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SPACE OF NONNEGATIVELY CURVED METRICS AND PSEUDOISOTOPIES

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Abstract

Let V be an open manifold with complete nonnegatively curved metric such that the normal sphere bundle to a soul has no section. We prove that the souls of nearby nonnegatively curved metrics on V are smoothly close. Combining this result with some topological properties of pseudoisotopies we show that for many V the space of complete nonnegatively curved metrics has infinite higher homotopy groups.

1. Introduction

Throughout the paper "smooth" means C^∞, all manifolds are smooth, and any set of smooth maps, such as diffeomorphisms, embeddings, pseudoisotopies, or Riemannian metrics, is equipped with the smooth compact-open topology.

Let $\mathfrak{R}_{K\geq 0}(V)$ denote the space of complete Riemannian metrics of nonnegative sectional curvature on a connected manifold V. The group $\mathrm{Diff}\, V$ acts on $\mathfrak{R}_{K\geq 0}(V)$ by pullback. Let $\mathfrak{M}_{K\geq 0}(V)$ be the associated *moduli space*, the quotient space of $\mathfrak{R}_{K\geq 0}(V)$ by the above $\mathrm{Diff}\, V$ action.

Many open manifolds V for which $\mathfrak{M}_{K\geq 0}(V)$ is not path-connected, or even has infinitely many path-components, were found in [**KPT05**, **BKS11**, **BKS15**, **Otta**]. On the other hand, it was shown in [**BH15**] that $\mathfrak{R}_{K\geq 0}(\mathbb{R}^2)$ is homeomorphic to the separable Hilbert space, and the associated moduli space $\mathfrak{M}_{K\geq 0}(\mathbb{R}^2)$ cannot be separated by a closed subset of finite covering dimension.

Recall that any open complete manifold V of $K \geq 0$ contains a compact totally convex submanifold without boundary, called a *soul*, such that V is diffeomorphic to the interior of a tubular neighborhood of the soul [**CG72**]. We call a connected open manifold *indecomposable* if it admits a complete metric of $K \geq 0$ such that the normal sphere bundle to a soul has no section.

Let N be a compact manifold (e.g., a tubular neighborhood of a soul). A key object in this paper is the map $\iota_N\colon P(\partial N) \to \mathrm{Diff}\, N$

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that extends a pseudoisotopy from a fixed collar neighborhood of ∂N to a diffeomorphism of N supported in the collar neighborhood. Here $P(\partial N)$ and $\mathrm{Diff}\, N$ are the topological groups of pseudoisotopies of ∂N and diffeomorphisms of N, respectively, see Section 5 for background. Let $\pi_j(\iota_N)$ be the homomorphism induced by ι_N on the jth homotopy group based at the identity. We prove the following:

Theorem 1.1. *Let N be a compact manifold with indecomposable interior. Then for every $h \in \mathfrak{R}_{K\geq 0}(\mathrm{Int}\, N)$ and each $k \geq 2$, the group* $\ker \pi_{k-1}(\iota_N)$ *is a quotient of a subgroup of $\pi_k(\mathfrak{R}_{K\geq 0}(\mathrm{Int}\, N), h)$.*

Prior to this result there has been no tool to detect nontrivial higher homotopy groups of $\mathfrak{R}_{K\geq 0}(V)$.

We make a systematic study of $\ker \pi_j(\iota_N)$ and find a number of manifolds for which $\ker \pi_j(\iota_N)$ is infinite and $\mathrm{Int}\, N$ admits a complete metric of $K \geq 0$. Here is a sample of what we can do:

Theorem 1.2. *Let U be the total space of one of the following vector bundles:*

1. *the tangent bundle to S^{2d}, CP^d, HP^d, $d \geq 2$, and the Cayley plane,*
2. *the Hopf $\mathbb{R}^4$ or $\mathbb{R}^3$ bundle over HP^d, $d \geq 1$,*
3. *any linear $\mathbb{R}^4$ bundle over S^4 with nonzero Euler class,*
4. *any nontrivial $\mathbb{R}^3$ bundle over S^4,*
5. *the product of any bundle in* (1), (2), (3), (4) *and any closed manifold of $K \geq 0$ and nonzero Euler characteristic.*

Then there exists m such that every path-component of $\mathfrak{R}_{K\geq 0}(U \times S^m)$ has some nonzero rational homotopy group.

It is well-known that each U in Theorem 1.2 admits a complete metric of $K \geq 0$: For bundles in (3), (4) this follows from [**GZ00**], and the bundles in (1), (2) come with the standard Riemannian submersion metrics, see Example 3.3 (2).

We can also add to the list in Theorem 1.2 some $\mathbb{R}^4$ and $\mathbb{R}^3$ bundles over S^5 and S^7 and an infinite family of $\mathbb{R}^3$ bundles over CP^2, which admit a complete metrics of $K \geq 0$ thanks to [**GZ00**] and [**GZ11**], respectively. Other computations are surely possible. In fact, we are yet to find N with indecomposable interior and such that ι_N is injective on all homotopy groups; the latter does happen when $N = D^n$, see Remark 4.5.

We are unable to compute m in Theorem 1.2[1] . Given U we find $k \geq 1$ such that for every $l \gg k$ there is $\sigma \in \{0, 1, 2, 3\}$ for which the group $\pi_k \mathfrak{R}_{K\geq 0}(U \times S^{l+\sigma}) \otimes \mathbb{Q}$ is nonzero. Here k and the bound "$l \gg k$" are explicit, but σ is not explicit. The smallest $k \geq 1$ for

[1]Explicit computations of m will appear in the upcoming work of the second author, Jiang Yi and Mauricio Bustamante.

which we know that the group is nonzero is $k = 7$, which occurs when U is the total space of a nontrivial $\mathbb{R}^3$ bundle over S^4.

We do not yet know how to detect nontriviality of $\pi_k \mathfrak{M}_{K\geq 0}(V)$, $k \geq 1$. The nonzero elements in $\pi_k \mathfrak{R}_{K\geq 0}(U \times S^m)$ given by Theorem 1.2 lie in the kernel of the π_k-homomorphism induced by the quotient map $\mathfrak{R}_{K\geq 0}(U \times S^m) \to \mathfrak{M}_{K\geq 0}(U \times S^m)$.

Structure of the paper. In Section 2, we outline geometric ingredients of the proof with full details given in Section 3. Theorem 1.1 is proved in Section 4. In Section 9, we derive the results on $\ker \pi_j(\iota_N)$, and prove Theorem 1.2. The proof involves various results on pseudoisotopy spaces occupying the rest of the paper; many of these results are certainly known to experts, but often do not appear in the literature in the form needed for our purposes. Theorems 9.4 and Proposition 9.17 are key ingredients in establishing nontriviality of $\ker \pi_j(\iota_N)$.

