4!C^8N^2}{M^6}\right)^2\left(\frac{3!4!N^2}{C^8M^6}\right)^{-2}\frac{6.3!^25!C^{12}N^3}{M^9}\\ &= \frac{24C^{79}N^4}{M^{11}}.\end{aligned}$$

Now we take $x_1 = m_k$, $x_2 = m_{k+1}$, $\ldots$, $x_6 = m_{k+5}$, where $1 \leqslant k \leqslant R - 5$. Let $L = x_6 - x_1$. The Vandermonde determinant is a product of 15 factors, each at most L, with

$$(x_2 - x_1)(x_3 - x_2)(x_4 - x_3)(x_5 - x_4)(x_6 - x_5) \leqslant \frac{L^5}{5^5}.$$

Since $|F(x_1, \ldots, x_6)|$ is a positive integer, we have

$$1 \leqslant |F(x_1, \ldots, x_6)| \leqslant \frac{24C^{79}L^{15}N^4}{5^5M^{11}} \leqslant \frac{C^{79}L^{15}N^4}{M^{11}}, \tag{2.2}$$

which implies

$$m_{k+5} - m_k = L \geqslant L_0 = \frac{M}{(C^{79}M^4N^4)^{1/15}}. \tag{2.3}$$

The numbers m_1, $\ldots$, m_{k+1} lie in an interval of length M, and so

$$R \leqslant 5\left(\frac{M}{L_0} + 1\right), \tag{2.4}$$

which is the result of Theorem 1. ■

Proof of the Corollary to Theorem 1. We subdivide the interval J. First we take the zeros of $H(f,x)$ as the endpoints of subintervals. The zeros of $F^{(3)}(x)$ and of $G(F,x)$, if any, are at internal points of these subintervals. We consider a neighbourhood of a zero of $F^{(3)}(x)$. Under a linear change of coordinates

$$X = ax + by, \qquad Y = cx + dy, \tag{2.6}$$

with non-zero determinant $D = ad - bc$, we have

$$\begin{aligned}
\frac{d^2Y}{dX^2} &= \frac{DF''}{(a+bF')^3},\\
\frac{d^3Y}{dX^3} &= \frac{aDF^{(3)} + bD(F'F^{(3)} - 3F''^2)}{(a+bF')^5},\\
\frac{d^4Y}{dX^4} &= \\
&\frac{a^2DF^{(4)} + abD(2F'F^{(4)} - 10F''F^{(3)}) + b^2D(F'^2F^{(4)} - 10F'F''F^{(3)} + 15F''^2)}{(a+bF')^7}.
\end{aligned}$$

Let $Y = \Phi(X)$. We can verify that at a point where $F''(x) \neq 0$, $F^{(3)}(x) = 0$, and $G(F,x) \neq 0$, that $\Phi^{(3)}(X)$ and $G(\Phi,X)$ are non-zero for general choices of a, b, c and d. Similarly at a point where $F''(x) \neq 0$, $F^{(3)} \neq 0$, but $G(F,x) = 0$, that $\Phi^{(3)}(X)$ and $G(\Phi,X)$ are non-zero for general choices of a, b, c and d. In fact in both cases, if

$$a = d = r\cos\theta, \qquad b = -c = r\sin\theta, \tag{2.7}$$

then $\Phi^{(3)}(X)$ and $G(\Phi,X)$ are non-zero for general choices of θ, independently of r by homogeneity.

For each zero $x = \xi$ of $F^{(3)}(x)$ or of $G(F,x)$, we choose a rotation θ so that $\tan\theta$ is a rational number b/a, where a and b are integers. The matrix entries c and d are given by (2.7). Under the change of coordinates (2.6) we have $\Phi^{(3)}(X)$ and $G(\Phi,X)$ non-zero on a neighbourhood of $x = \xi$. We choose a subinterval J' of J containing ξ so that $\Phi^{(3)}(X)$ and $G(\Phi,X)$ are bounded away from zero on J'. There is a constant C' such that the function $\Phi(X)$ satisfies (1.7), (1.8) and (1.9) with $M = N = 1$. Without calculation, we know that $\Phi''(X)$ does not vanish on J', because the vanishing of $\Phi''(X)$ is the condition for the curve to have an inflexion, a point where there is better-than-usual approximation by a straight line, and this property does not depend on the coordinate system. Similarly the vanishing of $H(f,x)$ is the condition for the curve to have a better-than-usual approximation by an algebraic curve of degree at most two, and this property does not depend on the coordinate system. We deduce that $H(\Phi,X)$ does not vanish on J'.

