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SOME PROPERTIES OF THE ANKENY-ONISHI FUNCTION

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In honor of the 60th birthday of our friend,
Jean-Marc Deshouillers

Abstract: We survey properties of the Ankeny-Onishi sieve function and establish inequalities for $j_\kappa(\kappa)$ and for $1 - j_\kappa(u)$ for $u \to \infty$.
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1. Introduction

The function $\sigma_\kappa(u)$ was first introduced by Ankeny and Onishi in their pioneering extension [1] of the Selberg sieve method, albeit in a different notational guise. It is given by

$$\sigma_\kappa(u) := j_\kappa(u/2), \quad \kappa \geqslant 1, \tag{1.1}$$

where

$$j(u) = j_\kappa(u) = \begin{cases} 0, & u \leqslant 0, \\ e^{-\gamma\kappa} u^\kappa / \Gamma(\kappa+1), & 0 < u \leqslant 1, \end{cases} \tag{1.2}$$

and j is continued forward as the continuous solution of

$$uj'(u) = \kappa j(u) - \kappa j(u-1) = \kappa \int_{u-1}^{u} j'(t)dt, \quad u > 1, \tag{1.3}$$

by means of the restatement

$$(u^{-\kappa} j(u))' = -\kappa u^{-\kappa-1} j(u-1), \quad u > 1, \tag{1.3$'$}$$

of (1.3); in fact (1.3) holds for all $u \geqslant 0$. It is a differential delay equation of a kind common in the study of sieves.

In this note we review basic information about j/σ and develop several interesting properties of these functions. In particular, we present simpler proofs that (i) $j_\kappa(\kappa) > 1/2$ for all $\kappa \geqslant 1$, and that (ii) for each fixed $c > 1$, $j_\kappa(c\kappa)$ tends

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to 1 from below as $\kappa \to \infty$ (both these results were first proved by Grupp and Richert in [2]); also, we show in explicit fashion that $j_\kappa(u) \to 1$ and $j'_\kappa(u) \to 0$ as $u \to \infty$, each at a rate that is faster than exponential.

We begin studying j with some observations about the continuity of its derivatives. If $u > 0$ and $\kappa \geqslant 1$, then $j'(u)$ is continuous for $u > 0$ by (1.3) and the continuity of j; more generally, by differentiating (1.3) we see that $j_\kappa^{(n)}(u)$ is continuous for $u \geqslant 0$ for all positive integers $n < \kappa$. If κ is a positive integer, then $j_\kappa^{(\kappa)}(u)$ has a jump discontinuity at $u = 0$, and $j_\kappa^{(\kappa+n)}(u)$ has jump discontinuities at $u = 1, \ldots, n$. If $\kappa > 1$ is not an integer, then $j^{([\kappa]+n)}(u)$ has infinite jump discontinuities from the right at $u = 0, 1, \ldots, n-1$ for each positive integer n. In each of the preceding cases, the function is continuous at all other values of $u > 0$.

We show next for each $\kappa \geqslant 1$ that $j_\kappa(u)$ is a positive, strictly increasing function of $u > 0$. By (1.2), $j'(u) > 0$ when $0 < u \leqslant 1$, and by (1.3) it remains positive for some distance to the right side of 1. Suppose there were a point $u_0 > 1$ with $j'(u_0) = 0$. By the continuity of j', we may assume that u_0 is the first such point, i.e. that $j'(u_0) = 0$ and $j'(t) > 0$ for $0 < t < u_0$. Upon evaluating the integral form of (1.3) at $u = u_0$ we obtain a contradiction, since the left side is 0 and the right side is κ times the integral of a positive function. Hence

$$j'(u) > 0, \quad u > 0; \tag{1.4}$$

and we deduce immediately that

$$j(u) > 0, \quad u > 0. \tag{1.5}$$

The higher derivatives of $j(u)$ also satisfy differential delay equations. Upon differentiating (1.3), and then once again, we obtain