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2. Geometric ingredients of Theorem 1.1

Open complete manifolds of $K \geq 0$ enjoy a rich structure theory. The soul construction of [**CG72**] takes as the input a basepoint of a complete open manifold V of $K \geq 0$, and produces a compact totally convex submanifold S without boundary, the so called *soul* of g, such that V is diffeomorphic to the total space of the normal bundle of S. The soul need not contain the basepoint.

Different basepoints sometimes produce different souls, yet any two souls can be moved to each other by an ambient diffeomorphism that restricts to an isometry on the souls, see [**Sha74**]. On the other hand, the diffeomorphism type and the ambient isotopy type of the soul may depends on the metric, see [**Bel03**, **KPT05**, **BKS11**, **BKS15**, **Otta**, **Ottb**].

The soul construction involves asymptotic geometry so there is no a priori reason to expect that the soul will depends continuously on the metric varying in the smooth compact-open topology. We resolve this by imposing the topological assumption that V is *indecomposable* meaning that V admits a complete metric of $K \geq 0$ such that the normal sphere bundle to a soul has no section. This occurs if the normal bundle to a soul has nonzero Euler class, see Section 3 for other examples. Also in Section 3, we explain that any indecomposable manifold V has the following properties:

(i) Any metric in $\mathfrak{R}_{K\geq 0}(V)$ has a unique soul, see [**Yim90**].
(ii) If two metrics lie in the same path-component of $\mathfrak{R}_{K\geq 0}(V)$, then their souls are diffeomorphic, see [**KPT05**], and ambiently isotopic [**BKS11**].
(iii) The souls of any two metrics in $\mathfrak{R}_{K\geq 0}(V)$ have nonempty intersection.
(iv) The normal sphere bundle to a soul S of any metric in $\mathfrak{R}_{K\geq 0}(V)$ has no section. In particular, $\dim(V) \leq 2\dim(S)$.

If Q is a compact smooth submanifold of V, we let $\mathrm{Emb}(Q,V)$ denote the space of all smooth embeddings of Q into V. By the isotopy extension theorem the $\mathrm{Diff}(V)$-action on $\mathrm{Emb}(Q,V)$ by postcomposition is transitive on each path-component, and its orbit map is a fiber bundle, see [**Pal60**, **Cer61**]. The fiber over the inclusion $Q \hookrightarrow V$ is $\mathrm{Diff}(V, \mathrm{rel}\, Q)$, the subgroup of the diffeomorphisms that fix Q pointwise.

The group $\mathrm{Diff}\, Q$ acts freely on $\mathrm{Emb}(Q,V)$ by precomposing with diffeomorphisms of Q. Let $\mathcal{X}(Q,V) = \mathrm{Emb}(Q,V)/\,\mathrm{Diff}\, Q$ with the quotient topology; the orbit map is a locally trivial principal bundle, see [**GBV14**]. Let $\mathcal{X}(V) = \coprod_Q \mathcal{X}(Q,V)$, the space of compact submanifolds of V with smooth topology. Here is the main geometric ingredient of this paper.

Theorem 2.1. *If V is indecomposable, the map $\mathfrak{R}_{K\geq 0}(V) \to \mathcal{X}(V)$ that associates to a metric its unique soul is continuous.*

The proof is a modification of arguments in [**KPT05**, **BKS11**]. What we actually use is the following version of Theorem 2.1 in which the soul is replaced by its tubular neighborhood whose size depends continuously on the metric. For an indecomposable V denote by i_g the normal injectivity radius of a unique soul of $g \in \mathfrak{R}_{K\geq 0}(V)$.

Corollary 2.2. *If V is indecomposable, the map $\mathfrak{R}_{K\geq 0}(V) \to (0,\infty]$ that associates i_g to g is continuous, and given a continuous function $\sigma\colon \mathfrak{R}_{K\geq 0}(V) \to \mathbb{R}$ with $0 < \sigma(g) < i_g$, the map $\mathfrak{R}_{K\geq 0}(V) \to \mathcal{X}(V)$ that associates to g the closed $\sigma(g)$-neighborhood of its soul is continuous.*

Proof. Continuity of $g \to i_g$ follows from Theorem 2.1 and Lemma 3.2 below. The other conclusion is immediate from Theorem 2.1. q.e.d.

3. Continuity of souls for indecomposable manifolds

Throughout this section we assume that V is indecomposable. Let us first justify claims (i)–(iv) of Section 2.

If a metric $g \in \mathfrak{R}_{K\geq 0}(V)$ has two distinct souls, then by a result of Yim [**Yim90**] the souls are contained in an embedded submanifold, the union of pseudosouls, that is diffeomorphic to $\mathbb{R}^l \times S$ where $l > 0$, where any soul is of the form $\{v\}\times S$. In particular, the normal bundle to any soul of g has a nowhere zero section, so V cannot be indecomposable. This implies (i).

Claim (ii) is proved in Lemma 3.1 and Remark 3.2 of [**BKS11**] building on an argument in [**KPT05**].

To prove (iii) and (iv) consider two vector bundles ξ, η with closed manifolds as bases and diffeomorphic total spaces. The associated unit sphere bundles $S(\xi)$, $S(\eta)$ are fiber homotopy equivalent, see [**BKS11**, Proposition 5.1]. By the covering homotopy property a homotopy section of a fiber bundle is homotopic to a section; thus having a section is a property of the fiber homotopy type. Hence if ξ has a nowhere zero section, then so does η. If the zero sections of ξ, η are disjoint in their common total space, then the zero section of η gives rise to a homotopy section of $S(\xi)$, and hence to a nowhere zero section of ξ. These remarks imply (iii) and (iv).

Proof of Theorem 2.1. Since $\mathfrak{R}_{K\geq 0}(V)$ is metrizable, it suffices to show that if the metrics g_j converge to g in $\mathfrak{R}_{K\geq 0}(V)$, then their (unique) souls converge in $\mathcal{X}(Q,V)$. Let S_j, S be souls of g_j, g, respectively. By Lemma 3.1 below it suffices to show that S_j converges to S in the C^0 topology. Arguing by contradiction pass to a subsequence such that each S_j lies outside some C^0 neighborhood of S. Let p_j, p denote the Sharafutdinov retractions onto S_j, S for g_j, g, and let $\breve{g}_j$, $\breve{g}$ denote the metric on S_j, S induced by g_j, g, respectively. By [**BKS11**, Lemma 3.1] $p_j|_S\colon S\to S_j$ is a diffeomorphism for all large j, and the pullback metrics $(p_j|_S)^*\breve{g}_j$ converge to $\breve{g}$ in the C^0 topology. In particular, the diameters of $\breve{g}_j$ are uniformly bounded. Note that each S_j intersects S else $p_j|S$ would give rise to a nowhere zero section of the normal bundle to S. Let U be a compact domain in V such that the interior of U contains the closure of $\cup_j S_j\cup S$.