For (m,n) on MJ', the integer point (k,ℓ) with $K = am+bn$, $\ell = -bm+an$ has $\ell = K\Phi(k/K)$ with $K = M\sqrt{a^2+b^2}$. This device is used in the "Gauss circle problem" of lattice points inside a curve, as in [4] chapter 18. We estimate the (possibly larger) number of integer points on the curve $Y = N\Phi(X/M)$, using Theorem 1 with M, N replaced by K, C replaced by C'.

In this process J is dissected into a finite partition by subintervals, J_1, $\ldots, J_T$ say. We have dealt with 'bad' subintervals $J_i = J'$ on which one of $F^{(3)}(x)$ or $G(F,x)$ vanishes, but not both. On the remaining 'good' subintervals $F^{(3)}(x)$ and $G(F,x)$ do not vanish, and $H(F,x)$ can vanish only at the endpoints. There is a constant C_i such that (1.7), (1.8) and (1.9) hold with $M = N = 1$, $C = C_i$. We estimate the number of integer points on the curve $y = MF(x/m)$ with x on J_i using Theorem 1. The partition, the coordinate changes, and the numbers C_i do not depend on the enlargement M, so we can collect terms to obtain (1.11). The partition argument also gives a bound for the Bombieri-Pila intersection number $B(2)$ for the curve $y = F(x)$, with x on I, as

$$B(2) \leqslant 5T + T + 1. \qquad \blacksquare$$

Proof of Theorem 2. We used a local scaling argument to prove Theorem 1. To prove Theorem 2 for δ very small, we add a perturbation argument. To extend the range of δ further, we prepare by subdividing the set of integer points, and we use a divisibility argument.

The perturbation principle is that if $g(x)$ is a function with

$$\frac{|g^{(r)}(x)|}{r!} \leqslant \frac{\epsilon N}{C^{r+1}M^r} \tag{2.8}$$

for $r = 2, \ldots, 5$, then $h(x) = f(x) - g(x)$ satisfies conditions of the type (1.7), (1.8) and (1.9), with C replaced by a larger constant

$$C' = C + O(\epsilon) \leqslant 2C$$

where ϵ is sufficiently small, for example $\epsilon = 1/100$.

We consider six consecutive integer points, putting $x_1 = m_k, \ldots, x_6 = m_{k+5}$ as in the proof of Theorem 1. Let $f(x_j) = \ell_j + \delta_j$, where $\ell_j = n_{k+j-1}$ is an integer, and $|\delta_j| \leqslant \delta$. The interpolation polynomial

$$g(x) = \sum_{i=1}^{6} \delta_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

takes the values $\delta_1, \ldots, \delta_6$ at $x = x_1, \ldots, x_6$. Let $L = x_6 - x_1$, $d = \min(x_{i+1} - x_i)$. Then for $x_1 \leqslant x \leqslant x_6$ and $r = 0, \ldots, 5$ we have

$$\begin{aligned} |g^{(r)}(x)| &\leqslant \frac{\delta}{d^5}\left(\frac{1}{5!} + \frac{1}{4!} + \frac{1}{3!2!} + \frac{1}{2!3!} + \frac{1}{4!} + \frac{1}{5!}\right) \max_i \left| \frac{d^r}{dx^r} \prod_{j \neq i} (x - x_j) \right| \\ &= \frac{4\delta}{15 d^5} \frac{5! L^{5-r}}{(5-r)!} \leqslant \frac{8\delta r! L^{5-r}}{3d^5}. \end{aligned} \tag{2.9}$$

If

$$d \geqslant Q(\epsilon) = \left(\frac{8C^6\delta}{3\epsilon N}\right)^{1/5} M, \tag{2.10}$$

then (2.9) implies (2.8) for $r = 0, \ldots, 5$ on the interval $x_1 \leqslant x \leqslant x_6$. If $Q(\epsilon) \leqslant 1$, then (2.8) is always true, and we follow the proof of Theorem 1 to obtain (2.3) and (2.4) with C replaced by C'.