$$uj''(u) = (\kappa - 1)j'(u) - \kappa j'(u-1) \tag{1.6}$$

and

$$uj'''(u) = (\kappa - 2)j''(u) - \kappa j''(u-1). \tag{1.7}$$

In (1.3) itself, if we integrate by parts on the right (which is valid, since j' is absolutely continuous), we obtain

$$uj'(u) = \kappa(t - \kappa + 1)j'(t)\Big|_{u-1}^{u} - \kappa \int_{u-1}^{u} (t - \kappa + 1)j''(t)dt$$

or

$$(u - \kappa)\{(\kappa - 1)j'(u) - \kappa j'(u-1)\} = \kappa \int_{u-1}^{u} (t - \kappa + 1)j''(t)dt;$$

hence by (1.6) (for all $\kappa \geqslant 1$ and $u > 0$),

$$u(u - \kappa)j''(u) = \kappa \int_{u-1}^{u} (t - \kappa + 1)j''(t)dt. \tag{1.8}$$

We use the last equation to show that j_κ has a unique inflection point u_κ (for $\kappa > 1$) and that it lies in the interval $(\kappa - 1, \kappa]$. A finer analysis (see [2]) would show that $\kappa - 1/2 < u_\kappa < \kappa$ for all $\kappa > 1$.

Lemma 1. *Suppose $\kappa > 1$. There exists a unique number, call it u_κ, between $\kappa - 1$ and κ, such that $j''(u) > 0$ for $0 < u < u_\kappa$ and $j''(u) < 0$ for all $u > u_\kappa$. For $\kappa = 1$, we have $j''(u) = 0$ for all $u < u_1 = \kappa = 1$ and $j''(u) < 0$ for all $u > 1$.*

Proof. For $\kappa = 1$, we have by (1.6) that $uj''(u) = -j'(u-1)$, an expression that is 0 for $u < 1$ and is negative for $u > 1$ by (1.5).

Now suppose $\kappa > 1$. On taking $u = \kappa$ in (1.8) we find that

$$\int_{\kappa-1}^{\kappa} (t - \kappa + 1) j''(t) dt = 0.$$

Since $t - \kappa + 1 > 0$ on $(\kappa - 1, \kappa)$ it follows that $j''(t)$ changes sign in this interval. By (1.2) $j''(u) > 0$ on $(0, 1]$ and it follows from (1.7) and the continuity of j' that j'' is continuous on $[0, \infty)$. Thus there exists some number u_κ, the smallest value of $u > 1$ at which $j''(u) = 0$. By (1.7) at $u = u_\kappa$

$$u_\kappa j'''(u_\kappa) = -j''(u_\kappa - 1) < 0$$

since $j''(u) > 0$ for $0 < u < u_\kappa$, whence u_κ is a simple zero of j''.

Suppose if possible that j'' has other zero beyond u_κ, and let v be the least of these. We claim that

$$v < u_\kappa + 1;$$

for if, on the contrary, $v \geqslant u_\kappa + 1$ then $j''(v) = 0$ and $j''(u) < 0$ when $u_\kappa < u < v$. But then, by (1.6) at $u = v$,

$$0 = vj''(v) = (\kappa - 1)j'(v) - \kappa j'(v - 1),$$

so that

$$0 < j'(v) = \kappa\{j'(v) - j'(v - 1)\} = \kappa j''(w)$$

for some w strictly between $v - 1$ $(\geqslant u_\kappa)$ and v, a contradiction.

Next suppose that $u_\kappa < v < u_\kappa + 1$. We know that $j''(u)$ is non-decreasing at $u = v$, so that $j'''(v) \geqslant 0$; yet by (1.7)

$$vj'''(v) = -\kappa j''(v - 1) < 0$$

since $v - 1 < u_\kappa$, also an impossibility.

