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TRACES ON FRACTALS OF FUNCTION SPACES WITH MUCKENHOUPT WEIGHTS

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Dedicated to Professor Hans Triebel
on the occasion of his 70th birthday

Abstract: The present paper is devoted to the study of traces on fractals of weighted Besov spaces $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ as well as weighted Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ with $w^\Gamma_\varkappa(x) = \operatorname{dist}(x,\Gamma)^\varkappa$, where Γ is some d-set, $0 < d < n$, $\varkappa > -(n-d)$.
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0. Introduction

The main purpose of this work is to present a solution of the trace problem for the weighted Besov spaces $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ and weighted Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ where the underlying weight $w^\Gamma_\varkappa$ is a function that measures the distance of a given point $x \in \mathbb{R}^n$ to a certain fractal set Γ, $w^\Gamma_\varkappa(x) = \operatorname{dist}(x,\Gamma)^\varkappa$, x near Γ. The precise definition of the weight $w^\Gamma_\varkappa$ will be stated in Section 1. The approach taken in this work is to a large extent based on Triebel's monograph [21, Section 18]. Recall that the classical trace operator $\operatorname{tr}_{\mathbb{R}^{n-1}}$ is a mapping given by

$$\operatorname{tr}_{\mathbb{R}^{n-1}}: \quad f(x) \longrightarrow f(x', 0) \tag{0.1}$$

for $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. In other words, $\operatorname{tr}_{\mathbb{R}^{n-1}}$ restricts functions on $\mathbb{R}^n$ to the hyperplane $H = \{x \in \mathbb{R}^n : \ x_n = 0\}$. Given a function space $X \subset \mathcal{D}'(\mathbb{R}^n)$, the trace problem consists in finding a space $Y \subset \mathcal{S}'(\mathbb{R}^{n-1})$ such that $\operatorname{tr}_{\mathbb{R}^{n-1}}$ is a bounded linear surjection from X to Y. There is quite an extensive literature concerning trace problems for classical Besov and Triebel-Lizorkin spaces, beginning with the work of H. Triebel [18] as well as of B. Jawerth [8]. The interested reader is referred to [20, Chapter 4.4] for a new approach to this topic using atomic decompositions and local means techniques. It is natural to try to replace the

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hyperplane H by appropriate fractal sets Γ. Possible candidates for fractal sets to consider are d-sets. The problem of characterizing traces on fractals attracted great attention rather recently, and important progress had been made in [21, Chapter 18]. The treatment of the fractal trace problem for weighted function spaces has been inspired by the unweighted results due to H. Triebel [21, Chapter 18]. The corresponding trace operator tr_Γ shall map weighted function spaces of type $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ and $F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ into suitable function spaces on Γ. For a deeper discussion of analysis on fractals we refer the reader to the Section 2. It may be worth reminding the reader that the weight function $w^\Gamma_\varkappa$ belongs to the Muckenhoupt class $\mathcal{A}_r$ if, and only if, $-(n-d) < \varkappa < (n-d)(r-1)$. A more complete theory may be obtained by considering trace problems for weighted function spaces of Besov and Triebel-Lizorkin type with weights from Muckenhoupt class, but we will not develop this point here. The basic idea is to investigate the interaction between the structure of fractals and the smoothness of the underlying functions by means of the corresponding weight function. The essential tool in proving our results will be atomic decomposition of function spaces with Muckenhoupt weights, which are proved in the greatest generality in the recent work [9].

The outline of this work is as follows. In the next section we introduce notation and certain preliminaries. Furthermore frequently used definitions and basic results on Muckenhoupt class, weighted function spaces and its atomic decompositions are discussed. Sections 2-4 are devoted to the trace problem, beginning with a heuristic approach. Our main result can be found in Section 3, Theorem 3.1. Precisely, we prove first a weighted counterpart of some recent results on traces parallel to the unweighted case in [21, Section 18] for B-case. Let $0 < d < n$, $\varkappa > -(n-d)$, $0 < p < \infty$, $0 < q \leqslant \min(1,p)$ and let Γ be a d-set. Then we have

$$\mathrm{tr}_\Gamma B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = L_p(\Gamma), \tag{0.2}$$

where $\mathrm{tr}_\Gamma f \in L_p(\Gamma)$ for any $f \in B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ and any $f^\Gamma \in L_p(\Gamma)$ is a trace of a suitable $g \in B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ on Γ. Furthermore

$$\|f^\Gamma \ |L_p(\Gamma)\| \sim \inf \left\| g \ |B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\|,$$

where the infimum is taken over all $g \in B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ such that $\mathrm{tr}_\Gamma g = f^\Gamma$. Furthermore, for d-set Γ with $0 < d < n$, $0 < p < \infty$ and $0 < q \leqslant \infty$ we obtain

$$\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathbb{B}^\sigma_{pq}(\Gamma) \qquad \text{with} \qquad \sigma = s - \frac{n-d}{p} - \frac{\varkappa}{p}$$

where $-(n-d) < \varkappa < sp - (n-d)$ and $\mathbb{B}^\sigma_{pq}(\Gamma)$ is the trace space according to the Definition 3.2.

Further consequences for F-spaces, are given in the last Section (Theorem 4.4). Let Γ d-set with $0 < d < n$, $0 < p < \infty$, $0 < q \leqslant \infty$, and

$-(n-d) < \varkappa < sp-(n-d)$. Then we achieve that

$$\mathrm{tr}_\Gamma F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathrm{tr}_\Gamma B^{s-\frac{\varkappa}{p}}_{pp}(\mathbb{R}^n) = \mathbb{B}^\sigma_{pp}(\Gamma) \quad \text{with} \quad \sigma = s - \frac{n-d}{p} - \frac{\varkappa}{p} \tag{0.3}$$

where $\mathbb{B}^\sigma_{pp}(\Gamma)$ is the trace space as above. Moreover for $0<p\leqslant 1$ and $0<q\leqslant\infty$ we have

$$\mathrm{tr}_\Gamma F^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = L_p(\Gamma). \tag{0.4}$$

At the end of this paper we characterize traces on $n-1$ dimensional hyperplanes of Sobolev spaces with special case of the weight function $w^\Gamma_\varkappa$ for $d=n-1$, $w_\alpha(x) = |x_n|^\alpha$. Recall that w_α belong to the Muckenhoupt class $\mathcal{A}_p$ if, and only if, $-1<\alpha<p-1$, (see [9]). Let $1<p<\infty$ and $-1<\alpha<p-1$. Then for any $k\in\mathbb{N}$

$$\mathrm{tr}_{\mathbb{R}^{n-1}} W^k_p(\mathbb{R}^n, w_\alpha) = \mathbb{B}^{k-\frac{\alpha+1}{p}}_{pp}(\Gamma) = \mathbb{B}^{k-\frac{\alpha+1}{p}}_{pp}(\mathbb{R}^{n-1}).$$

1. Weights and function spaces

This section collects basic notations and concepts. Let $\mathbb{N}$ be the collection of all natural numbers. Furthermore, let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ stand for non-negative integers . In the sequel let $\mathbb{N}^n_0$ denote the set of all multi-indices $\alpha = (\alpha_1,\ldots,\alpha_n)$ with $\alpha_i\in\mathbb{N}_0$, $|\alpha| = \alpha_1+\ldots+\alpha_n$, and

$$\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}, \qquad \alpha\in\mathbb{N}^n_0.$$

Let $\mathbb{C}$ stand for the complex numbers, and $\mathbb{Z}^n$ denote the lattice of all points in $\mathbb{R}^n$. Let for $m\in\mathbb{Z}^n$ and $\nu\in\mathbb{N}_0$, $Q_{\nu m}$ denote the n-dimensional cube with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$. We denote by $\mathcal{D}(\Omega)$ the class of infinitely differentiable functions with compact support in Ω. $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class of all complex-valued, rapidly decreasing C^∞-functions. The space of continuous linear functionals on $\mathcal{D}$ and $\mathcal{S}$ will be denoted by $\mathcal{D}'$ and $\mathcal{S}'$ respectively. If $\varphi\in\mathcal{S}(\mathbb{R}^n)$ then

$$\mathcal{F}\varphi(\xi) := (2\pi)^{\frac{-n}{2}}\int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)\,\mathrm{d}x, \qquad \xi\in\mathbb{R}^n, \tag{1.1}$$

is the Fourier transform of φ where $x\xi$ denotes the scalar product in $\mathbb{R}^n$. As usual, $\mathcal{F}^{-1}\varphi$ stands for the inverse Fourier transform, given by the right-hand side of (1.1) with i in place of $-i$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'$ in standard way. To obtain a smooth dyadic resolution of unity $\{\varphi_j\}_{j=0}^\infty$ we choose a function $\varphi_0 = \varphi\in\mathcal{S}(\mathbb{R}^n)$ such that

$$\mathrm{supp}\,\varphi\subset\{y\in\mathbb{R}^n\ :|y|<2\} \quad \text{and} \quad \varphi(x)=1 \ \text{ if } \ |x|\leqslant 1,$$

and for each $j\in\mathbb{N}$, let φ_j be given by $\varphi_j(x) = \varphi(2^{-j}x)-\varphi(2^{-j+1}x)$.