The embedding $p_j|_S\colon (S,\breve{g})\to(V,g)$ can be written as the composition of $\mathrm{id}\colon (S,\breve{g})\to (S,(p_j|_S)^*\breve{g}_j)$, the isometric embedding of $(S,(p_j|_S)^*\breve{g}_j)$ onto a convex subset of (V,g_j), and $\mathrm{id}\colon (V,g_j)\to(V,g)$. Recall that C^0 convergence of metrics implies Gromov–Hausdorff, and hence Lipschitz convergence. Hence the above identity map of S has bi-Lipschitz constants approaching 1 as $j\to\infty$. Also there are compact domains U_j in V and homeomorphisms $(U_j,g_j)\to(U,g)$ that converge to the identity and have bi-Lipschitz constants approaching 1, and hence the same is true for $p_j|_S\colon (S,\breve{g})\to(V,g)$.

By the Arzelà–Ascoli theorem $p_j|_S$ subconverge to $p_\infty\colon (S,\breve{g})\to (V,g)$, which is an isometry onto its image (equipped with the metric obtained by restricting the distance function of g). Compactness of S implies that p_∞ is homotopic to p_j for large j. Since p is 1-Lipschitz map $(V,g)\to(S,\breve{g})$ that is homotopic to the identity of V, we conclude that $p\circ p_\infty$ is a 1-Lipschitz homotopy self-equivalence of $(S,\breve{g})$.

Homotopy self-equivalences of closed manifolds are surjective, so $p \circ p_\infty$ is surjective, and hence compactness of S implies that $p \circ p_\infty$ is an isometry.

Set $f = p_\infty \circ (p \circ p_\infty)^{-1}$. Then $f(S) = p_\infty(S)$ and $p \circ f$ is the identity of S. Note that $f(S)$ and S intersect, else f would give rise to a section of the normal sphere bundle to S. Fix $x \in f(S) \cap S$. Since every S_j lies outside a C^0 neighborhood of S, there is $y \in f(S) \backslash S$. Let u be a unit vector at $p(y)$ that is tangent to a segment from $p(y)$ to y. Parallel translate u along a segment joining x and $p(y)$. By [**Per94**] this vector field exponentiate to an embedded flat totally geodesic strip, where x lies on one side of the strip and y, $p(y)$ lie on the other side. Finally, $d(p(y), x) = d(y, x)$ contradicts $y \neq p(y)$. q.e.d.

Lemma 3.1. *Given $k \in [1, \infty]$, let g_i be a sequence of complete Riemannian metrics on a manifold M that C^k-converge on compact sets to a metric g. Suppose S_i, S are totally geodesic compact submanifolds of (M, g_i), (M, g), respectively. If S_i converges to S in C^0-topology, then it converges in C^{k-1}-topology.*

Proof. Fix $x \in S$ and pick r such that $\exp_g|_x$, $\exp_{g_i}|_x$ are diffeomorphisms on the $2r$-ball centered at the origin of T_xM for all sufficiently large i. Since S_i, S are totally geodesic, they are equal to the images under $\exp_{g_i}$, $\exp_g$ of some linear subspaces L_i, L of $T_{x_i}M$, T_xM, respectively, where x_i is near x. Since $k \geq 1$, the maps $\exp_{g_i}$, $\exp_g$ are C^0-close, so that C^0-closeness of S_i, S implies that $\exp_g(L_i)$ is C^0-close to S in $B_g(x, r)$. Thus L_i, L are C^0-close in the r-disk tangent bundle over $B(x, r)$, but then they must be C^∞-close because C^0-close linear subspaces are C^∞-close. Thus $\exp_g(L_i)$, $\exp_g(L) = S$ are C^∞-close in $B(x, r)$. Since $\exp_{g_i}$ is C^{k-1}-close to $\exp_g$, we conclude that S_i is C^{k-1}-close to $\exp_g(L_i)$, and hence to S. q.e.d.

The following lemma generalizes the well-known fact that the injectivity radius depends continuously on a point of a Riemannian manifold.

Lemma 3.2. *Let g_j be a sequence of Riemannian metrics on a manifold M that converges smoothly to a Riemannian metric g. Let $S_j \to S$ be a smoothly converging sequence of compact boundaryless submanifolds of M. Then normal injectivity radii of S_i in (M, g_i) converge to the normal injectivity radius of S in (M, g).*

Sketch of the Proof. Denote by i_{g_j}, i_g the normal injectivity radii of S_j in (M, g_j), and S in (M, g), respectively. Recall that i_g equals the supremum of all t such that $d(\gamma(t), S) = t$ for every unit speed geodesic γ with $\gamma(0) \in S$ and $\gamma'(0)$ orthogonal to S.

Arguing by contradiction suppose i_{g_j} does not converge to i_g. By passing to a subsequence we can assume that $i_{g_j} \to I \in [0, \infty]$ and $I \neq i_g$. A standard rescaling argument implies that $I > 0$. We shall only treat the case $I < \infty$; the case $I = \infty$ is similar. Then there is an $\varepsilon > 0$ such that either $I - \varepsilon > i_g$ or $I + \varepsilon < i_g$.

If $I - \varepsilon > i_g$, there is a unit speed geodesic γ starting on S and orthogonal to S at $\gamma(0)$ such that $d(\gamma(I-\varepsilon), S) < I - \varepsilon$. This geodesic is the limit of unit speed g_j-geodesics γ_j starting on S_j. Since i_{g_j} tends to $I > I - \varepsilon$ we get $d_j(\gamma_j(I-\varepsilon), S_j) = I - \varepsilon$ for all large j. The distance functions $d_j(\cdot, S_j)$ converge to $d(\cdot, S)$, hence passing to the limit gives $d(\gamma(I-\varepsilon), S) = I - \varepsilon$. This contradiction rules out the case $I - \varepsilon > i_g$.

Assume $I + \varepsilon < i_g$. Smooth convergence of metrics implies convergence of Jacobi fields, so the g_j-focal radii of S_j are $> I + \varepsilon$ for large j. Then by the well-known dichotomy there is a g_j-geodesic γ_j of length $2i_{g_j}$ starting and ending on S_j which is orthogonal to S_j at end points. Geodesics γ_j subconverge to a g-geodesic of length $2I$ orthogonal to S at the endpoints. Therefore, $i_g \le I$ which is a contradiction. q.e.d.