Hence v does not exist, and j'' has just the one zero u_κ, which is simple and lies in $(\kappa - 1, \kappa)$. ■

The most rapid rate of increase of j occurs at u_κ. How fast is the function rising here? It was shown by Wheeler ([3], [4]) that $j'_\kappa(u_\kappa) \sim 1/\sqrt{\pi\kappa}$ as $\kappa \to \infty$.

2. The adjoint function

We introduce next the so-called "adjoint" of j, a function $r(u) = r_\kappa(u)$ defined for $\kappa > 0$ by

$$(ur(u))' = \kappa r(u+1) - \kappa r(u), \quad u > 0, \tag{2.1}$$

and normalized so that

$$\lim_{u\to\infty} u\,r(u) = 1. \tag{2.2}$$

A normalized solution of (2.1) is provided by

$$r_\kappa(u) = \int_0^\infty \exp(-ut + \kappa\,\mathrm{Ein}\,t)dt, \tag{2.3}$$

where

$$\mathrm{Ein}\,t := \int_0^t (1-e^{-s})\frac{ds}{s} = \sum_{n=1}^\infty (-1)^{n-1}\frac{t^n}{n!\,n}, \quad t \in \mathbb{C}, \tag{2.4}$$

an entire function. With $\log t$ denoting the principal value of $\log t$,

$$\mathrm{Ein}\,t = \log t + \gamma + \int_t^\infty \frac{e^{-s}}{s}ds, \quad |\arg t| < \pi. \tag{2.5}$$

To see that the integral (2.3) satisfies (2.1), first integrate it by parts, next multiply by u, and then differentiate with respect to u.

The behavior of $r(u)$ as $u \to \infty$ is no harder to derive: by (2.4) we have

$$0 \leqslant \mathrm{Ein}\,t \leqslant t, \quad t \geqslant 0,$$

whence

$$\int_0^\infty \exp(-ut)dt < r_\kappa(u) < \int_0^\infty \exp(-ut+\kappa t)dt,$$

and it follows at once that

$$u^{-1} < r_\kappa(u) \quad (u>0) \quad \text{and} \quad r_\kappa(u) < (u-\kappa)^{-1} \quad (u>\kappa).$$

Together, the last two inequalities imply that the normalization (2.2) holds.

The integral representation (2.3) of $r(u)$ shows that $(-1)^\nu r^{(\nu)}(u) > 0$ for $\nu = 0,1,2,\dots,$ and in particular, that $r'(u) < 0$ and $r''(u) > 0$ for all $u > 0$; also, that $(ur(u))' < 0$ by (2.1) and $((u-\kappa)r(u))' > 0$. The last inequality holds since

$$((u-\kappa)r(u))' = \kappa\{r(u+1) - r(u) - r'(u)\} = \kappa r''(u+\theta)/2 > 0$$

for some θ in $(0,1)$, by Taylor's expansion. It follows that

$$\frac{u+1}{u} < \frac{r(u)}{r(u+1)} < \frac{u-\kappa+1}{u-\kappa}\,,$$

the latter for $u > \kappa$.

The Iwaniec "inner product"

$$\langle j, r\rangle(u) := uj(u)r(u) - \kappa \int_{u-1}^{u} j(t)r(t+1)dt, \quad u > 0,$$

is constant, as one can verify by differentiating and using the defining equations of r and j. To evaluate this constant let $u \to 0+$; by (2.3) and (2.5)

$$\begin{aligned} r(u) &= \int_0^\infty \exp\{-ut + \kappa \log t + \gamma\kappa + o(1)\}dt \\ &\sim e^{\gamma\kappa} \int_0^\infty \exp(-ut)t^\kappa dt, \quad u \to 0+, \\ &= e^{\gamma\kappa}\Gamma(\kappa+1)u^{-\kappa-1}. \end{aligned}$$