For a real number t let $[t] = \max\{a \in \mathbb{Z} : \ a \leqslant t\}$, i.e. the greatest integer less than or equal to t. The positive part of a real function f is given by $f_+(x) = \max(f(x), 0)$. In the sequel let both $\mathrm{d}x$ and $|\cdot|$ stand for the Lebesgue measure.

A positive function $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the Muckenhoupt class $\mathcal{A}_p$, $1 < p < \infty$ if there exists a constant $0 < A < \infty$ such that for all balls B holds

$$\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)^{1/p} \cdot \left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\mathrm{d}x\right)^{1/p'} \leqslant A, \tag{1.2}$$

where p' is the dual exponent to p given by $1/p' + 1/p = 1$.

Recall that these classes are monotonically ordered, $A_{p_1} \subset A_{p_2}$ for $p_1 < p_2$. Moreover for $p = \infty$, the Muckenhoupt class $\mathcal{A}_\infty$ is given by

$$\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p. \tag{1.3}$$

We define

$$r_0 := \inf\{r \ : w \in \mathcal{A}_r\} < \infty, \quad w \in \mathcal{A}_\infty. \tag{1.4}$$

These classes of weights function were introduced by B. Muckenhoupt in [11]. A systematic study of Muckenhoupt classes was initiated by García-Cuerva and Rubio de Francia in [7] and Strömberg, Torchinsky in [17]. We refer the reader also to [16, Ch. V], for more details. Furthermore, Rychkov considered in [15] a more general class of weights, the class $\mathcal{A}_p^{\mathrm{loc}}$ that contains Muckenhoupt class as well locally regular weights. But we shall not deal with this class in the present paper.

Definition 1.1. Let $0 < d < n$. A set $\Gamma \subset \mathbb{R}^n$ is called *d-set*, if there exists a Borel measure μ in $\mathbb{R}^n$ such that $\operatorname{supp}\mu = \Gamma$ and there are constants $c_1, c_2 > 0$ such that for arbitrary $\gamma \in \Gamma$ and all $0 < r < 1$ holds

$$c_1 r^d \leqslant \mu(B(\gamma, r) \cap \Gamma) \leqslant c_2 r^d.$$

Note that self-similar fractals, like the famous Cantor or Sierpinski triangle, are special examples of d-sets. It is well-known that $\mu \sim \mathcal{H}^d$, the d-dimensional Hausdorff measure, see [21, Chapter 1].

Definition 1.2. Let Γ be a non-empty Borel set in $\mathbb{R}^n$ with $|\Gamma| = 0$. We say that Γ satisfies the *ball condition* if there is a number $0 < \eta < 1$ such that for any ball $B(x, r)$ centered at $x \in \Gamma$ and of radius $0 < r < 1$ there is a ball $B(y, \eta r)$ centered at some $y \in \mathbb{R}^n$, and of radius ηr with

$$B(y, \eta r) \subset B(x, r) \qquad \text{and} \qquad B(y, \eta r) \cap \overline{\Gamma} = \emptyset. \tag{1.5}$$

Note that any d- set possesses this feature, see [22, Proposition 9.18].

The Lebesgue space $L_p(\Gamma)$ on the d-set Γ with $0 < p < \infty$ is the usual complex L_p-space with respect to the d-dimensional Hausdorff measure μ equipped with the quasi-norm

$$\|f|L_p(\Gamma)\| = \left(\int_\Gamma |f(\gamma)|^p \mu(\mathrm{d}\gamma)\right)^{1/p} < \infty.$$

For a weight $w \in \mathcal{A}_\infty$ and $0 < p \leqslant \infty$ we define the weighted Lebesgue space $L_p(\mathbb{R}^n, w)$ as collection of all measurable functions such that

$$\|f\ |L_p(\mathbb{R}^n, w)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\mathrm{d}x\right)^{1/p} \tag{1.6}$$

is finite. Obviously, for $p = \infty$ one obtains the unweighted Lebesgue space, normed by

$$\|f\ |L_\infty(\mathbb{R}^n)\| = \operatorname{esssup}|f(x)|.$$

Thus we shall assume $p < \infty$ in the sequel.

Definition 1.3. Let $0 < q \leqslant \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $\{\varphi_j\}_{j=0}^\infty$ a smooth dyadic resolution of unity. Assume $w \in \mathcal{A}_\infty$. *The weighted Besov space* $B^s_{pq}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\|f\ |B^s_{pq}(\mathbb{R}^n, w)\right\| = \left(\sum_{j=0}^\infty 2^{jsq}\left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)|L_p(\mathbb{R}^n, w)\right\|^q\right)^{1/q} \tag{1.7}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

Remark 1.4. The discussion on weighted Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^n, w)$ with $0 < q \leqslant \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty$ will be postponed to the last section. Note that for $w \equiv 1$ we get classical Besov spaces $B^s_{pq}(\mathbb{R}^n)$. For a systematic treatment of the unweighted spaces we refer the interested reader to monographs of H. Triebel [18], [19] and [20]. It turns out that the space $B^s_{pq}(\mathbb{R}^n, w)$ does not depend on the particular choice of the resolution of unity (φ_j), see [13], [2]. This space is a quasi-Banach space, and if $p \geqslant 1$ and $q \geqslant 1$ then $B^s_{pq}(\mathbb{R}^n, w)$ becomes a Banach space. For $p < \infty$, $q < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n, w)$.

We recall the counterpart of atomic decomposition in Besov spaces.

Definition 1.5. a) Suppose that $K \in \mathbb{N}_0$ and $b > 1$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an 1_K-*atom* (or simply an 1-*atom*) if the following assumptions are satisfied

i) $\operatorname{supp} a \subset bQ_{0m}$ for some $m \in \mathbb{Z}^n$,

ii) $|\mathrm{D}^\alpha a(x)| \leqslant 1$ for $|\alpha| \leqslant K$, $x \in \mathbb{R}^n$.

b) Suppose that $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$ and $b > 1$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s,p)_{K,L}$-*atom* (or simply an (s,p) -atom) if for some $\nu \in \mathbb{N}_0$ the following assumptions are satisfied

i) $\operatorname{supp} a \subset bQ_{\nu m}$ for some $m \in \mathbb{Z}^n$,

ii) $|\mathrm{D}^\alpha a(x)| \leqslant 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu}$ for $|\alpha| \leqslant K$, $x \in \mathbb{R}^n$,

iii) $\int_{\mathbb{R}^n} x^\beta a(x)\,\mathrm{d}x = 0$ for $|\beta| \leqslant L$.

We will write $a_{\nu m}(x)$ instead of $a(x)$ to indicate the localization and size of an atom. Note that the moment condition b) (iii) can be reformulated as $\mathrm{D}^\beta(\mathcal{F}a)(0) = 0$ for $|\beta| \leqslant L$ which shows that a sufficiently strong decay of $\mathcal{F}a$ at the origin is required. If $L = -1$ then b) (iii) must be interpreted in the sense that there is no moment condition.

Let $\chi^{(p)}_{\nu m}(x)$ the p-normalized characteristic function of the cube $Q_{\nu m}$ defined by

$$\chi^{(p)}_{\nu m}(x) = 2^{\frac{\nu n}{p}}\, \chi_{\nu m}(x) = \begin{cases} 2^{\frac{\nu n}{p}} & \text{for} \quad x \in Q_{\nu m} \\ 0 & \text{for} \quad x \notin Q_{\nu m}, \end{cases} \tag{1.8}$$

such that $\|\chi^{(p)}_{\nu m}\ |L_p(\mathbb{R}^n)\| = 1$.

Definition 1.6. Let $0 < p < \infty$, $0 < q \leqslant \infty$, $w \in \mathcal{A}_\infty$, and put $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$. We define

$$\begin{aligned} &b_{pq}(w) \qquad\qquad (1.9)\\ &= \left\{ \lambda = \{\lambda_{\nu m}\} \ : \big\|\lambda\ |b_{pq}(w)\big\| = \left(\sum_{\nu=0}^\infty \Big\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi^{(p)}_{\nu m}\ |L_p(\mathbb{R}^n, w)\Big\|^q \right)^{1/q} < \infty \right\}. \end{aligned}$$

Recall that for $w \equiv 1$ we have $b_{pq}(w) = b_{pq}$, see []Example 3.9(1)]nasza. In [9] the following example of a weight function was considered.