Example 3.3. Here are some examples of indecomposable manifolds.

(1) If the normal bundle to a soul has nonzero Euler class with $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients, then V is indecomposable because Euler class is an obstruction to the existence of a nowhere zero section.

(2) The simplest method to produce open complete manifolds of $K \ge 0$ is to start with a compact connected Lie group G with a bi-invariant metric, a closed subgroup $H \le G$, and a representation $H \to O_m$ so that the Riemannian submersion metric on the quotient $(G \times \mathbb{R}^m)/H$ is a complete metric of $K \ge 0$ with soul $(G \times \{0\})/H$. Any G-equivariant Euclidean vector bundle over G/H is isomorphic to a bundle of this form: the representation is given by the H-action on the fiber over eH. This applies to the tangent bundle $T(G/H)$ with the Euclidean structure induced by the G-invariant Riemannian metric on G/H. When G/H is orientable we conclude that $T(G/H)$ is indecomposable if and only if G/H has nonzero Euler characteristic (because for orientable $\mathbb{R}^n$ bundles over n-manifolds the Euler class is the only obstruction to the existence of a nowhere zero section).

(3) To be indecomposable the normal bundle to a soul need not have a nontrivial Euler class. For example, all $\mathbb{R}^3$ bundles over S^4, S^5, S^7 admit a complete metric of $K \ge 0$, see [**GZ00**] and so do many $\mathbb{R}^3$ bundles over CP^2 [**GZ11**]. Their Euler classes lie in $H^3(\text{ base}; \mathbb{Z}) = 0$, yet their total spaces are often indecomposable:

(3a) Nontrivial rank 3 bundles over S^n, $n \ge 3$ do not have a nowhere zero section else the bundle splits as a Whitney sum of a bundle of ranks 1 and 2 which must be trivial. Thus all nontrivial rank 3 bundles over S^4, S^5, S^7 have indecomposable total spaces.

(3b) By [**DW59**] oriented isomorphism classes of rank 3 vector bundles over CP^2 are in a bijection via (w_2, p_1) with the subset of

$$H^2(CP^2; \mathbb{Z}_2) \times HP^4(CP^2) \cong \mathbb{Z}_2 \times \mathbb{Z},$$

given by the pairs $(0, 4k)$, $(1, 4l+1)$, $k, l \in \mathbb{Z}$, and such a bundle has a nowhere zero section if and only if p_1 is a square of the integer that reduces to w_2 mod 2. It follows from [**GZ11**, Theorem 3] that the total space of such a bundle is indecomposable with three exceptions: k is odd, k is a square, or l is the product of two consecutive integers.

(3c) According to [**GZ00**, Corollary 3.13] there are 88 oriented isomorphism classes of $\mathbb{R}^4$ bundles over S^7 that admit complete metrics of $K \geq 0$. If the total space of such a bundle is not indecomposable, then it is the Whitney sum of an $\mathbb{R}$ bundle and an $\mathbb{R}^3$ bundle (as any $\mathbb{R}^2$ bundle over S^7 is trivial). Since there are only 12 oriented isomorphism classes of $\mathbb{R}^3$ bundles over S^7, we conclude that there are at least 76 oriented isomorphism classes of $\mathbb{R}^4$ bundles over S^7 with indecomposable total spaces. In fact, there are precisely 76 such bundles because the inclusion $SO(3) \to SO(4)$ is injective on homotopy groups. Similarly, [**GZ00**, Proposition 3.14] implies that there are 2 oriented isomorphism classes of $\mathbb{R}^4$ bundles over S^5 with indecomposable total spaces.

(4) The product of any indecomposable manifold with a closed manifold of $K \geq 0$ is indecomposable. Indeed, suppose V is indecomposable with a soul S and B is closed. If $V \times B$ were not indecomposable, then the normal bundle to $S \times B$ in $V \times B$ would have a nowhere zero section. Restricting the section to a slice inclusion $S \times \{*\}$ gives a section of the normal bundle of S in V.

Remark 3.4. The product of indecomposable manifolds need not be indecomposable. Indeed, let ξ, η be oriented nontrivial rank two bundle over S^2, RP^2 classified by Euler classes in $H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$ and $H^2(RP^2; \mathbb{Z}) \cong \mathbb{Z}_2$. Suppose $e(\xi)[S^2]$ is even. Then the Euler class of $\xi \times \eta$ equals $e(\xi \times \eta) = e(\xi) \times e(\eta)$, which vanishes because the cross product is bilinear. By dimension reasons the Euler class is the only obstruction to the existence of a nowhere zero section of $\xi \times \eta$, so the total space of $\xi \times \eta$ is not indecomposable. In view of (1) above in order to show that ξ, η have indecomposable total spaces it is enough to give them complete metrics of $K \geq 0$. The case of ξ is well-known: Any plane bundle over S^2 can be realized as $(S^3 \times \mathbb{R}^2)/S^1$, see (2) above, so it carries a complete metric of $K \geq 0$. To prove the same for η we shall identify it with the quotient of $S^2 \times \mathbb{R}^2$ by the involution $i(x, v) = (-x, -v)$ which is isometric in the product of the constant curvature metrics. The quotient can be thought of $\gamma \oplus \gamma$ where γ is the canonical line bundle over RP^2, so its total Stiefel–Whitney class equals $(1 + w_1(\gamma))^2 = 1 + w_1(\gamma)^2 \neq 1$. Thus $\gamma \oplus \gamma$ is orientable and nontrivial, and hence it is isomorphic to η which is the only orientable nontrivial plane bundle over RP^2.

4. Topological restrictions on indecomposable manifolds

In this section, we prove Theorem 1.1. Let $V = \operatorname{Int} N$ be an indecomposable manifold. Fix a homeomorphism $\rho\colon (0,\infty] \to (0,1]$ with $\rho(s) < s$, e.g., $\rho(s) := \frac{s}{s+1}$.

Fix an arbitrary metric $h \in \mathfrak{R}_{K\geq 0}(V)$ with soul S_h of normal injectivity radius i_h. By a slight abuse of notation we identify N with the $\rho(i_h)$-neighborhood of S_h. Let θ_h be the orbit map of a metric $h \in \mathfrak{R}_{K\geq 0}(V)$ under the pullback (left) action of $\operatorname{Diff} V$ given by $\theta_h(\phi) := \phi^{-1*}h$; we sometimes denote $\phi^{-1*}h$ by h_ϕ. Consider the diagram

$$\begin{tikzcd}
 & * \simeq \operatorname{Diff}(V, \operatorname{rel} N) \arrow[r] & \operatorname{Diff} V \arrow[r, "\theta_h"] \arrow[d] & \mathfrak{R}_{K\geq 0}(V) \arrow[d, "\delta"] & \\
\Omega\mathcal{X}(N,V) \arrow[r, "\Omega f"] & \operatorname{Diff} N \arrow[r] & \operatorname{Emb}(N,V) \arrow[r, "q"] & \mathcal{X}(N,V) \arrow[r, "f"] & B\operatorname{Diff} N
\end{tikzcd}$$

The map q takes an embedding to its image. Note that q is a principal bundle [**GBV14**], and f denotes its classifying map.