Hence, by (1.2), $uj(u)r(u) \to 1$ as $u \to 0+$ and so

$$uj(u)r(u) - \kappa \int_{u-1}^{u} j(t)r(t+1)dt = 1, \quad u > 0. \tag{2.6}$$

In the same vein

$$ur(u) - \kappa \int_{u-1}^{u} r(t+1)dt$$

is constant by (2.1), and since $ur(u) \to 1$ as $u \to \infty$, we see that

$$ur(u) - \kappa \int_{u-1}^{u} r(t+1)dt = 1. \tag{2.7}$$

Lemma 2. *Suppose $\kappa \geqslant 1$ and $u \geqslant \kappa$. Then each of the functions*

$$(j(u) - j(t))r(t+1), \ (1 - j(t))r(t+1)$$

is convex in t on the interval $u - 1 \leqslant t \leqslant u$.

Proof. The argument is the same for each function, so focus on the first and call it $J(t)$. Then, by (2.3),

$$\begin{aligned} J''(t) &= -r(t+1)j''(t) + 2(-j'(t))r'(t+1) + (j(u) - j(t))r''(t+1) \\ &= \int_0^\infty \big\{ - j''(t) + 2(-j'(t))(-u) + (j(u) - j(t))u^2 \big\} \\ &\quad \times \exp(-(t+1)u + \kappa \ \mathrm{Ein}\, u)\, du. \end{aligned}$$

By (1.6) the expression within the curly brackets is equal to

$$\begin{aligned} &- \frac{1}{t}((\kappa-1)j'(t) - \kappa j'(t-1)) + 2uj'(t) + (j(u) - j(t))u^2 \\ &\quad = \Big(2u - \frac{\kappa-1}{t}\Big)j'(t) + \frac{\kappa}{t}j'(t-1) + (j(u) - j(t))u^2; \end{aligned}$$

The second and third terms here are positive, and the coefficient of $j'(t)$ is at least

$$2u - \frac{\kappa - 1}{u - 1} \geqslant 2u - 1 > 0$$

since $u \geqslant \kappa$. Hence, $J'' > 0$. ■

We next consider the limiting behavior of $j(u)$ as $u \to \infty$. When we multiply (2.7) by $j(u)$ and subtract it from (2.6) we obtain

$$1 - j(u) = \kappa \int_{u-1}^{u} \{j(u) - j(t)\} r(t+1) dt; \tag{2.8}$$

if we simply subtract (2.6) from (2.7) this time we find

$$\{1 - j(u)\} u r(u) = \kappa \int_{u-1}^{u} \{1 - j(t)\} r(t+1) dt. \tag{2.9}$$

There is much to be learned from these two relations. First, the integral on the right of (2.8) is positive and therefore

$$j(u) < 1, \quad u > 0,$$

as we reported earlier. Next, by (2.9), since $r(\cdot)$ is positive and decreasing and $j(\cdot) > 0$, we obtain at once

$$\{1 - j(u)\}\, u r(u) < \kappa \int_{u-1}^{u} r(t+1) dt < \kappa r(u),$$

so that

$$0 < 1 - j(u) < \kappa / u$$

and therefore

$$\lim_{u \to \infty} j_\kappa(u) = 1. \tag{2.10}$$

We apply Lemma 2 to the right side of (2.9) and obtain

$$\{1 - j(u)\} u r(u) < \frac{\kappa}{2}\big(r(u+1)\{1 - j(u)\} + r(u)\{1 - j(u-1)\}\big), \quad u \geqslant \kappa; \tag{2.11}$$

this inequality can be rewritten in two ways, which lead to different lines of development, one an iteration and the other a differential inequality.