Example 1.7. Let Γ be a d-set with $0 < d < n$ introduced in Definition 1.1 and let $\varkappa \in \mathbb{R}$. We study the weight $w^\Gamma_\varkappa(x)$, $x \in \mathbb{R}^n$, given by

$$w^\Gamma_\varkappa(x) = \begin{cases} \operatorname{dist}(x,\Gamma)^\varkappa & \operatorname{dist}\,(x,\Gamma) \leqslant 1 \\ 1 & \text{otherwise}, \end{cases} \tag{1.10}$$

[9, Proposition 2.8, Corollary 2.10]. In the same paper we have shown that $w^\Gamma_\varkappa \in \mathcal{A}_r$, $1 < r < \infty$ if, and only if,

$$-(n-d) < \varkappa < (n-d)(r-1), \tag{1.11}$$

thus $w^\Gamma_\varkappa \in \mathcal{A}_\infty$ if, and only if,

$$\varkappa > -(n-d). \tag{1.12}$$

For the proof and more details we refer to [9, Proposition 2.8 (ii)].

Atomic representation results provide an essential tool for studying trace problems. For our purpose, we need an atomic decomposition of weighted Besov spaces $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$. It is proved in full generality in [9, Theorem 3.10], see also [1].

We use utilize the following abbreviation in the sequel

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ .$$

Theorem 1.8. *Let* $0 < d < n$ *and let* Γ *be a* d*-set in* $\mathbb{R}^n$ *in the sense of Definition 1.1. Moreover let* $w^\Gamma_\varkappa$ *be the weight introduced in* (1.10) *with* $\varkappa > -(n-d)$, *and* $r_0 = \max\left(\frac{\varkappa}{n-d}+1, 1\right)$. *Assume* $0 < p < \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$. *Let* $K, L+1 \in \mathbb{N}_0$ *with*

$$K \geqslant (1+[s])_+, \quad \textit{and} \quad L \geqslant \max\,(-1, [\sigma_{p/r_0} - s]). \tag{1.13}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \qquad \textit{converging in} \quad \mathcal{S}'(\mathbb{R}^n), \tag{1.14}$$

where $a_{\nu m}(x)$ *are* 1_K*-atoms* ($\nu = 0$) *or* $(s,p)_{K,L}$*-atoms* ($\nu \in \mathbb{N}$) *and* $\lambda \in b_{pq}(w_\varkappa)$. *Furthermore, taking the infimum over all admissible representations* (1.14) *with*

$$\|\lambda | b_{pq}(w^\Gamma_\varkappa)\| \sim \left(\sum_{\nu=0}^{\infty}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w^\Gamma_\varkappa(x^{\nu,m})\right)^{q/p}\right)^{1/q}, \tag{1.15}$$

we obtain an equivalent quasi-norm in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$, *where* $x^{\nu,m} \sim 2^{-\nu}m$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ *are chosen such that* $w^\Gamma_\varkappa(x^{\nu,m}) \sim \|\chi^{(p)}_{\nu m} | L_p(\mathbb{R}^n, w^\Gamma_\varkappa)\|$. *In particular, for* $-(n-d) < \varkappa \leqslant 0$ *we can replace* (1.13) *by its unweighted counterpart,*

$$K \geqslant (1+[s])_+, \quad \text{and} \quad L \geqslant \max\,(-1, [\sigma_p - s]), \tag{1.16}$$

such that for $s > \sigma_p$ *no moment conditions are necessary for the corresponding atoms in* (1.14).

2. Traces of Besov spaces on fractals: a heuristic approach

There is a variety of literature on traces on $\mathbb{R}^n$ both for Besov and Triebel-Lizorkin spaces, but the systematic study of trace problems in the framework of fractal sets started rather recently in [21] only. This section contains results on traces of Besov spaces with Muckenhoupt weights, on fractals. Let us start by summarizing unweighted results in this direction. Recall that for $x = (x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$ the mapping

$$\mathrm{tr}_{\mathbb{R}^{n-1}} : f(x) \mapsto f(x', 0) \tag{2.1}$$

is called trace of f on $\mathbb{R}^{n-1}$ introduced in the standard way, see also the argument below. The following theorem gives the complete answers to the trace problem in the case of a hyperplane $\mathbb{R}^{n-1}$.

Theorem 2.1. i) *Let* $0 < p, q \leqslant \infty$ *and* $s - \frac{1}{p} > (n-1)(\frac{1}{p} - 1)_+$. *Then we get*

$$\mathrm{tr}_{\mathbb{R}^{n-1}} B^s_{pq}(\mathbb{R}^n) = B^{s-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}). \tag{2.2}$$

ii) *Let* $n \geqslant 2$, $0 < p < \infty$ *and* $0 < q < \min(1, p)$. *Then we get*

$$\mathrm{tr}_{\mathbb{R}^{n-1}} B^{\frac{1}{p}}_{pq}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}). \tag{2.3}$$

Classical references for trace problems in that case are [20, 4.4.1 and 4.4.2]. We shall now extend assertions of type (2.3) to the case of suitable compact d-sets instead of hyperplanes in $\mathbb{R}^{n-1}$. In the sequel any function $f^\Gamma \in L_p(\Gamma)$, $1 \leqslant p \leqslant \infty$, will be interpreted as a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$f(\varphi) = \int_\Gamma f^\Gamma(\gamma)(\varphi|\Gamma)(\gamma)\mu(\mathrm{d}\gamma), \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where the restriction $\varphi|\Gamma$ of φ is understood pointwise and μ is a Radon measure on Γ. We explain the fractal counterpart of (2.1) now.

Let us temporarily consider a closed set $\Gamma \subset \mathbb{R}^n$ with $|\Gamma| = 0$ and assume that there exists a Radon measure μ on $\mathbb{R}^n$ with $\mathrm{supp}(\mu) = \Gamma$. Therefore the restriction $\mathrm{tr}_\Gamma \varphi = \varphi|\Gamma$ understood pointwise is well-defined for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Moreover let us suppose that for $s > 0$ and $0 < p, q < \infty$ there is a constant $c > 0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\mathrm{tr}_\Gamma \varphi | L_p(\Gamma)\| \leqslant c \|\varphi | B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\|. \tag{2.4}$$

Since the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$, the inequality (2.4) may be extended by completion to all $f \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$. The resulting limit of $\mathrm{tr}_\Gamma \varphi$ will be denoted by $\mathrm{tr}_\Gamma f$. Note that it is independent of the approximation of $f \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ by $\mathcal{S}(\mathbb{R}^n)$-functions due to (2.4).

We first recall what is known on traces of unweighted Besov spaces on a d-set Γ.

Theorem 2.2. *Let* Γ *be a* d*-set with* $0 < d < n$. *Moreover let* $0 < p < \infty$ *and* $0 < q \leqslant \min(1, p)$. *Then*

$$\mathrm{tr}_\Gamma B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma). \tag{2.5}$$

The interpretation of the equality (2.5) *is that* $\mathrm{tr}_\Gamma f \in L_p(\Gamma)$ *for any* $f \in B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$, *and that any* $f^\Gamma \in L_p(\Gamma)$ *is a trace of a suitable* $g \in B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ *on* Γ *in the above described sense with*

$$\|f^\Gamma | L_p(\Gamma)\| \sim \inf \|g \, | B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ *such that* $\mathrm{tr}_\Gamma g = f^\Gamma$.

For a complete discussion and proof we refer to [21, Theorem 18.6, Corollary 18.12] in connection with [22, Remark 9.19]. The interested reader will find there also further references.

From now on let $0 < p < \infty$, $0 < q \leqslant \infty$, $\sigma \in \mathbb{R}$. We will work in the framework of a d-set Γ as introduced in Definition 1.1 with $0 < d < n$. Moreover let $w_\varkappa$ be the weight according to Definition 1.10 and $\varkappa > -(n-d)$. Recall that by Theorem 1.8 the question wheter a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the weighted Besov space $B^\sigma_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ can be equivalently expressed in terms of sequence spaces, $\lambda \in b_{pq}(w^\Gamma_\varkappa)$, where we use the appropriate atomic decomposition in the form

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x) \tag{2.6}$$

with suitable coefficients $\lambda_{\nu m}$ and (σ, p)-atoms $a_{\nu m}$. In the sequel we shall divide the summation over $m \in \mathbb{Z}^n$ in (2.6) with respect to the following "remainder" set

$$\mathrm{I}_{\Gamma,\nu} = \left\{ m \in \mathbb{Z}^n : \operatorname{dist}(\Gamma, \operatorname{supp} a_{\nu m}) > b2^{-\nu} \right\}, \quad \nu \in \mathbb{N}_0, \tag{2.7}$$

i.e. for $m \in \mathrm{I}_{\Gamma,\nu}$ the supports of the corresponding atoms have an empty intersection with Γ. To shorten the notation we utilize the following abbreviations for respective sums,

$$\sum_{m\in\mathbb{Z}^n\setminus \mathrm{I}_{\Gamma,\nu}} = \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \qquad \text{and} \qquad \sum_{m\in \mathrm{I}_{\Gamma,\nu}} = \sum_{\Gamma,\nu \atop m\in\mathbb{Z}^n},$$

such that $\sum^{\Gamma,\nu}$ collects all atoms with a support near to Γ, and $\sum_{\Gamma,\nu}$ the remaining ones, that are less important for trace problems on Γ. This notation allows us to write (2.6) as