The leftmost vertical arrow is given by restricting to N, which is a fiber bundle due to the parametrized isotopy extension theorem. Its fiber over the inclusion $\operatorname{Diff}(V, \operatorname{rel} N)$ is contractible by the Alexander trick towards infinity. (The fibers over other components of $\operatorname{Emb}(N,V)$ might not be contractible, but we will only work in the component of the inclusion).

The map δ taking g to the closed $\rho(i_g)$-neighborhood of S_g is continuous by Corollary 2.2.

The above diagram commutes because the isometry $\phi\colon (V,h) \to (V,h_\phi)$ takes the $\rho(i_h)$-neighborhood of S_h to the $\rho(i_{h_\phi})$-neighborhood of S_{h_ϕ}.

Let $\pi_k(\theta_h)$ be the homomorphism induced by θ_h on the kth homotopy groups based at the identity map of V, and similarly, let $\pi_k(q)$, $\pi_k(f)$, $\pi_k(\Omega f)$ be the induced maps of homotopy groups based at inclusions. With these notations the commutativity of the diagram implies that $\operatorname{Im}\pi_k(q)$ is a quotient of a subgroup of $\operatorname{Im}\pi_k(\theta_h)$.

In the bottom row of the diagram every two consecutive maps form a fibration, up to homotopy. This gives isomorphisms $\operatorname{Im}\pi_k(q) \cong \ker\pi_k(f) \cong \ker\pi_{k-1}(\Omega f)$ for each $k \geq 1$.

A collar neighborhood of ∂N defines the inclusion $\iota_N\colon P(\partial N) \to \operatorname{Diff} N$ extending a pseudoisotopy on the collar neighborhood by the identity outside the neighborhood. In Theorem 6.1 below we identify the homomorphisms $\pi_{k-1}(\Omega f)$ and $\pi_{k-1}(\iota_N)$ for each $k \geq 2$, where $\pi_{k-1}(\iota_N)$ is the map induced by ι_N on the $(k-1)$th homotopy group with identity maps as the basepoints.

In summary, $\ker\pi_{k-1}(\iota_N)$ is quotient of a subgroup of $\operatorname{Im}\pi_k(\theta_h)$ for $k \geq 2$, which completes the proof of Theorem 1.1.

Remark 4.1. For $k = 0$ and 1 the proof of Theorem 1.1 shows that $\operatorname{Im}\pi_k(q)$ is a quotient of a subgroup of $\operatorname{Im}\pi_k(\theta_h)$.

Notation. *If* $\pi_j(X)$ *is abelian, we let* $\pi_j^{\mathbb{Q}}(X) := \pi_j(X) \otimes \mathbb{Q}$ *and denote the dimension of this rational vector space by* $\dim \pi_j^{\mathbb{Q}}(X)$.

Remark 4.2. Tensoring with the rationals immediately implies that under the assumptions of Theorem 1.1 any subspace of $\operatorname{Im}\pi_k^{\mathbb{Q}}(q)$ embeds into $\operatorname{Im}\pi_k^{\mathbb{Q}}(\theta_h)$.

Remark 4.3. One may hope to use Theorem 1.1 to produce infinitely generated subgroups of $\operatorname{Im}\pi_k(\theta_h)$. This is somewhat of an illusion because $\ker \pi_{k-1}(\iota_N)$ is a finitely generated abelian group if $\pi_1(\partial N)$ is finite and $\max\{2k+7, 3k+4\} < \dim N$. Indeed, $\ker \pi_{k-1}(\iota_N)$ can be identified with a subgroup of $\pi_{k+1}A(\partial N)$, see (7.1) below, which is finitely generated [**Dwy80**, **Bet86**]. Note that all known computations of $\ker \pi_{k-1}(\iota_N)$ are in the above stability range.

Remark 4.4. An integral cohomology class is called *spherical* if it does not vanish on the image of the Hurewicz homomorphism. In many of our examples of indecomposable V the normal bundle to the soul has spherical Euler class, which forces the soul of any metric in $\mathfrak{R}_{K\geq 0}(V)$ to have infinite normal injectivity radius by a result of Guijarro–Schick–Walschap [**GSW02**]. For such V the proof of Theorem 1.1 simplifies: we need not consider ρ or i_g, and instead can let $\delta(g)$ be the 1-neighborhood of S_g and identify $\delta(h)$ with N.

Remark 4.5. The map ι_{D^n} is injective for all homotopy groups. Indeed, by Theorem 6.1 the map f in the diagram below is a delooping of ι_{D^n} provided both maps are restricted to the identity components. The leftmost horizontal arrow is given by precomposing with the inclusion, the downward arrow is the inclusion, and the slanted arrow is their composition

$$\begin{tikzcd}
O(n) \arrow[d] \arrow[dr] & & & \\
\operatorname{Diff} D^n \arrow[r] & \operatorname{Emb}(D^n, \mathbb{R}^n) \arrow[r, "q"] & \mathcal{X}(D^n, \mathbb{R}^n) \arrow[r, "f"] & B\operatorname{Diff} D^n
\end{tikzcd}$$

The slanted arrow is a homotopy equivalence: deform an embedding e so that it fixes 0 via $t \to te(x) + (1-t)e(0)$, then deform it to the its differential at 0 via $s \to \frac{e(sx)}{s}$, and, finally, apply a deformation retraction $GL(n,\mathbb{R}) \to O(n)$. Hence the left bottom arrow has a section which makes q trivial on the homotopy groups, so by exactness f is injective on homotopy groups.

5. Pseudoisotopy spaces, stability, and involution

A *pseudoisotopy* of a compact smooth manifold M is a diffeomorphism of $M\times I$ that is the identity on a neighborhood of $M\times\{0\}\cup\partial M\times I$. Pseudoisotopies of M form a topological group $P(M)$. Let $P_\partial(M)$ denote the topological subgroup of $P(M)$ consisting of diffeomorphisms of $M\times I$ that are the identity on a neighborhood of $\partial(M\times I)$.