First, we have

$$\begin{aligned} 1 - j(u) &< \frac{\kappa r(u)}{2ur(u) - \kappa r(u+1)} \{1 - j(u-1)\} \\ &= \frac{\kappa}{2u - \kappa r(u+1)/r(u)} (1 - j(u-1)) \end{aligned}$$

and since $r(u+1)/r(u) < u/(u+1)$ from above, we derive the recurrence

$$1 - j(u) < \frac{u+1}{u}\,\frac{\kappa/2}{u+1-\kappa/2}\,\{1 - j(u-1)\}, \quad u \geqslant \kappa. \tag{2.12}$$

This inequality plainly lends itself to iteration and leads, for any $v \geqslant \kappa - 1$ and positive integer n, to

$$\begin{aligned} &1 - j_\kappa(v+n) \\ &\quad< \frac{v+1-\kappa/2}{v+1}\,\frac{v+n+1}{v+n+1-\kappa/2}\,\frac{(\kappa/2)^n\,\Gamma(v+1-\kappa/2)}{\Gamma(v+n+1-\kappa/2)}\,(1 - j_\kappa(v)) \\ &\quad< \Gamma(v+1-\kappa/2)\left\{\frac{(\kappa/2)^n}{\Gamma(v+n+1-\kappa/2)}\right\}(1 - j_\kappa(v)). \end{aligned} \tag{2.13}$$

If u is a number near $\kappa + n$ for some positive integer n, then the factor in curly brackets shows that $j_\kappa(u)$ does indeed tend to 1 faster than exponentially as $u \to \infty$. In the next section we shall show that $1 - j_\kappa(\kappa) < 1/2$, which in combination with (2.13) yields a quite sharp inequality for $1 - j_\kappa(u)$.

To conclude this section, we return to (2.11) and deduce from it a differential inequality. We begin by writing the relation in the form

$$\{1 - j(u)\}ur(u) < \frac{\kappa}{2}\{1 - j(u)\}(r(u+1) + r(u)) + \frac{\kappa}{2}(\{j(u) - j(u-1)\}r(u)$$

and, after applying (1.3) and a little rearrangement, this becomes

$$\begin{aligned} 1 - j(u) &< \frac{\kappa}{2}(1 - j(u))\Big(\frac{r(u+1)}{ur(u)} + \frac{1}{u}\Big) + \frac{1}{2}j'(u) \\ &< \frac{\kappa}{2}(1 - j(u))\Big(\frac{1}{u+1} + \frac{1}{u}\Big) + \frac{1}{2}j'(u), \end{aligned}$$

or

$$(1 - j(u))' + \Big\{2 - \kappa\Big(\frac{1}{u} + \frac{1}{u+1}\Big)\Big\}(1 - j(u)) < 0;$$

in other words, for $u \geqslant \kappa$,

$$\{(1 - j(u)\exp(2u - \kappa\log u(u+1))\}' < 0.$$

Upon integrating, we find for $u \geqslant \kappa$ that

$$(1 - j(u))\exp(2u - \kappa\log u(u+1)) \leqslant (1 - j(\kappa))\exp(2\kappa - \kappa\log\kappa(\kappa+1)).$$

Here then we have come to a curious pass: starting from (2.11) and adding extra information – application of (1.3) – we have derived the inequality

$$1 - j_\kappa(u) \leqslant (1 - j_\kappa(\kappa))\Big(\frac{u(u+1)}{\kappa(\kappa+1)}\Big)^\kappa \exp(-2u + 2\kappa), \quad u \geqslant \kappa, \tag{2.14}$$

which is perhaps more pleasing to the eye, and not without interest, but yields only exponential decay of $1 - j_\kappa(u)$ towards 0 as $u \to \infty$! We cannot understand why, apparently, the second approach is inferior to the first.

It should be said at this point that [1] derives a slightly weaker inequality than (2.14) valid for $u \geqslant \kappa + 1$. This is implicit in their formula (2.9) on p. 40.

In the next section we shall simplify (2.13) and (2.14) by determining a lower bound for $j_\kappa(\kappa)$.