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(x) + \sum_{\nu=0}^{\infty} \sum_{\Gamma,\nu \atop m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x). \tag{2.8}$$

Subsequently, we simplify the writing by denoting by f^Γ and f_Γ the first and second sum, respectively, i.e.

$$f^\Gamma = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m} \quad \text{and} \quad f_\Gamma = \sum_{\nu=0}^{\infty} \sum_{\Gamma,\nu \atop m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}.$$

A careful look at (2.7) shows that f_Γ has no influence on the trace problem on Γ. It implies that $\mathrm{tr}_\Gamma a_{\nu m}(x) = 0$ for $m \in \mathrm{I}_{\Gamma,\nu}$. For $m \in \mathbb{Z}^n \setminus \mathrm{I}_{\Gamma,\nu}$ we obtain that $w^\Gamma_\varkappa(x^{\nu,m}) \sim 2^{-\nu\varkappa}$. Consequently, f and f^Γ possess the same trace on Γ,

$$\mathrm{tr}_\Gamma f = \mathrm{tr}_\Gamma f^\Gamma.$$

Assume for the moment that $f \in \mathcal{S}(\mathbb{R}^n)$ and the trace is taken pointwise; recall that $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ for $q < \infty$.

Let us now consider the following reformulation of f^Γ,

$$f^\Gamma = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \left(\lambda_{\nu m} 2^{-\nu\frac{\varkappa}{p}}\right)\left(2^{\nu\frac{\varkappa}{p}} a_{\nu m}(x)\right) = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \widetilde{\lambda}_{\nu m} \widetilde{a}_{\nu m}(x), \tag{2.9}$$

where $\widetilde{\lambda}_{\nu m} = \lambda_{\nu m} 2^{-\nu\frac{\varkappa}{p}}$ are new coefficients and $\widetilde{a}_{\nu m} = 2^{\nu\frac{\varkappa}{p}} a_{\nu m}$ are $(\sigma - \frac{\varkappa}{p}, p)_{K,L}$-atoms, accordingly. Let $\widetilde{f}^\Gamma$ be given by (2.9) and let $\tilde{\lambda}_{\nu m} = 0$, $m \in \mathrm{I}_{\Gamma,\nu}$, $\nu \in \mathbb{N}_0$.

Applying Theorem 1.8 jointly with its unweighted counterpart for $w \equiv 1$, see also, [21, Theorem 3.8 p.75], to (2.9) yields

$$\begin{aligned} \left\| \widetilde{f}^\Gamma | \, B^{\sigma-\frac{\varkappa}{p}}_{pq}(\mathbb{R}^n)\right\| &\leqslant \left\|\widetilde{\lambda} | \, b_{pq}\right\| = c \left(\sum_{\nu=0}^{\infty} \left(\sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} |\lambda_{\nu m}|^p 2^{-\nu\varkappa}\right)^{q/p}\right)^{1/q} \\ &\leqslant c' \left\|\lambda | \, b_{pq}(w^\Gamma_\varkappa)\right\| \leqslant c'' \left\| f | \, B^\sigma_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\|, \end{aligned} \tag{2.10}$$

for suitably chosen $\{\lambda_{\nu m}\}$,i.e. $f \in B^\sigma_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ implies $\widetilde{f}^\Gamma \in B^{\sigma-\frac{\varkappa}{p}}_{pq}(\mathbb{R}^n)$.

Assume for the moment that $\sigma - \frac{\varkappa}{p} = \frac{n-d}{p}$, i.e. $\sigma = \frac{\varkappa+n-d}{p} > 0$, and $\mathrm{tr}_\Gamma f = f^\Gamma = \mathrm{tr}_\Gamma \widetilde{f}$. Then $f \in B^\sigma_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ leads to $\mathrm{tr}_\Gamma f \in L_p(\Gamma)$,that is $\mathrm{tr}_\Gamma B^{\frac{\varkappa+n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) \subset L_p(\Gamma)$, see Theorem 3.1 below.

3. Traces on fractals of weighted Besov spaces

We can formulate the first main result of this paper, which extends Theorem 2.2 to the weighted case.

Theorem 3.1. *Let $0 < d < n$, $\varkappa > -(n-d)$, $0 < p < \infty$, $0 < q \leqslant \min(1, p)$ and let Γ be a d-set. Then we have*

$$\mathrm{tr}_\Gamma B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = L_p(\Gamma), \tag{3.1}$$

in the sense, that $\mathrm{tr}_\Gamma f \in L_p(\Gamma)$ for any $f \in B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ and any $f^\Gamma \in L_p(\Gamma)$ is a trace of a suitable $g \in B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ on Γ and

$$\|f^\Gamma \, | L_p(\Gamma)\| \sim \inf \left\| g \, | B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\|,$$

where the infimum is taken over all $g \in B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ such that $\mathrm{tr}_\Gamma g = f^\Gamma$.

Proof. Our proof is based upon ideas found in [21, Theorem 18.6]. We essentially make use of the atomic decomposition techniques from [9].

Step 1. Let us assume that $0<p<\infty$, $0<d<n$ and $0<q\leqslant \min(1,p)$. We first prove that

$$\mathrm{tr}_\Gamma B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma) \subset L_p(\Gamma). \tag{3.2}$$

We start with $\varphi \in \mathcal{S}(\mathbb{R}^n)$. This causes no loss of generality, since the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)$, see [2]. We recall that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ the restriction operator $\mathrm{tr}_\Gamma\varphi = \varphi|\Gamma$ it meant pointwise. We consider an optimal atomic decomposition according to Theorem 1.8 of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ in $B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)$,

$$\varphi = \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \tag{3.3}$$

such that

$$\left\| \varphi| \ B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\right\| \sim \left\|\lambda| \ b_{pq}(w_\varkappa^\Gamma)\right\|. \tag{3.4}$$

Here the coefficients $\lambda_{\nu m}$ and the $\left(\frac{n-d+\varkappa}{p}, p\right)$-atoms $a_{\nu m}$ have the same meaning as explained in Definitions 1.5 and 1.6. In particular, according to Definition 1.5 we have that $\operatorname{supp} a_{\nu m} \subset bQ_{\nu m}$ and

$$|a_{\nu m}(x)| \leqslant 2^{-\nu\left(\frac{n-d+\varkappa}{p}-\frac{n}{p}\right)} = 2^{\frac{\nu(d-\varkappa)}{p}}, \quad m\in\mathbb{Z}^n, \nu\in\mathbb{N}_0. \tag{3.5}$$

Proceeding exactly as in Section 2 let us consider a decomposition $\varphi = \varphi^\Gamma + \varphi_\Gamma$, such that φ^Γ collects all atoms with a non-empty intersection of their support with Γ, and φ_Γ the rest.

Assume first that $0<p\leqslant 1$. In view of (3.4), to prove (3.2) we have to find an estimate from above of the quasi-norm

$$\|\mathrm{tr}_\Gamma\varphi| \ L_p(\Gamma)\|^p = \int_\Gamma |\varphi^\Gamma(\gamma)|^p\mu(\mathrm{d}\gamma) + \int_\Gamma |\varphi_\Gamma(\gamma)|^p\mu(\mathrm{d}\gamma) \tag{3.6}$$

by the quasi-norm $\|\lambda| \ b_{pq}(w_\varkappa)\|$. Taking into account that $a_{\nu m}\cap\Gamma = \emptyset$ for all atoms belonging to the representation of φ_Γ, we immediately get that the last integral in (3.6) vanishes, since then $\int_\Gamma |\varphi_\Gamma(\gamma)|^p\mu(\mathrm{d}\gamma) = 0$. Hence we have

$$\begin{aligned}\|\mathrm{tr}_\Gamma\varphi| \ L_p(\Gamma)\|^p &\leqslant \sum_{\nu=0}^{\infty}\int_\Gamma \left|\sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(\gamma)\right|^p \mu(\mathrm{d}\gamma) \\ &\leqslant c\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} |\lambda_{\nu m}|^p \int_\Gamma |a_{\nu m}(\gamma)|^p\mu(\mathrm{d}\gamma)\end{aligned} \tag{3.7}$$

Recall that

$$\sum_{m\in\mathbb{Z}^n\setminus \mathrm{I}_{\Gamma,\nu}} = \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu}$$

i.e. we consider only atoms with a support near Γ. The rest of atoms play no rle for a trace problem on Γ.