Igusa in [**Igu88**] discussed a number of inequivalent definitions of pseudoisotopy, e.g., a pseudoisotopy is often defined as a diffeomorphism of $M\times I$ that restricts to the identity of $M\times\{0\}\cup\partial M\times I$. Igusa in [**Igu88**, Chapter 1, Proposition 1.3] establishes a weak homotopy equivalence of pseudoisotopy spaces arising from various definitions, and in particular, the inclusion

$$P(M)\to \mathrm{Diff}(M\times I, \mathrm{rel}\, M\times\{0\}\cup\partial M\times I)$$

is a weak homotopy equivalence. The co-domain of the inclusion is homotopy equivalent to a CW complex; in fact, for any compact manifold L with boundary and any closed subset X of L, the space $\mathrm{Diff}(L,\mathrm{rel}\, X)$ is a Fréchet manifold [**Yag**, Lemma 4.2(ii)] and hence is homotopy equivalent to a CW complex [**Yag**, Lemma 2.1]. By contrast, we do not know if $P(M)$ is homotopy equivalent to a CW complex which necessitates some awkward arguments in Section 6.

Defining a pseudoisotopy as an element of $P(M)$ is convenient for our purposes because it allows for easy gluing: *A codimension zero embedding of closed manifolds $M_0\to M$ induces a continuous homomorphism $P(M_0)\to P(M)$ given by extending a diffeomorphism by the identity on $(M\setminus M_0)\times I$.* Similarly, the map ι_N defined in the introduction is a continuous homomorphism.

By Igusa's stability theorem [**Igu88**] the stabilization map

$$\Sigma\colon\ P(M)\to P(M\times I) \tag{5.1}$$

is k-connected if $\dim M\geq \max\{2k+7, 3k+4\}$. Thus the iterated stabilization is eventually a π_i-isomorphism for any given i. The *stable pseudoisotopy space* $\mathscr{P}(M)$ is the direct limit $\lim\limits_{m\to\infty} P(M\times I^m)$.

It is known, see the proof of [**Hat78**, Proposition 1.3], that $\mathscr{P}(-)$ is a functor from the category of compact manifolds and continuous maps to the category of topological spaces and homotopy classes of continuous maps. Also homotopic maps $M\to M'$ induce the same homotopy classes $\mathscr{P}(M)\to\mathscr{P}(M')$. Every k-connected map $M\to M'$ induces a $(k-2)$-connected map $\mathscr{P}(M)\to\mathscr{P}(M')$ [**Igu**, Theorem 3.5].

The space $P(M)$ has an involution given by $f\to\bar f$, where

$$\bar f(x,t)=r(f(f^{-1}(x,1),1-t))\quad\text{and}\quad r(x,t)=(x,1-t),$$

see [**Vog85**, p. 296]. We write the induced involution of $\pi_iP(M)$ as $x\to\bar x$.

Since $P(M)$ is a topological group, the sum of two elements in $\pi_i P(M)$ is represented by the pointwise product of the representatives of the elements [**Spa66**, Corollary 1.6.10]. Hence the endomorphism of $\pi_i P(M)$ induced by the map $f \to f \circ \bar{f}$ is given by $x \to x + \bar{x}$.

Note that the image of the map $f \to f \circ \bar{f}$ lies in $P_\partial(M)$. It follows that any element $x+\bar{x} \in \pi_i P(M)$ is in the image of the inclusion induced homomorphism $\pi_i P_\partial(M) \to \pi_i P(M)$ for if f represents x, then $x + \bar{x}$ is represented by $f \circ \bar{f}$. For future use we record the following lemma:

Lemma 5.2. *Let M be a compact manifold with boundary, let i be an integer with $\dim(M) \geq \max\{2i+9, 3i+7\}$, and let η_i^m be the endomorphism of $\pi_i^{\mathbb{Q}} P(M \times I^m)$ induced by the map $f \to f \circ \bar{f}$.*

(1) *If $x \in \pi_i P(M)$ has infinite order, then $x + \bar{x} \in \pi_i P_\partial(M)$ and $\Sigma x + \overline{\Sigma x} \in \pi_i P_\partial(M \times I)$ cannot both have finite order.*
(2) *$\pi_i^{\mathbb{Q}} \mathscr{P}(M)$ embeds into $\operatorname{Im} \eta_i^m \oplus \operatorname{Im} \eta_i^{m+1}$. In particular, there is $\varepsilon \in \{0,1\}$ such that $2 \dim \operatorname{Im} \eta_i^{m+\varepsilon} \geq \dim \pi_i^{\mathbb{Q}} \mathscr{P}(M)$.*

Proof. (1) The map $f \to \bar{f}$ homotopy anti-commutes with the stabilization map (5.1), as proved in [**Hat78**, Appendix I]. By assumption i is below Igusa's stability range so Σ is a π_i-isomorphism, and $\pi_i P(M)$ contains an infinite order element x. Then either $x + \bar{x}$ or $\Sigma x + \overline{\Sigma x}$ has infinite order for otherwise

$$2\Sigma x = \Sigma x + \overline{\Sigma x} + \Sigma x - \overline{\Sigma x} = \Sigma x + \overline{\Sigma x} + \Sigma(x + \bar{x}),$$

would have finite order, contradicting π_i-injectivity of Σ.

(2) Let $\Sigma \ker \eta_i^m$ denote the image of $\ker \eta_i^m$ under the $\pi_i^{\mathbb{Q}}$-isomorphism induced by Σ. The intersection of $\ker \eta_i^{m+1}$ and $\Sigma \ker \eta_i^m$ is trivial, for if $x = -\bar{x}$ and $\Sigma x = -\overline{\Sigma x}$, then $\Sigma x = -\Sigma \bar{x} = \overline{\Sigma x}$ so that $\Sigma x = 0$. Thus $\ker \eta_i^m$ injects into $\operatorname{Im} \eta_i^{m+1}$, and the claim follows by observing that $\pi_i^{\mathbb{Q}} \mathscr{P}(M) \cong \ker \eta_i^m \oplus \operatorname{Im} \eta_i^m$. q.e.d.

6. Pseudoisotopies and the space of submanifolds

Let $\mathrm{Diff}_0 M$, $P_0(M)$ denote the identity path-components of $\mathrm{Diff}\, M$, $P(M)$, respectively. Given a submanifold X of Y let $\mathrm{Emb}_0(X,Y)$ denote the component of the inclusion in the space of embeddings of $X \to Y$, and let $\Omega_0 \mathcal{X}(X,Y)$ be the component of the constant loop based at the inclusion.

If $f\colon E \to B$ is a continuous map and $E_f \to B$ is the corresponding standard fibration with a fiber F, then *the associated homotopy fiber map* $F \to E$ is the composition of the inclusion $F \to E_f$ with the standard homotopy equivalence $E_f \to E$.