If $Q(\epsilon) \geqslant 1$, let q be the positive integer with

$$Q(\epsilon) \leqslant q < Q(\epsilon) + 1. \tag{2.11}$$

We divide the original set S of integer points (m_i, n_i) into q subsequences S_a of sizes R_a for $a = 0, \ldots, q-1$ according to $m_i \equiv a \pmod q$. Let $x_1 = m_k, \ldots, x_6 = m_{k+5}$ correspond to integer points in S_a. If $a = 0$, then $F(x_1, \ldots, x_6)$ is an integer divisible by q^4. The determinant $F(x_1, \ldots, x_6)$ is unchanged by linear shifts $x \to x + a$, so we must have $q^4 | F(x_1, \ldots, x_6)$ irrespective of a. Hence we can sharpen (2.2) to

$$q^4 \leqslant |F(x_1, \ldots, x_6)| \leqslant \frac{C'^{79} L^{15} N^4}{M^{11}}.$$

The analogue of (2.4) is

$$R = \sum_0^{q-1} R_a \leqslant 5q\left(\frac{M}{L_0} + 1\right) = 5q\left(\left(\frac{C'^{79} M^4 N^4}{q^4}\right)^{1/15} + 1\right), \tag{2.12}$$

and we obtain (1.13) of Theorem 2 using the inequality (2.11), which also holds trivially in the case $Q(\epsilon) < 1$, $q = 1$. The choice of q in (2.10) and (2.11) ensures that L_0/M is bounded, so we may drop the second term in (2.12) at the cost of multiplying the upper bound by a constant. ∎

3. The general case

We prove the assertions of Theorem 3. Let $(m_1, n_1), \ldots, (m_{k+1}, n_{k+1})$ be $k+1$ solutions of (1.15) with $m_1 < m_2 < \ldots < m_{k+1} = m_1 + L$. Let D be the determinant

$$D = \det(z_j(m_i, n_i))_{(k+1)\times(k+1)}. \tag{3.1}$$

Applying Theorem A1 directly and expanding by the first $d+1$ columns gives the same result as the simpler but less elegant method of applying Lemma A2 $d+1$ times, then Theorem A1. When $n_i = f(m_i)$ we have

$$\frac{D}{V(m_1, \ldots, m_{k+1})} = \left(\prod_{j=d+1}^{k} \frac{1}{j!}\right) \prod_{r=1}^{n} \prod_{i=1}^{r} \frac{E_{n-r-1}(a_{ir}) E_{n-r+1}(a_{ir})}{E_{n-r}(a_{ir})^2},$$

in the notation of Theorem A1 with $n = k - d$ and

$$f_j(x) = (z_{j+d}(x, f(x))^{(d+1)},$$

for certain points a_{ir} in the open interval (m_1, m_{k+1}).

We note that in the notation of Theorem 3

$$E_s(x) = E_s(x, f(x)) = M^{A_s} N^{B_s} E_s\left(\frac{x}{M}, F\left(\frac{x}{M}\right)\right) \tag{3.2}$$

for some exponents A_s and B_s. We recall that $E_0(x)$ and $E_{-1}(x)$ are defined to be one, and we observe that, for $s = 1, \ldots, k-d-1$, $E_s(x)$ occurs $2n - 2s$ times in the numerator and denominator, with

$$\left|\frac{E_s(\xi)}{E_s(\eta)}\right| \leqslant \frac{b_s}{a_s}$$

for any ξ and η in J. The big determinant $E_{k-d}(x)$ occurs once, in the numerator only. It obeys the scaling law (3.2) with

$$\begin{aligned} B_{k-d} &= \frac{d(d+1)(d+2)}{6} = B, \\ A_{k-d} &= \frac{d(d+1)(d+2)}{6} - \frac{k(k+1)}{2} = B - K, \end{aligned}$$

say. The integer D cannot be zero, so

$$1 \leqslant |D| \ll (MN)^B M^{-K} V(m_1, \ldots, m_{k+1}) \ll \left(\frac{L}{M}\right)^K (MN)^B \tag{3.3}$$

which implies

$$m_{k+1} - m_1 = L \geqslant L_0 \gg \frac{M}{(MN)^{B/K}}.$$

Then

$$R \leqslant k\left(\frac{M}{L_0} + 1\right) \ll (MN)^{\frac{B}{K}} = (MN)^{\frac{4}{3(d+3)}}. \tag{3.4}$$

Here we use Vinogradov's order of magnitude notation $U \ll V$ for $|U| \leqslant cV$, where c is some constant that does not depend on the size parameters, and $U \gg V$ similarly. The implied constants c in (3.3) and (3.4) are constructed from d and the numbers a_r and b_r. This establishes part (i) of Theorem 3.