Let us turn our attention to the last integral in (3.7). Since $\mu(\Gamma \cap Q_{\nu m}) \sim 2^{-\nu d}$ by Definition 1.1, we obtain by (3.5),

$$\int_\Gamma |a_{\nu m}(\gamma)|^p \mu(\mathrm{d}\gamma) \leqslant c 2^{\nu(d-\varkappa)} \mu(\Gamma \cap Q_{\nu m}) \sim c 2^{-\nu\varkappa}.$$

Plugging the above estimate into last term in (1.15) yields

$$\|\mathrm{tr}_\Gamma \varphi |\ L_p(\Gamma)\|^p \leqslant c' \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} |\lambda_{\nu m}|^p 2^{-\nu\varkappa} \ \leqslant c'' \left\|\lambda |\ b_{pq}(w_\varkappa^\Gamma)\right\|^p, \tag{3.8}$$

where the last inequality holds by virtue of $q \leqslant p$ and (1.15). Consequently, by (3.4) for $0 < p \leqslant 1$ and $q \leqslant p$ we have

$$\|\mathrm{tr}_\Gamma \varphi\ |L_p(\Gamma)\| \leqslant c' \left\|\lambda \mid b_{pq}(w_\varkappa^\Gamma)\right\| \leqslant c'' \left\|\varphi\ |B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\right\|. \tag{3.9}$$

For $p > 1$ we use the triangle inequality to get

$$\begin{aligned}\|\mathrm{tr}_\Gamma \varphi |\ L_p(\Gamma)\| &\leqslant c' \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} |\lambda_{\nu m}|^p 2^{-\nu\varkappa} \right)^{1/p} \leqslant c' \left\|\lambda\ |b_{p1}(w_\varkappa^\Gamma)\right\| \\ &\leqslant c''\|\lambda\ |b_{pq}(w_\varkappa^\Gamma)\|.\end{aligned} \tag{3.10}$$

Again, the last inequality holds by virtue of $q \leqslant 1$. Finally, we arrive at

$$\|\mathrm{tr}_\Gamma \varphi |\ L_p(\Gamma)\| \leqslant c \left\|\varphi|\ \ B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\right\| \quad \text{with} \quad 0 < p < \infty, 0 < q \leqslant \min(p,1), \tag{3.11}$$

which proves the inclusion (3.2).

Step 2. Let $0 < q \leqslant \min(p,1)$ and $\max\left(\frac{d-\varkappa}{n}, 0\right) = \left(\frac{d-\varkappa}{n}\right)_+ < p < \infty$. We give a proof of the reverse inclusion

$$L_p(\Gamma) \subset \mathrm{tr}_\Gamma B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n). \tag{3.12}$$

We shall adapt the arguments used in Step 2 of the proof of Theorem 18.2 of [21]. It is known that $\mathcal{D}|_\Gamma$ is dense in $L_p(\Gamma)$. Thus, we may work without loss of generality with $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Moreover assume that $\varphi|_\Gamma \neq 0$ and consider the neighborhood of Γ given by

$$\Gamma_k = \left\{x \in \mathbb{R}^n :\ \mathrm{dist}\,(x,\Gamma) < 2^{-k}\right\}.$$

By compactness of Γ together with properties of Hausdorff measure, there are open balls $B(x_j, r)$ with $j = 1, \ldots, N$ centered at Γ with the same radius $r > 0$

depending on the covering, that cover Γ. Note that $\overline{\Gamma}_h \subset \bigcup_{j=1}^N B(x_j, r)$, where h depends on given covering.

Now, let $\{\varphi_j\}_{j=1}^N$ be a smooth resolution of unity in a neighborhood Γ_k of $\Gamma \cap \operatorname{supp} \varphi$ adapted to $(B(x_j,r))_{j=1}^N$. In particular, we have $\sum_{j=1}^N \varphi_j(x) = 1$ for $x \in \operatorname{supp}\varphi$ and $\operatorname{supp}\varphi_j \subset B(x_j,r)$. Let us now put $\lambda_j = \max_{x\in B(x_j,r)} |\varphi(x)|$. Then, by the properties of the above defined resolution of unity we get

$$\varphi(x) = \sum_{j=1}^N \varphi(x)\varphi_j(x) = \sum_{j=1}^N \lambda_j r^{\frac{d-\varkappa}{p}} \left[r^{-\frac{d-\varkappa}{p}} \lambda_j^{-1} \varphi(x)\varphi_j(x)\right], \tag{3.13}$$

where terms with $\lambda_j = 0$ are omitted. Let us define

$$\widetilde{\lambda}_j = \lambda_j r^{\frac{d-\varkappa}{p}} \quad \text{and} \quad a_j(x) = r^{-\frac{d-\varkappa}{p}} \lambda_j^{-1}\varphi(x)\varphi_j(x).$$

We obtain that $\operatorname{supp} a_j \subset B(x_j,r)$. Furthermore, choosing $r>0$ small enough, we get

$$|a_j(x)| = \frac{|\varphi(x)|}{\lambda_j} r^{-\frac{d-\varkappa}{p}} |\varphi_j(x)| \leqslant c' r^{\frac{n-d}{p}+\frac{\varkappa}{p}-\frac{n}{p}}$$

and analogous estimates for all $\mathrm{D}^\alpha a_j$. We thus can consider a_j as $\left(\frac{n-d+\varkappa}{p}, p\right)_{K,L}$-‑atoms according to Definition 1.5. It follows from the assumption $p > \left(\frac{d-\varkappa}{n}\right)_+$ that $\frac{n-d+\varkappa}{p} > n\left(\frac{1}{p}-1\right)_+$. Therefore, moment conditions as needed in (1.16) may be omitted. Once again, using the atomic decomposition method together with properties of the weight $w_\varkappa^\Gamma$ we may estimate the quasi-norm of (3.13) as follows,

$$\begin{aligned} \|\varphi\ |B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\| &\leqslant \|\tilde{\lambda}\ |b_{pq}(w_\varkappa^\Gamma)\| \\ &\leqslant c \left(\sum_{j=1}^N |\lambda_j|^p r^{d-\varkappa} \left\| \chi_{B(x_j,r)}^{(p)}\ |L_p(\mathbb{R}^n, w_\varkappa^\Gamma)\right\| \right)^{1/p}. \end{aligned} \tag{3.14}$$

Let us again choose $r>0$ arbitrarily small. Straightforward computation shows that $\|\chi_{B(x_j,r)}^{(p)}\ |L_p(\mathbb{R}^n, w_\varkappa^\Gamma)\| \sim r^\varkappa$. Moreover we have $\mu(B(x_j,r)) \sim r^d$ by Definition 1.1. Proceeding further as in the Riemann integral construction we arrive at

$$\left(\sum_{j=1}^N |\lambda_j|^p r^{d-\varkappa} \left\| \chi_{B(x_j,r)}^{(p)}\ |L_p(\mathbb{R}^n, w_\varkappa^\Gamma)\right\| \right)^{1/p} \leqslant c\, \|\mathrm{tr}_\Gamma \varphi\ |L_p(\Gamma)\|\,. \tag{3.15}$$

Hence, we have proved that

$$\|\varphi\ |B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\| \leqslant c \|\mathrm{tr}_\Gamma \varphi\ |L_p(\Gamma)\|, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{3.16}$$

The rest of the proof goes through as for [21, Theorem 18.6], with hardly any changes: for convenience, we include the argument here. It follows from density of $\mathcal{D}|_\Gamma$ in $L_p(\Gamma)$ that any $f \in L_p(\Gamma)$ can be represented in the form

$$f(\gamma) = \sum_{j=1}^{\infty} f_j(\gamma), \quad \gamma \in \Gamma, \quad f_j \in \mathcal{D}(\mathbb{R}^n) \tag{3.17}$$

with

$$0 < \|\mathrm{tr}_\Gamma f_j \ |L_p(\Gamma)\| \leqslant c\ 2^{-j} \|f|\ L_p(\Gamma)\|, \quad j \in \mathbb{N}. \tag{3.18}$$

Thus by (3.16) we have

$$\|f_j \ |B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\| \leqslant c' \|\mathrm{tr}_\Gamma f_j \ |L_p(\Gamma)\|. \tag{3.19}$$

Now we may define an extension operator in the following way,

$$\mathrm{ext} f = \sum_{j=1}^{\infty} f_j \in B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma), \quad \mathrm{tr}_\Gamma \mathrm{ext} f = f. \tag{3.20}$$

By virtue of (3.17) and (3.18) we obtain

$$\|\mathrm{ext} f \ |B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\| \leqslant c' \|f \ |L_p(\Gamma)\|. \tag{3.21}$$

This finishes the proof of (3.12).