Theorem 6.1. *Let M be a compact manifold with nonempty boundary. Suppose U is obtained by attaching $\partial M \times [0,1)$ to M via the* *identity map of the boundary. Let* $l\colon \Omega\mathcal{X}(M,U) \to \operatorname{Diff} M$ *be the homotopy fiber map associated with the map* $\operatorname{Diff} M \to \operatorname{Emb}(M,U)$ *given by postcomposing diffeomorphisms with the inclusion. Then there is a weak homotopy equivalence* $\phi\colon \Omega_0\mathcal{X}(M,U) \to P_0(\partial M)$ *such that* $\iota_M \circ \phi$ *is homotopic to the restriction of* l *to* $\Omega_0\mathcal{X}(M,U)$.

Proof. Let M_0 be the complement of an open collar of ∂M in M. Consider the following commutative diagram:

$$\begin{tikzcd} \operatorname{Diff}_0(M) \arrow[r, "i"] \arrow[d, "r"'] & \operatorname{Emb}_0(M,U) \arrow[d, "s"'] \\ \operatorname{Emb}_0(M_0, \operatorname{Int} M) \arrow[r, "j"'] & \operatorname{Emb}_0(M_0,U). \end{tikzcd}$$

Here r and s are given by restriction to M_0, while i and j is induced by precomposing with the inclusion $M \hookrightarrow U$, and postcomposing with the inclusion Int $M \hookrightarrow U$.

First we show that s is a homotopy equivalence. Let us factor the restriction $\operatorname{Diff}_0 U \to \operatorname{Emb}_0(M_0,U)$ as the restriction $\operatorname{Diff}_0 U \to \operatorname{Emb}_0(M,U)$ followed by s. By the parametrized isotopy extension theorem [**Pal60**, **Cer61**] the above restrictions are fiber bundles with fibers $\operatorname{Diff}_0(U, \operatorname{rel} M_0)$, $\operatorname{Diff}_0(U, \operatorname{rel} M)$, respectively. The fibers are contractible by the Alexander trick towards infinity, so s is a homotopy equivalence.

The map j is also a homotopy equivalence. Note that the space of smooth embeddings of a compact manifold into an open manifold is an ANR because it is an open subset of a Fréchet manifold of all smooth maps between the manifolds. Hence the domain and codomain of j are homotopy equivalent to CW complexes and it suffices to show that j is a weak homotopy equivalence. This easily follows from the existence of an isotopy of U that pushes a given compact subset into $\operatorname{Int} M$, e.g., given a map $S^k \to \operatorname{Emb}_0(M_0,U)$ based at the inclusion we can use the isotopy to push the adjoint $S^k \times M_0 \to U$ of the above map into $\operatorname{Int} M$ relative to the inclusion, so j is π_k-surjective, and injectivity is proved similarly.

By the parametrized isotopy extension theorem the map r is a fiber bundle, and its fiber F_r over the inclusion equals the space of diffeomorphisms of $M \setminus \operatorname{Int}(M_0)$ that restrict to the identity of ∂M_0 and lie in $\operatorname{Diff}_0 M$. The inclusion

$$P(\partial M) \cap \operatorname{Diff}_0 M \to F_r \tag{6.2}$$

is a weak homotopy equivalence [**Igu88**, Chapter 1, Proposition 1.3]. The space F_r is a Fréchet manifold, see [**Yag**, Lemma 4.2(ii)], hence it is an ANR. Therefore, the CW-approximation theorem gives a weak homotopy equivalence

$$h_r\colon\ F_r \to P(\partial M) \cap \mathrm{Diff}_0\, M,$$

whose composition with the inclusion (6.2) is homotopic to the identity of F_r.

Since s and j are homotopy equivalences, the homotopy fibers F_i, F_r of i, r are homotopy equivalent, i.e., there is a homotopy equivalence $h\colon F_i \to F_r$ which together with the homotopy fiber maps $f_i\colon F_i \to \mathrm{Diff}_0\, M$, $f_r\colon F \to \mathrm{Diff}_0\, M$ forms a homotopy commutative triangle. This gives homotopies $\iota_M \circ h_r \circ h \sim f_r \circ h \sim f_i$.

Look at the map of fibration sequences

$$\begin{tikzcd}
\Omega\,\mathrm{Diff}_0\, M \arrow[r] \arrow[d] & \Omega\mathrm{Emb}_0(M,U) \arrow[r] \arrow[d] & F_i \arrow[r, "f_i"'] \arrow[d, "g"'] & \mathrm{Diff}_0\, M \arrow[r] \arrow[d] & \mathrm{Emb}_0(M,U) \arrow[d] \\
\Omega\,\mathrm{Diff}\, M \arrow[r] & \Omega\mathrm{Emb}(M,U) \arrow[r] & \Omega\mathcal{X}(M,U) \arrow[r, "l"] & \mathrm{Diff}\, M \arrow[r] & \mathrm{Emb}(M,U)
\end{tikzcd}$$

where the maps in the rightmost and the leftmost squares are inclusions, and g is the associated map of homotopy fibers. The two rightmost vertical arrows are inclusions of path-components. Hence the unlabeled vertical arrows induce π_k-isomorphisms for $k > 0$, and so does g by the five lemma. The space $\mathcal{X}(M,U)$ is a Fréchet manifold [**GBV14**], and hence its loop space is homotopy equivalent to a CW complex [**Mil59**]. Thus the restriction of g to the identity component is a homotopy equivalence whose homotopy inverse we denote by g'. For the map $\phi := h_r \circ h \circ g'$ we have homotopies $\iota_M \circ \phi \sim f_i \circ g' \sim l \circ g \circ g' \sim l|_{\Omega_0\mathcal{X}(M,U)}$ as claimed. q.e.d.

Remark 6.3. We do not know if the groups $\pi_0 P(\partial M)$, $\pi_0\Omega\mathcal{X}(M,U)$ are isomorphic. Theorem 6.1 implies that any two path-components of $P(\partial M)$, $\Omega\mathcal{X}(M,U)$ are weakly homotopy equivalent. (If X is an H-space whose H-multiplication induces a group structure on $\pi_0(X)$, then all path-components of X are homotopy equivalent. This applies to topological groups and loop spaces.)

7. Rational homotopy of the pseudoisotopy space

In this section, we review how to compute $\pi_*^{\mathbb{Q}}\mathscr{P}(M)$, work out the cases when M is S^n, HP^d, $S^4 \times S^4$, $S^4 \times S^7$, and explain that every 2-connected rational homotopy equivalence induces an isomorphism on $\pi_*^{\mathbb{Q}}\mathscr{P}(-)$.

It turns out that if M is simply-connected, the computation of $\pi_*^{\mathbb{Q}}\mathscr{P}(M)$ reduces to a problem in the rational homotopy theory.