Proof of part (ii). As in the proof of Theorem 2 we use a perturbation argument for δ very small. To extend the range of δ further, we prepare by subdividing the set of integer points, and we use a divisibility argument.

The perturbation principle is that if $g(x)$ is a function with

$$\frac{|g^{(r)}(x)|}{r!} \leqslant \frac{\epsilon N}{M^r} \tag{3.5}$$

for $r = 1, \dots, k$ with ϵ sufficiently small, then

$$H(x) = \frac{1}{N}(f(Mx) - g(Mx))$$

satisfies conditions of the form (1.16) and (1.17), with a_r, b_r replaced by a'_r, b'_r satisfying

$$a'_r = a_r + O(\epsilon) \geqslant \frac{1}{2}a_r,$$
$$b'_r = b_r + O(\epsilon) \leqslant 2b_r.$$

Let (m_i, n_i) for $i = 1, \dots, k+1$ be consecutive integer points in S, numbered in order of m_i increasing. Then $f(m_i) = n_i + \delta_i$, where $|\delta_j| \leqslant \delta$. The interpolation polynomial

$$g(x) = \sum_{i=1}^{k+1} \delta_i \prod_{j \neq i} \frac{x - m_j}{m_i - m_j}$$

takes the values $\delta_1, \dots, \delta_{k+1}$ at $x = m_1, \dots, m_{k+1}$. Let $L = m_{k+1} - m_1$, $\ell = \min(m_{i+1} - m_i)$. Then for $m_1 \leqslant x \leqslant m_{k+1}$ and $r = 0, \dots, k$ we have

$$\frac{|g^{(r)}(x)|}{r!} \leqslant \frac{(k+1)\delta}{\ell^k} {}_kC_r L^{k-r} \leqslant \frac{2^k(k+1)\delta L^{k-r}}{\ell^k}. \tag{3.6}$$

If

$$d \geqslant Q(\epsilon) = 2\left(\frac{(k+1)\delta}{\epsilon N}\right)^{1/k} M, \tag{3.7}$$

then (3.6) implies (3.5) for $r = 0, \dots, k$ on the interval $m_1 \leqslant x \leqslant m_{k+1}$. If $Q(\epsilon) \leqslant 1$, then (3.5) is always true, and we follow the proof of Theorem 3 part (i). The determinant D in (3.1) is the functional determinant formed with $h(x) = f(x) - g(x)$ in place of $f(x)$, so it can be estimated in the same way. In particular, D is non-zero. We obtain the bound (1.18) with a different constant $C(F)$.

If $Q(\epsilon) \geqslant 1$, then we take q in (2.11) to be the least integer with $Q(\epsilon) \leqslant q$. Hence

$$q < Q(\epsilon) + 1 \ll \left(\frac{\delta}{N}\right)^{1/k} M + 1; \tag{3.8}$$

in particular $q \ll M$. We divide the original set S of integer points (m_i, n_i) into q subsequences S_a of sizes R_a for $a = 0, \dots, q-1$ according to $m_i \equiv a \pmod q$. The determinant D of (3.1), when formed with $k+1$ points in S_0, is divisible by q^B with $B = d(d+1)(d+2)/6$. The determinant D is unchanged by linear shifts, so $Q^B | D$ whenever the $k+1$ points lie in the same subsequence S_a.