Step 3. To complete our proof we have to extend the result of Step 2 to $p > 0$, i.e. for $\varkappa < d$. Let us assume now that $0 < q < p \leqslant \frac{d-\varkappa}{n}$. Analysis similar to that in the proof of [21, Corollary, 18.12]Triebel97 shows that for $\left(\frac{n-d+\varkappa}{p}, p\right)_{K,L}$-atoms we do not have moment conditions for $\varphi\varphi_j$ in (3.13) by property (1.13). Let $B(y_j, \eta r)$ be a ball with the condition (1.5) which can be written, after easy reformulation, in the following form

$$\mathrm{dist}(B(y_j, \eta r), \overline{\Gamma}) \geqslant \eta r. \tag{3.22}$$

We follow the argument in Step 2 replacing $\varphi\varphi_j$ by the function

$$\psi_j(x) = (\varphi\varphi_j)(x) + \chi_j(x)$$

where supp $\chi_j \subset B(y_j, \eta r)$ and ψ_j is an $\left(\frac{n-d+\varkappa}{p}, p\right)$-atom with moment conditions according to Definition 1.5 with $L \geqslant \max(-1, \sigma_{p/r_0} - s)$. This is a somewhat tricky construction and can be found in the proof of [24, Theorem 3.6]. The atoms $\varphi\varphi_j$ and ψ_j coincide in a neighbourhood of Γ due to (3.22). Now we can use the argument of Step 2 again. The proof of Theorem 3.1 is thus complete. ■

In the concluding part of this section we shall work with Besov spaces introduced in terms of traces on fractals, and recall their definition first.

Definition 3.2. Let Γ be a d-set in $\mathbb{R}^n$ according to Definition 1.1 with $0 < d < n$. Let $s > 0$, $0 < p \leqslant \infty$, and $0 < q \leqslant \infty$. Let us define

$$\mathbb{B}^s_{pq}(\Gamma) = \mathrm{tr}_\Gamma B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n). \tag{3.23}$$

We equip this space with the quasi-norm

$$\left\| f \;|\mathbb{B}^s_{pq}(\Gamma)\right\| = \inf \left\| g| \; B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\right\|, \tag{3.24}$$

where the infimum ranges over all $g \in B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ with $\mathrm{tr}_\Gamma g = f$.

In a natural way we extend this notation to weighted spaces $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$: by $\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ we mean the collection of all $f \in L_p(\Gamma)$ such that there exists some $g \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ with $\mathrm{tr}_\Gamma g = f$, and $\|f \;|\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\| = \inf \|g \;|B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\|$, the infimum is taken over all $g \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ such that $\mathrm{tr}_\Gamma g = f$. To study the fractal trace problem we get the following statement.

Theorem 3.3. *Let $0 < d < n$, $0 < p < \infty$, $0 < q \leqslant \infty$ and $-(n-d) < \varkappa < sp-(n-d)$. Then*

$$\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathbb{B}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}_{pq}(\Gamma)$$

Proof. Step 1. The idea of the proof is to use Definition 3.2 together with the observation that

$$\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathrm{tr}_\Gamma B^{s-\frac{\varkappa}{p}}_{pq}(\mathbb{R}^n), \tag{3.25}$$

with the parameters given above. Afterwards we apply (3.23) to $s' = s - \frac{\varkappa}{p} - \frac{n-d}{p} > 0$, i.e. such that $s' + \frac{n-d}{p} = s - \frac{\varkappa}{p}$. This leads to

$$\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathbb{B}^{s'}_{pq}(\Gamma),$$

that is, the desired result. Moreover, as will be clear from the argument below, it is sufficient to deal with the inclusion

$$\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) \hookrightarrow \mathrm{tr}_\Gamma B^{s-\frac{\varkappa}{p}}_{pq}(\mathbb{R}^n) \tag{3.26}$$

only, the converse assertion follows by parallel observations.

We consider some $f \in \mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$. Let $\varepsilon > 0$. By the definition of this space there is some $g \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ such that $\mathrm{tr}_\Gamma g = f$ and

$$\left\|g| \; B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| \leqslant \left\|f| \; \mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| + \frac{\varepsilon}{2}. \tag{3.27}$$

We take the atomic decomposition of g in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$,

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \tag{3.28}$$

where $\lambda_{\nu m} \in \mathbb{C}$ are coefficients and $a_{\nu m}(x)$ are $(s,p)_{K,L}$-atoms in the sense of Definition 1.5. In view of Theorem 1.8 we have to choose $K > s$, $L \geqslant \max(-1, [\sigma_{p/r_0} - s])$ with $r_0 = \max(\frac{\varkappa}{n-d}+1, 1)$; so let us assume

$$K > \max(s, s - \frac{\varkappa}{p})$$

and (3.29)

$$L > \max\left(-1, \left[\sigma_{p/r_0} - s\right], \left[\sigma_p - s + \frac{\varkappa}{p}\right]\right).$$

Thus Theorem 1.8 implies that we find a corresponding atomic decomposition (3.28) with (3.29) and

$$\left\|\lambda | \ b_{pq}(w_\varkappa^\Gamma)\right\| \leqslant \left\|g | \ B^s_{pq}(\mathbb{R}^n, w_\varkappa^\Gamma)\right\| + \frac{\varepsilon}{2}. \tag{3.30}$$

We now proceed similar to Section 2. Recall our notation

$$\mathrm{I}_{\Gamma,\nu} = \left\{m \in \mathbb{Z}^n : \ \mathrm{dist}(\Gamma, \mathrm{supp} a_{\nu m}) > b2^{-\nu}\right\}, \qquad \nu \in \mathbb{N}_0,$$

and

$$\sum_{m\in\mathbb{Z}^n\setminus \mathrm{I}_{\Gamma,\nu}} = \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu}, \qquad \sum_{m\in \mathrm{I}_{\Gamma,\nu}} = \sum_{\Gamma,\nu \atop m\in\mathbb{Z}^n}.$$

We decompose

$$g = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(x) + \sum_{\nu=0}^{\infty} \sum_{\Gamma,\nu \atop m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x) := g^\Gamma + g_\Gamma$$

with $\mathrm{tr}_\Gamma g = \mathrm{tr}_\Gamma g^\Gamma$, $\mathrm{tr}_\Gamma g_\Gamma = 0$. We extend g^Γ by 0 outside,

$$\begin{aligned} \widetilde{g} &= \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \left(\lambda_{\nu m} 2^{-\nu\frac{\varkappa}{p}}\right)\left(2^{\nu\frac{\varkappa}{p}} a_{\nu m}(x)\right) + \sum_{\nu=0}^{\infty} \sum_{\Gamma,\nu \atop m\in\mathbb{Z}^n} 0 \cdot \left(2^{\nu\frac{\varkappa}{p}} a_{\nu m}(x)\right) \\ &= \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \widetilde{\lambda}_{\nu m} \widetilde{a}_{\nu m}(x), \end{aligned} \tag{3.31}$$

obtaining an atomic decomposition of $\widetilde{g}$ with

$$\widetilde{\lambda}_{\nu m} = \begin{cases} \lambda_{\nu m} 2^{-\nu\frac{\varkappa}{p}} & \text{for } m \in \mathbb{Z}^n \setminus \mathrm{I}_{\Gamma,\nu}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.32}$$

Moreover $\widetilde{a}_{\nu m} = 2^{\nu\frac{\varkappa}{p}} a_{\nu m}$ are $\left(s - \frac{\varkappa}{p}, p\right)_{K,L}$-atoms. We benefit from our assumption (3.29) and can apply the unweighted version of Theorem 1.8 ($\varkappa = 0, r_0 = 1$), see [20, Theorem 3.10], to obtain

$$\left\|\widetilde{g} \ |B^{s-\frac{\varkappa}{p}}_{pq}(\mathbb{R}^n)\right\| \leqslant \left\|\widetilde{\lambda}_{\nu m} \ |b_{pq}\right\|. \tag{3.33}$$

On the other hand, $\mathrm{tr}_\Gamma \widetilde{g} = \mathrm{tr}_\Gamma g^\Gamma = \mathrm{tr}_\Gamma g = f$, and

$$\left\| \widetilde{\lambda}_{\nu m} \; | b_{pq} \right\| \leqslant \left\| \lambda_{\nu m} \; | b_{pq}(w_\varkappa^\Gamma) \right\| \tag{3.34}$$

by (3.32) and (1.9), recall $\left\| \chi_{\nu m}^{(p)} \; | L_p(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| \sim 2^{\nu \frac{\varkappa}{p}}$, $m \in \mathbb{Z}^n \backslash \mathrm{I}_{\Gamma\nu}$, $\nu \in \mathbb{N}_0$. Combining (3.27), (3.30), (3.33) and (3.34) we obtain

$$\left\| \widetilde{g} \; | B_{pq}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n) \right\| \leqslant c \left\| f \; | \mathrm{tr}_\Gamma B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| + \varepsilon,$$

that is, we have found some $\widetilde{g} \in B_{pq}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n)$ with $\mathrm{tr}_\Gamma \widetilde{g} = f$ and the above norm estimate. Hence, $f \in \mathrm{tr}_\Gamma B_{pq}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n)$, and for $\varepsilon \searrow 0$,

$$\left\| f \; | \mathrm{tr}_\Gamma B_{pq}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n) \right\| \leqslant c \left\| f \; | \mathrm{tr}_\Gamma B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| .$$

This proves (3.26). ∎

In view of (1.12) it is clear that the theory of Besov spaces with Muckenhoupt weights covers only weights $w_\varkappa^\Gamma$ from (1.10) with $\varkappa > -(n-d)$. Theorem 3.1 above concerns weights $w_\varkappa^\Gamma$ with $\varkappa < sp - (n-d)$, $s > 0$, $0 < p < \infty$, where $f \in B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$ possesses a trace $\mathrm{tr}_\Gamma f \in \mathbb{B}_{pq}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}(\Gamma)$.