3. A lower bound for $j_\kappa(\kappa)$

We learn from (1.2) that $j_1(1) = e^{-\gamma} = 0.56145\ldots$ and from numerical computations that $j_{1.5}(1.5) = 0.55179\ldots$ and $j_2(2) = 0.54454\ldots$. In fact, it was proved in [2] that for any constant $c \geqslant 0$, $j_\kappa(\kappa + c)$ decreases in $\kappa \geqslant 1$ and tends to $1/2$ as $\kappa \to \infty$; also that $j_\kappa(c'\kappa) \to 1$ as $\kappa \to \infty$ for any constant $c' > 1$. Also, it was shown by Wheeler ([3], [4]) that, for $\kappa \geqslant 1$,

$$j_\kappa(\kappa) \;=\; 1/2 \;+\; 1/(9\sqrt{\pi\kappa}) \;+\; O\big(\kappa^{-3/2}\big).$$

Here we show by a Laplace inversion method that

Proposition 1. *For $\kappa \geqslant 1$,*

$$j_\kappa(\kappa) > 1/2.$$

Proof. Since $1 - j_\kappa(u)$ vanishes rapidly at infinity, it has a Laplace transform whose integral converges for Res $\geqslant 0$. By a calculation analogous to that which identified $r(u)$ as a Laplace transform, we have

$$\int_0^\infty e^{-su}(1 - j_\kappa(u))du = \frac{1}{s}(1 - \exp(-\kappa \;\operatorname{Ein} s)), \quad \Re s \geqslant 0.$$

It follows by Fourier inversion (Laplace inversion on the imaginary axis) that, for $u > 0$,

$$1 - j_\kappa(u) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} e^{iuy}\{1 - \exp(-\kappa \;\operatorname{Ein} iy)\}\frac{dy}{iy}.$$

Since j is real valued, we have at $u = \kappa$

$$\begin{aligned} 1 - j_\kappa(\kappa) &= \Re\Big\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i\kappa y}(1 - \exp\{-\kappa \operatorname{Ein}(iy)\})\frac{dy}{y}\Big\} \\ &= \frac{1}{\pi}\int_0^\infty \sin\kappa y \,\frac{dy}{y} - \Re\Big\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-\kappa(\operatorname{Ein}(iy) - iy)}\frac{dy}{y}\Big\}. \end{aligned}$$

The first expression on the right is well known to be equal to $1/2$. In the second expression,

$$\begin{aligned} \operatorname{Ein}(iy) - iy &= \int_0^y \frac{1 - \cos t}{t}\,dt + i\int_0^y \frac{\sin t - t}{t}\,dt \\ &= C(y) + iS(y), \end{aligned}$$

say, where $C(y)$ is an even function of y and $S(y)$ an odd function. Hence

$$j_\kappa(\kappa) - \frac{1}{2} = \frac{1}{\pi}\int_0^\infty e^{-\kappa C(y)}\sin(-\kappa S(y))\frac{dy}{y}\,. \tag{3.1}$$

We complete the proof by showing that the integral on the right is positive.

Since

$$\sin(-\kappa S(y)) = \Big(\kappa\frac{\sin y - y}{y}\Big)^{-1}\frac{d}{dy}\cos(-\kappa S(y)),$$

the integral equals, after integrating by parts,

$$\frac{1}{\kappa}\Big\{\frac{e^{-\kappa C(y)}}{y-\sin y}(1-\cos\{\kappa S(y)\})\Big\}\Big|_0^\infty - \frac{1}{\kappa}\int_0^\infty(1-\cos\{\kappa S(y)\})\frac{d}{dy}\Big(\frac{e^{-\kappa C(y)}}{y-\sin y}\Big)dy\,.$$

The integrated term vanishes at infinity since $C(y)\sim\log y$ as $y\to\infty$, and it vanishes also at 0 since

$$1-\cos(\kappa S(y)) \sim \frac{1}{2!}(\kappa S(y))^2 \sim \frac{\kappa^2}{648}\,y^6 \quad \text{as } y\to 0$$

whereas

$$y-\sin y \sim \frac{1}{6}\,y^3 \quad \text{as } y\to 0\,.$$

As for the integral, we observe that each of $e^{-\kappa C(y)}$ and $(y-\sin y)^{-1}$ is positive and decreasing as y increases, so that

$$-\frac{d}{dy}\Big(\frac{e^{-\kappa C(y)}}{y-\sin y}\Big) > 0.$$

Since $1-\cos(\kappa S(y))>0$, this completes the proof that the integral on the right side of (3.1) is positive. ∎

The estimate of the Proposition appears to be quite sharp: it is likely, on the basis of the two asymptotic estimates of Wheeler that we have cited, that $j_\kappa(\kappa-1)<1/2$. However, we have not investigated this question.