The analogue of (3.4) is

$$R_a \ll \left(\frac{MN}{q}\right)^{B/K} + k \ll \left(\frac{MN}{q}\right)^{\frac{4}{3d(d+3)}}, \tag{3.9}$$

where we have used $q \ll M$. We obtain (1.19) on summing $a \pmod q$, and then estimating q by (3.8). ∎

Appendix. A determinant mean value theorem

Let $f_1(x), \ldots, f_n(x)$ be real functions, and let $a_1, \ldots, a_n$ be distinct real numbers in increasing order. We consider the determinant of the function values

$$D(f_1, \ldots, f_n; a_1, \ldots, a_n) = \det\left(f_j(a_i)\right)_{n\times n}. \qquad (A1)$$

The Vandermonde determinant is the special case

$$V(a_1, \ldots, a_n) = D(1, x, \ldots, x^{n-1}; a_1, \ldots, a_n). \qquad (A2)$$

Suppose that the functions $f_j(x)$ are $n-1$ times continuously differentiable on an interval. Taylor's theorem suggests that the determinant in (A1) should be approximately proportional to $V(a_1, \ldots, a_n)$ multiplied by a value of the determinant

$$E(f_1, \ldots, f_n; x) = \det\left(f_j^{(i-1)}(x)\right)_{n\times n}. \qquad (A3)$$

However, the case $f_1(x) = \cos x$, $f_2(x) = \sin x$, $a_1 = 0$, $a_2 = \pi$ shows that the determinant in (A1) can vanish whilst the determinant in (A3) does not. We must also consider the minor determinants

$$E_r(x) = E(f_1, \ldots, f_r; x) \qquad (A4)$$

for $r = 1, \ldots, n-1$.

Theorem A1. *Suppose that $f_1(x), \ldots, f_n(x)$ are $n-1$ times continuously differentiable on an interval I, and the determinants $E_r(x)$ for $r = 1, \ldots, n-1$ do not vanish on I. Then for distinct points $a_1 < a_2 < \ldots < a_n$ on I, there are points a_{ir} with*

$$a_{in} = a_i, \qquad a_1 < a_{ir} < a_n \ \text{ for } \ 1 \leqslant r \leqslant n-1 \qquad (A5)$$

for which

$$\frac{D(f_1, \ldots, f_n; a_1, \ldots, a_n)}{V(a_1, \ldots, a_n)} = \prod_{r=1}^{n} \frac{1}{(r-1)!} \prod_{i=1}^{r} \frac{E_{n-r-1}(a_{ir}) E_{n-r+1}(a_{ir})}{E_{n-r}(a_{ir})^2}, \qquad (A6)$$

where $E_0(x)$ and $E_{-1}(x)$ are interpreted as 1.

This question arose in the approximation of real numbers by values of rational functions in [7,8].

Theorem A2. *Let $f(x)$ be a real function, $2d+1$ times continuously differentiable on an interval I. Define the determinants $B_r(n, x)$ for $1 \leqslant r \leqslant n$, $n + r \leqslant 2d+2$ by*

$$B_r(n, x) = \det\left(\frac{f^{(n+i-j)}(x)}{(n+i-j)!}\right)_{r\times r}. \qquad (A7)$$

Suppose that for each $r = 1, \ldots, d+1$, *the function* $B_r(d+1, x)$ *does not vanish on* I. *Then if* $u(x)$, $v(x)$ *are any polynomials of degree at most* d, *the equation*

$$f(x) = \frac{u(x)}{v(x)} \tag{A8}$$

has at most $2d+1$ *distinct roots on* I, *and for* $\delta 0$, *the points on* I *which satisfy*

$$\left| f(x) - \frac{u(x)}{v(x)} \right| \leqslant \delta \tag{A9}$$

form at most $2d+2$ *disjoint subintervals of* I.

The case $d = 1$ of Theorem A2 was established in [6], where we considered the rational points $(m_i/n_i, r_i/q_i)$ that lie close to the curve $y = f(x)$. Under certain conditions, if four such points lie close together, then they are of the form $(x, u(x)/v(x))$, where $u(x)$ and $v(x)$ are linear functions. Rational functions with $u(x)$ and $v(x)$ of higher degrees are discussed in [7] and [8].

The prototype for our arguments is Cauchy's mean value theorem (article 128 of [2]).

Lemma A1 (Cauchy's Mean Value Theorem)**.** *For* $f(x)$, $g(x)$ *continuously differentiable on an interval* $I = [a, b]$ *with* $g'(x)$ *non-zero on* I, *there is a point* c *in* $a < c < b$ *for which*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

By repeated application of Lemma A1 we obtain the next step (Lemma 3.1 of [6]).