Similarly for $\varkappa = sp-(n-d)$, $0 < q \leqslant \min(1,p)$, see Theorem 3.1. A natural question to ask is what happens for stronger weights, that is, $\varkappa > sp - (n-d)$ or $\varkappa = sp - (n-d)$ with $q > \min(1,p)$, respectively? The final answer to this question in the unweighted case is due H. Triebel [23, Theorem 1.174], see also [23, Corollary 7.21]. Roughly speaking, the result given there states that for $s < \frac{n-d}{p}$, $0 < p, q \leqslant \infty$, or $s = \frac{n-d}{p}$, $q > \min(p,1)$, the trace space $\mathrm{tr}_\Gamma B_{pq}^s(\mathbb{R}^n)$ does not exist. Below we show how to transfer this observation to our situation.

Corollary 3.4. *Let $0 < d < n$, $s > 0$, $1 < p < \infty$, $0 < q < \infty$ and $\varkappa > -(n-d)$. Then $\mathrm{tr}_\Gamma B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$ exists if, and only if,*

$$\varkappa < sp - (n-d) \qquad \text{or} \quad \varkappa = sp - (n-d) \;\; \text{and} \;\; 0 < q \leqslant 1.$$

Moreover, if $\varkappa > sp - (n-d)$, then $\mathcal{D}(\mathbb{R}^n \backslash \Gamma)$ is dense in $B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$.

Proof. The sufficiency follows from Theorems 3.1 and 3.3, concerning the necessity we refer to [23, Corollary 7.21] for the unweighted case and (3.25). Note that the additional assumption $\varkappa < sp-(n-d)$ or $0 < q \leqslant \min(p,1)$ when $\varkappa = sp-(n-d)$ are needed only later on to determine the trace space explicitly.

It remains to show the density of $\mathcal{D}(\mathbb{R}^n \backslash \Gamma)$ in $B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$ when $\varkappa > sp - (n-d)$. Clearly, by the embeddings

$$B_{pp}^{s+\varepsilon}(\mathbb{R}^n, w_\varkappa^\Gamma) \hookrightarrow B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \hookrightarrow B_{pp}^{s-\varepsilon}(\mathbb{R}^n, w_\varkappa^\Gamma)$$

for all $0 < q < \infty$, and $\varepsilon > 0$ small, it is enough to deal with spaces $B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)$ only, where $\varkappa > sp-(n-d)$ and $0 < p < \infty$. Then $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)$ and we can restrict ourselves to show that for all $\varepsilon > 0$ and all $\psi \in \mathcal{S}(\mathbb{R}^n)$ there is some $\varphi \in \mathcal{D}(\mathbb{R}^n \backslash \Gamma)$, i.e. $\varphi \in C^\infty_0(\mathbb{R}^n)$ with $\operatorname{supp}(\varphi) \subset \mathbb{R}^n \backslash \Gamma$, such that

$$\left\| \psi - \varphi \mid B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| < \varepsilon. \tag{3.35}$$

We continue by assuming that $\operatorname{supp} \psi \cap \Gamma \neq \emptyset$. Otherwise, $\operatorname{dist}(\Gamma, \operatorname{supp} \psi) = \delta > 0$ and we can take $\varphi = \psi$, appropriately modified if $\operatorname{supp} \psi$ is not compact. Let Γ_k be some neighbourhood of $\Gamma \cap \operatorname{supp} \psi$. For $j \in \mathbb{N}$, consider a covering of Γ_k with balls centered at Γ and with radius 2^{-j}. Since Γ is a compact d-set one needs $M_j \sim 2^{jd}$ balls to cover it. Let $\{\varphi_r\}_{r=1}^{M_j}$ be an associated smooth partition of unity such that $\varphi_r \in C^\infty_0(\mathbb{R}^n)$, $\operatorname{supp} \varphi_r \subset B_{r,j} = B(\gamma_r, 2^{-j})$, $\gamma_r \in \Gamma$ and $\sum_{r=1}^{M_j} \varphi_r(x) = 1$ with $x \in \Gamma_k$. Recall that $\left\| \chi^{(p)}_{B_{r,j}} \, | L_p(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| \sim 2^{-j\frac{\varkappa}{p}}$. Let $\gamma \in \mathcal{D}(\mathbb{R}^n)$ with $\gamma = 1$ on $\Gamma_{k/2}$ and $\operatorname{supp} \gamma \subset \Gamma_k$. Taking into account Definition 1.5 and Theorem 1.8 we obtain

$$\gamma = \sum_{r=1}^{M_j} (\varphi_r \gamma)(x) = \sum_{r=1}^{M_j} 2^{j(s-\frac{n}{p})} 2^{-j(s-\frac{n}{p})} (\varphi_r \gamma)(x), \qquad x \in \Gamma_k. \tag{3.36}$$

The sum on the right-hand side of (3.36) may be viewed as an atomic decomposition of γ in $B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)$ with atoms given by $2^{-j(s-\frac{n}{p})}(\varphi_r \gamma)(x)$ and coefficients $\lambda_r = 2^{j(s-\frac{n}{p})}$. For convenience let us assume once more that we do not need moment conditions, otherwise (3.36) has to be modified. Then Theorem 1.8 and Definition 1.6 imply

$$\begin{aligned} \left\| \gamma \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| &\leqslant \left(\sum_{r=1}^{M_j} 2^{j(s-\frac{n}{p})p} \left\| \chi^{(p)}_{B_{r,j}} | \; L_p(\mathbb{R}^n, w^\Gamma_\varkappa) \right\|^p \right)^{1/p} \\ &\leqslant 2^{j(s-\frac{n}{p})-j\frac{\varkappa}{p}} \left(\sum_{r=1}^{M_j} 1 \right)^{1/p} = c 2^{j(s-\frac{n-d}{p}-\frac{\varkappa}{p})}. \end{aligned}$$

It follows from the assumption $s < \frac{\varkappa+n-d}{p}$ that

$$\left\| \gamma \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| < \varepsilon,$$

choosing in our construction j sufficiently large. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we thus arrive at

$$\begin{aligned} \left\| \psi \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| &= \left\| \psi\gamma + (1-\gamma)\psi \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| \\ &\leqslant \left\| \psi\gamma \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| + \left\| (1-\gamma)\psi \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| \\ &\leqslant \left\| \psi \, | C^k(\mathbb{R}^n) \right\| \left\| \gamma \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| + \left\| (1-\gamma)\psi \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\| \\ &< \varepsilon' + \left\| (1-\gamma)\psi \, | B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \right\|, \end{aligned}$$

where $k \in \mathbb{N}$ is chosen large enough. On the other hand, we obtain $(1-\gamma)\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{dist}(\operatorname{supp}((1-\gamma)\psi), \Gamma) > 0$. Hence, there exists some $\varphi \in \mathcal{D}(\mathbb{R}^n \backslash \Gamma)$ with

$$\left\|(1-\gamma)\psi - \varphi \;|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| < \varepsilon.$$

This concludes the proof of (3.35). ■

Remark 3.5. The Corollary 3.4 explains, at least in some cases, the impossibility to have a trace of $f \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$, $\varkappa > sp-(n-d)$ in the sense of $L_p(\Gamma)$. We only get the trivial counterpart of (2.4), i.e. for the dense subset $\mathcal{D}(\mathbb{R}^n\backslash\Gamma)$ in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ the left-hand side in (2.4) always vanishes unlike the right-hand side. But then it is not possible to explain $\operatorname{tr}_\Gamma f$ in a reasonable (standard) way, as the independence of the approximating sequence fails. One would like to have a real alternative in the sense that either $\operatorname{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ exists or $\mathcal{D}(\mathbb{R}^n\backslash\Gamma)$ is dense in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$. But this remains open so far - as in the unweighted case.

4. Traces on fractals of weighted Triebel-Lizorkin spaces and applications

In this section we discuss traces on fractals of weighted Triebel-Lizorkin spaces. Our main aim here is to extend known results on traces of unweighted Triebel-Lizorkin spaces to the weighted case. The last part of this section is devoted to give an application of our results for F-spaces to traces of weighted Sobolev spaces on $(n-1)$-dimensional hyperplanes. Let us start by recalling needed definitions. The best references here are [2] and [9].