The Proposition allows us to derive from (2.13) and (2.14)

Theorem 1. *For* $u\geqslant\kappa$

$$j_\kappa(u) \geqslant 1 - \frac{1}{2}\Big(\frac{u(u+1)}{\kappa(\kappa+1)}\Big)^\kappa\exp(2\kappa-2u), \tag{3.2}$$

and for any positive integer n,

$$\begin{aligned} j_\kappa(n+\kappa) &> 1-\frac{1}{2}\Big(1-\frac{\kappa}{2\kappa+2}\Big)\Gamma(\frac{\kappa}{2})\Big(1+\frac{\kappa}{2n+2+\kappa}\Big)\frac{(\kappa/2)^{n+1}}{\Gamma(n+1+\kappa/2)} \\ &> 1-\frac{\Gamma(\kappa/2)\,(\kappa/2)^{n+1}}{2\,\Gamma(n+1+\kappa/2)}. \end{aligned} \tag{3.3}$$

Corollary 1. *Let* $c > 1$ *be a constant. Then* $j_\kappa(c\kappa) \to 1$ *from below as* $\kappa \to \infty$.

Proof. Let $c = 1 + \delta$, $\delta > 0$. By (3.2)

$$1 > j_\kappa(c\kappa) > 1 - \frac{1}{2}(1+\delta)^{2\kappa} \exp(-2\kappa\delta) = 1 - \frac{1}{2}\Big(\frac{1+\delta}{e^\delta}\Big)^{2\kappa} \to 1$$

as $\kappa \to \infty$. ∎

The theorem is most effective when u is large. As an illustration of its use, we have $\sigma_\kappa(3.5\kappa) = j_\kappa(1.75\kappa) > 0.99995$ for $\kappa \geqslant 25$. In an earlier paper, we had been able to show only that $\sigma_\kappa(3.5\kappa) > 0.99994$ when $\kappa \geqslant 200$.

The following examples illustrate the accuracy – and the limitations – of formulas (3.2) and (3.3) for κ and u of modest size. For $\kappa = 2$ and $u = 6$ we have

$$\begin{aligned} 1 - j_2(6) &< 0.00821\ldots \ \text{(using (3.2))} \\ &< 0.00324\ldots \ \text{(using (3.3) – first form)} \\ &= 0.000908\ldots \ \text{(calculation)} \end{aligned}$$

We had remarked earlier that the differential inequality for j gave poorer estimates than did the recurrence. We note in conclusion that estimates of $j'(u)$ as $u \to \infty$ of the quality of (3.3) are easy to achieve. By (1.3)

$$\begin{aligned} uj'(u) &= \kappa(1 - j(u-1)) - \kappa(1 - j(u)) \\ &< \kappa(1 - j(u-1)), \qquad u > 1. \end{aligned}$$

In light of (2.12), little has been lost by omitting the term involving $1 - j(u)$ when u is large. Thus when $n \geqslant 1$, we have

$$j_\kappa'(n+1+\kappa) < \frac{\kappa}{n+1+\kappa}(1 - j_\kappa(n+\kappa)),$$

and we may apply (3.3) to estimate the last factor.

Added in proof. At the end of Section 2, we observed that the asymptotic estimate (2.14) for $1 - j(u)$ produced by using the differential equation was worse than that found by using the recursion (2.13). We have now obtained an estimate for $1 - j(u)$ having the size predicted by the recursion. The method is based on establishing a monotonicity of j''/j'. The details will be given in our forthcoming monograph on sieves.

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