Lemma A2. *For* $f_1(x) = 1$ *and* $f_2(x), \ldots, f_r(x)$ *continuously differentiable on an interval* I, *and for* $a_1 < a_2 < \ldots < a_r$ *distinct points in* I, *there are points* $b_1, \ldots, b_{r-1}$ *with* $a_1 < b_1 < b_2 < \ldots < b_{r-1} < a_r$ *for which*

$$\frac{D(f_1, \ldots, f_r; a_1, \ldots, a_r)}{V(a_1, \ldots, a_r)} = \frac{D(f_2', \ldots, f_r'; b_1, \ldots, b_{r-1})}{(r-1)! V(b_1, \ldots, b_{r-1})}. \tag{A10}$$

We prove Theorem A1 by iterating Lemma A2. We set up a family of functions $f_{j,k}(x)$ for $j + k \leqslant n$, with

$$f_{j,0} = f_j, \qquad f_{j,k+1} = \left(\frac{f_{j+1,k}}{f_{1,k}} \right)'. \tag{A11}$$

We shall show that the denominators in (A11) are non-zero on I. Since

$$\begin{aligned} &D(f_{1,k}, \ldots, f_{r,k}; a_1, \ldots, a_r) \\ &\quad = D\left(1, \frac{f_{2,k}}{f_{1,k}}, \ldots, \frac{f_{r,k}}{f_{1,k}}; a_1, \ldots, a_r\right) \prod_1^r f_{1,k}(a_i), \end{aligned}$$

Lemma A2 gives

$$\begin{aligned}&\frac{D(f_{1,k},\ldots,f_{r,k};a_1,\ldots,a_r)}{V(a_1,\ldots,a_r)}\\&\quad=\frac{D(f_{1,k+1},\ldots,f_{r-1,k+1};b_1,\ldots,b_{r-1})}{(r-1)!V(b_1,\ldots,b_{r-1})}\prod_{i=1}^{r}f_{1,k}(a_i).\end{aligned}$$

We iterate to get

$$\frac{D(f_1,\ldots,f_n;a_1,\ldots,a_n)}{V(a_1,\ldots,a_n)}=\prod_{r=1}^{n}\frac{1}{(r-1)!}\prod_{i=1}^{r}f_{1,n-r}(a_{ir}) \qquad (A12)$$

for some real numbers a_{ir} in the range (A5). To approach Theorem A1, we shall express the right-hand side of (A12) in terms of $f_1,\ldots,f_n$, by unpacking the iteration (A10).

Lemma A3. *For $f_1(x),\ldots,f_r(x)$ continuously differentiable on an interval I, with $f_1(x)$ non-vanishing on I, in the notation (A3), (A4) we have*

$$E_r(x)=f_1(x)^rE\left(\left(\frac{f_2}{f_1}\right)',\left(\frac{f_3}{f_1}\right)',\ldots,\left(\frac{f_r}{f_1}\right)';x\right). \qquad (A13)$$

Proof. We write $f_j=f_1g_j$. Then

$$f_j^{(i-1)}=\sum_{k=1}^{i}{}_{i-1}C_{k-1}f_1^{(i-k)}g_j^{(k-1)},$$

and we have the matrix product

$$\left(f_j^{(i-1)}\right)_{r\times r}=T_{r\times r}\left(g_j^{(i-1)}\right)_{r\times r},$$

where T is the lower triangular matrix with entries

$$t_{ik}={}_{i-1}C_{k-1}f_1^{(i-k)}$$

for $i\geqslant k$, whose diagonal entries are all $f_1(x)$.