Definition 4.1. Let $0 < p < \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty$. Moreover let $\{\varphi_j\}_{j=0}^\infty$ be a smooth partition of unity as introduced in Section 3. *The weighted Triebel - Lizorkin space* $F^s_{pq}(\mathbb{R}^n, w)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\|f \;|F^s_{pq}(\mathbb{R}^n, w)\right\| = \left\| \left(\sum_{j=0}^\infty 2^{jsq} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n, w) \right\| \tag{4.1}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

Note that in this case for $1 < p < \infty$, $s \in \mathbb{N}_0$ and $q = 2$ we obtain classical Sobolev spaces, i.e.

$$F^s_{p2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n),$$

see [19, Section 2], [20, Section 1.2.5] and [21, Section 10.5].

The unweighted trace result due to H. Triebel [21, Corollary 18.12] reads as follows.

Theorem 4.2. *Let Γ be a d-set, $0 < d < n$. Let $0 < p \leqslant 1$ and $0 < q \leqslant \infty$. Then we get*

$$\operatorname{tr}_\Gamma F_{pq}^{\frac{n-d}{p}}(\mathbb{R}^n) = L_p(\Gamma)$$

with the usual interpretation.

Next we define the corresponding Triebel-Lizorkin sequence spaces.

Definition 4.3. Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $w \in \mathcal{A}_\infty$. Furthermore let $\chi_{\nu m}^{(p)}$ denote the p-normalized characteristic function of the cube $Q_{\nu m}$ defined by (1.8). Then

$$f_{pq}(w) = \Big\{\lambda = \{\lambda_{\nu m}\} \; : \tag{4.2}$$

$$\big\|\lambda \;|f_{pq}(w)\big\| = \Big\|\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\Big|\lambda_{\nu m}\chi_{\nu m}^{(p)}(\cdot)\Big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^n, w)\Big\| < \infty\Big\}$$

(usual modification for $q = \infty$).

In the sequel, we again consider the weight $w_\varkappa^\Gamma$ as introduced in (1.10). We now present a generalization of Theorem 4.2 to the weighted case.

Theorem 4.4. *Let Γ be a d-set, $0 < d < n$. Let $0 < p < \infty$, $0 < q \leqslant \infty$, $-(n-d) < \varkappa < sp - (n-d)$, or $\varkappa = sp - (n-d)$ if $0 < p \leqslant 1$. Then*

$$\operatorname{tr}_\Gamma F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) = \operatorname{tr}_\Gamma B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n). \tag{4.3}$$

In particular,

$$\operatorname{tr}_\Gamma F_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma) = L_p(\Gamma) \tag{4.4}$$

for $0 < p \leqslant 1$, $0 < q \leqslant \infty$, and

$$\operatorname{tr}_\Gamma F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) = \mathbb{B}_{pp}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}(\Gamma), \tag{4.5}$$

provided that $\varkappa < sp - (n-d)$, $0 < p < \infty$ and $0 < q \leqslant \infty$.

Proof. The proof is based on the argument given in the proof of Theorem 3.3 combined with [23, Proposition 9.22]. We only outline the main ideas of the proof for

$$\operatorname{tr}_\Gamma F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \subset \operatorname{tr}_\Gamma B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n). \tag{4.6}$$

The proof of the converse inclusion is done analogously. Let $f \in \operatorname{tr}_\Gamma F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$. Following the same consideration as in Step 1 of the proof of Theorem 3.3 we arrive at the atomic decomposition of g in $F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$ and its reformulation for $\widetilde{g}$ as in (3.31). We conclude that $\widetilde{g} \in B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n)$, since

$$\Big\|\widetilde{g}|\ B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n)\Big\| \leqslant \Big\|\widetilde{\lambda}|\ b_{pp}\Big\| \leqslant c\Big\|\widetilde{\lambda}|\ f_{pq}\Big\| \leqslant c\,\big\|\lambda|\ f_{pq}(w_\varkappa^\Gamma)\big\|, \tag{4.7}$$

where the equation $\left\|\widetilde{\lambda}|\ b_{pp}\right\| \sim \left\|\widetilde{\lambda}|\ f_{pq}\right\|$ follows from [23, Proposition 9.22 (i)] since d-sets satisfy the ball condition what means that they are porous in the notation used in [23]. Consequently we have for $\widetilde{g}$ with $\mathrm{tr}_\Gamma \widetilde{g} = f$,

$$\left\|\widetilde{g}|\ B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n)\right\| \leqslant c\left\|g|\ F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| + \frac{\varepsilon}{2} \leqslant \left\|f|\ \mathrm{tr}_\Gamma F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| + \varepsilon, \qquad (4.8)$$

which completes the proof. ■

Remark 4.5. It turns out that the index q plays no rle in the consideration of traces on d-sets of $F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$. More precisely, for $0 < q_0 < q_1 < \infty$ we get

$$\mathrm{tr}_\Gamma F^s_{pq_0}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathrm{tr}_\Gamma F^s_{pq_1}(\mathbb{R}^n, w^\Gamma_\varkappa),$$

as in the unweighted case, see [22, Theorem 9.21].

We have the following counterpart of Corollary 3.4 due to (4.3).

Corollary 4.6. *Let $0 < d < n$, $s > 0$, $1 < p < \infty$, $0 < q \leqslant \infty$ and $\varkappa > -(n-d)$. Then $\mathrm{tr}_\Gamma F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ exists if, and only if, $\varkappa < sp-(n-d)$. Moreover, if $\varkappa > sp-(n-d)$, and $1 < p, q < \infty$, then $\mathcal{D}(\mathbb{R}^n \backslash \Gamma)$ is dense in $F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$.*

We conclude our paper with a well-known example for Sobolev spaces and a d-set Γ with $d = n-1$. We characterize traces on $n-1$-dimensional hyperplanes of Sobolev spaces. We first discuss a special case of the weight function $w^\Gamma_\varkappa$ for $d = n-1$.

Example 4.7. Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Note that for $d = n-1$ and taking $\varkappa = \alpha$ the weight $w^\Gamma_\varkappa$ transforms into

$$w_\alpha(x) = \begin{cases} |x_n|^\alpha & |x_n| < 1 \\ 1 & \text{otherwise}\ . \end{cases} \qquad (4.9)$$

As shown in [9], $w_\alpha(x)$ belongs to the Muckenhoupt class $\mathcal{A}_r$ if, and only if, $-1 < \alpha < r-1$.

We recall briefly the definition of Sobolev spaces.

Definition 4.8. Let $k \in \mathbb{N}$ and $1 \leqslant p < \infty$. The *Sobolev Space* $W^k_p(\mathbb{R}^n, w_\alpha)$ is the collection of all $f \in L_p(\mathbb{R}^n, w_\alpha)$ such that the norm

$$\left\|f|\ W^k_p(\mathbb{R}^n, w_\alpha)\right\| = \left(\sum_{|\beta|\leqslant k} \left\|\mathrm{D}^\beta f|\ L_p(\mathbb{R}^n, w_\alpha)\right\|^p\right)^{1/p}$$

is finite.

It is well-known that for $k \in \mathbb{N}_0$, $1 < p < \infty$, and $w_\alpha \in \mathcal{A}_p$, i.e. $-1 < \alpha < p-1$, we have

$$F^k_{p,2}(\mathbb{R}^n, w_\alpha) = W^k_p(\mathbb{R}^n, w_\alpha). \qquad (4.10)$$

This can be found, for instance in [15, Proposition 1.9]. We are now in a position to state the last result of this paper.

Proposition 4.9. *Let $1 < p < \infty$ and $-1 < \alpha < p-1$. Then for any $k \in \mathbb{N}$*

$$\mathrm{tr}_{\mathbb{R}^{n-1}} W_p^k(\mathbb{R}^n, w_\alpha) = \mathbb{B}_{pp}^{k-\frac{\alpha+1}{p}}(\Gamma).$$

Proof. Using (4.10) and Remark 4.5 combined with Theorems 3.3 and 4.4, we obtain

$$\begin{aligned}\mathrm{tr}_{\mathbb{R}^{n-1}} W_p^k(\mathbb{R}^n, w_\alpha) &= \mathrm{tr}_{\mathbb{R}^{n-1}} F_{p,2}^k(\mathbb{R}^n, w_\alpha) = \mathrm{tr}_{\mathbb{R}^{n-1}} B_{pp}^{k-\frac{\alpha}{p}}(\mathbb{R}^n) \\ &= \mathbb{B}_{pp}^{k-\frac{\alpha}{p}-\frac{1}{p}}(\Gamma) = \mathbb{B}_{pp}^{k-\frac{\alpha+1}{p}}(\Gamma).\end{aligned}$$

Note that our assumption for α to imply $w_\alpha \in \mathcal{A}_p$, i.e. $\alpha < p-1$, already ensures $\alpha < kp-1$, $k \in \mathbb{N}$, needed in Theorem 4.4. ■

Remark 4.10. This result was first proved in [18, Section 3.6] using tricky interpolation techniques.

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