Since $g_1(x)=1$, the first column of the second matrix has an entry one followed by $r-1$ zeros, so the determinant is given by the minor determinant of rows and columns 2 to r, which is $E(g_2',\ldots,g_r';x)$. ■

Proof of Theorem A1. Lemma A3 applied to the iteration (A11) gives

$$E(f_{1,k},\ldots,f_{r,k})=f_{1,k}^rE(f_{1,k+1},\ldots,f_{r-1,k+1}),$$

so we have

$$\begin{aligned} E_r &= E(f_{1,0}, \ldots, f_{r,0}) = f_{1,0}^r E(f_{1,1}, \ldots, f_{r-1,1}) \\ &= \ldots = f_{1,0}^r f_{1,1}^{r-1} \ldots f_{1,r-1}. \end{aligned} \tag{A14}$$

We see that $f_{1,0} = E_1$, $f_{1,1} = E_2/E_1^2$, and for $r \geqslant 3$ we verify by induction that

$$f_{1,j} = \frac{E_{j-1}E_{j+1}}{E_j^2} \tag{A15}$$

for $j \geqslant 2$. We may include the cases $j = 0$ and $j = 1$ in (A15) by defining $E_0 = E_{-1} = 1$. The determinants $E_1, \ldots, E_{n-1}$ do not vanish by assumption, so $f_{1,j}$ is well-defined, and $f_{1,j} \neq 0$ for $j = 1, \ldots, n-1$, as we required to set up the iteration (A11). We obtain Theorem A1 by substituting (A15) into (A12). ■

Proof of Theorem A2. The condition for $2d+2$ points (x_i, y_i) to satisfy some relation

$$y = \frac{u(x)}{v(x)} = \frac{a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0}{b_d x^d + b_{d-1}x^{d-1} + \cdots + b_0}$$

is that the determinant

$$D = \begin{vmatrix} 1 & x_1 & \cdots & x_1^d & y_1 & x_1 y_1 & \cdots & x_1^d y_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{2d+2} & \cdots & x_{2d+2}^d & y_{2d+2} & x_{2d+2}y_{2d+2} & \cdots & x_{2d+2}^d y_{2d+2} \end{vmatrix}$$

should vanish. If $y = f(x)$, then the determinant D is

$$D(1, x, \ldots, x^d, f, xf, \ldots, x^d f; x_1, \ldots, x_{2d+2}).$$

Repeated use of Lemma A2 gives

$$\begin{aligned} &\frac{D}{V(x_1, \ldots, x_{2d+2})} \\ &\quad = \frac{2!3!\cdots d!}{(d+1)!\cdots(2d+1)!} \cdot \frac{D\big(f^{(d+1)}, (xf)^{(d+1)}, \ldots, (x^d f)^{(d+1)}; \xi_1, \ldots, \xi_{d+1}\big)}{V(\xi_1, \ldots, \xi_{d+1})} \end{aligned}$$

for some $\xi_1, \ldots, \xi_{d+1}$ in $x_1 < \xi_1 < \ldots < \xi_{d+1} < x_{2d+2}$. Let $n = d+1$, and let $f_j = (x^{j-1}f)^{(d+1)}$.

The first assertion of Theorem A2 will follow from Theorem A1 when we show that the determinants $E_1, \ldots, E_n$ are non-vanishing on I. The (i,j) entry of E_r is

$$(x^{j-1}f)^{(d+i)} = \sum_{k=1}^{j} \frac{(d+i)!(j-1)!}{(k-1)!(d+i+1-k)!(j-k)!} f^{(d+i+1-k)} x^{j-k},$$

which corresponds to a matrix product ST, where

$$S = \left(\frac{(d+i)!}{(d+i+1-j)!} f^{(d+1+i-j)} \right)_{r \times r},$$

and $T_{r\times r}$ is the upper triangular matrix with entries

$$t_{kj} = {}_{j-1}C_{k-1} x^{j-k}$$

for $k \leqslant j$. The diagonal entries are all one, so the determinant of T is one. The determinant of S is

$$\left(\prod_{i=1}^{r} (d+i)! \right) B_r(d+1, x)$$

in the notation (A7) of Theorem A2. Hence each determinant E_r is non-zero under the hypotheses of Theorem A2, and the first part of the theorem follows.

For the second part of Theorem A2, if the solutions of (A9) form at least $2d+3$ subintervals, then there are $4d+4$ internal endpoints at which

$$f(x) - \frac{u(x)}{v(x)} = \pm\delta.$$

Some value $\epsilon = \pm\delta$ occurs $2d+2$ times, which contradicts the first assertion with $u(x)$ replaced by $u(x) + \epsilon v(x)$, which is also a polynomial of degree at most d. We deduce the second part of Theorem A2. ∎

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Received: 12 December 2006; **revised:** 29 June 2007