There is a fundamental relationship between $\mathscr{P}(M)$ and the Waldhausen algebraic K-theory $A(M)$. For our purposes a definition of $A(-)$ is not important, and it is enough to know that $A(-)$ is a functor from the category of continuous maps of topological spaces into itself, see [**Wal78**]. Let $A_f\colon A(X)\to A(Y)$ denote a map induced by a map $f\colon X\to Y$. For each $i\geq 0$ there is a natural isomorphism

$$\pi_{i+2}A(M)\cong \pi^S_{i+2}(M_+)\oplus \pi_i\mathscr{P}(M). \tag{7.1}$$

This result was envisioned in works of Hatcher and Waldhausen in 1970s, and a complete proof has, finally, appeared in [**WJR13**, Theorem 0.3], where the notations are somewhat different, see [**Rog**, section 1.15] and [**HS82**, p.227] for relevant background.

Here $\pi^S_{i+2}(M_+)$ is the $(i+2)$th stable homotopy group of the disjoint union of M and a point, which after tensoring with the rationals becomes naturally isomorphic to the homology of M, i.e., $\pi^S_{i+2}(M_+)\otimes\mathbb{Q}\cong H_{i+2}(M;\mathbb{Q})$, see, e.g., [**tD08**, section 20.9].

Dwyer [**Dwy80**] showed that if X is simply-connected and each $\pi_i(X)$ is finitely generated, then each $\pi_iA(X)$ is finitely generated. Since compact simply-connected manifolds have finitely generated homotopy groups [**Spa66**, Corollary 9.6.16], it follows from (7.1) that $\mathscr{P}(M)$ have finitely generated homotopy groups for each compact simply-connected manifold M.

The constant map $M\to *$ induces retractions $\mathscr{P}(M)\to\mathscr{P}(*)$ and $A(M)\to A(*)$, which give isomorphisms:

$$\begin{aligned}\pi_i\mathscr{P}(M)&\cong \pi_i\mathscr{P}(*)\oplus\pi_i(\mathscr{P}(M),\mathscr{P}(*)),\\ \pi_iA(M)&\cong\pi_iA(*)\oplus\pi_i(A(M),A(*)).\end{aligned} \tag{7.2}$$

Waldhausen computed the rational homotopy groups of $A(*)$, the algebraic K-theory of a point [**Wal78**, p. 48], which gives

$$\pi^{\mathbb{Q}}_q\mathscr{P}(*)\cong\pi^{\mathbb{Q}}_{q+2}A(*)=\begin{cases}\mathbb{Q} & \text{if } q\equiv 3\,(\operatorname{mod} 4),\\ 0 & \text{else.}\end{cases} \tag{7.3}$$

Thus the Poincaré series of $\pi^{\mathbb{Q}}_*\mathscr{P}(*)$ is $t^3(1-t^4)^{-1}$. Recall that the *Poincaré series* of a graded vector space $\underset{i}{\oplus}W_i$ is $\sum\limits_i t^i\dim W_i$.

The Poincaré series of $\pi^{\mathbb{Q}}_*(\mathscr{P}(M),\mathscr{P}(*))$, where $M=S^k$ with $k>1$, was computed in [**HS82**] as

$$\frac{t^{3n-4}}{1-t^{2n-2}}\quad\text{if } M=S^n \text{ where } n\geq 2 \text{ is even,} \tag{7.4}$$

$$\frac{t^{4n-5}}{1-t^{2n-2}}\quad\text{if } M=S^{2n-1} \text{ where } n\geq 2 \text{ is an integer.} \tag{7.5}$$

More precisely, [**HS82**, pp. 227–229] gives the Poincaré series of $\pi_*A(S^k)$ and (7.4)–(7.5) is obtained from the series by subtracting the Poincaré series for $H_*(S^k;\mathbb{Q})$ and $\pi^{\mathbb{Q}}_*A(*)$, and shifting dimensions by two.

The range of spaces X for which $\pi^{\mathbb{Q}}_*A(X)$ is readily computable was greatly extended after the discovery of a connection between $\pi^{\mathbb{Q}}_*A(X)$

and $HC_*(X;\mathbb{Q})$, the rational cyclic homology, see [**Goo86**], and references therein.

By [**Goo85**, Theorem V.1.1] or [**BF86**, Theorem A] there is a natural isomorphism between $HC_*(X;\mathbb{Q})$ and the *equivariant rational homology* $H_*^{S^1}(LX;\mathbb{Q})$. The latter is defined as $H_*(LX\times_{S^1} ES^1;\mathbb{Q})$, where $LX\times_{S^1} ES^1$ is the Borel construction and LX is free loop space of X, i.e., the space of continuous maps $S^1\to X$ with the compact-open topology. Note that LX comes with the circle action by pre-composition, and the post-composition with a continuous map $f\colon X\to Y$ induces the S^1-equivariant continuous map $L_f\colon LX\to LY$.

The free loop space of a point is a point, so $H_*^{S^1}(*;\mathbb{Q}) = H_*(BS^1)$. The map $X\to *$ induces a retraction $LX\times_{S^1} ES^1\to *\times_{S^1} ES^1 = BS^1$, which gives an isomorphism:

$$H_i^{S^1}(LX;\mathbb{Q})\cong H_i^{S^1}(*;\mathbb{Q})\oplus H_i^{S^1}(LX,*;\mathbb{Q}). \tag{7.6}$$

In many cases $H_*^{S^1}(LX;\mathbb{Q})$ can be computed due to

- the Künneth formula for rational cyclic homology $HC_*(X;\mathbb{Q})$ of [**BF86**];
- a Sullivan minimal model for $LX\times_{S^1} ES^1$ developed in [**VPB85**] for any simply-connected X such that $\dim \pi_i^{\mathbb{Q}}(X)$ is finite for every i.

To state a result in [**Goo86**] we need a notation. Given a functor F that associates to a continuous map $g\colon X\to Y$ a sequence of linear maps of rational vector spaces $g_i\colon F_i(X)\to F_i(Y)$ indexed by $i\in\mathbb{N}$, we let $F_i(g)$ denote a rational vector space that fits into an exact sequence

$$\dots\longrightarrow F_i(X)\xrightarrow{g_i} F_i(Y)\longrightarrow F_i(g)\longrightarrow F_{i-1}(X)\xrightarrow{g_{i-1}}\dots \tag{7.7}$$

so that $F_i(g)$ is isomorphic to direct sum of $\ker g_{i-i}$ and $F_i(Y)/\mathrm{Im}\, g_i$. We apply the above when F_i is the rational homotopy $\pi_i^{\mathbb{Q}}(-)$ or equivariant rational homology $H_i^{S^1}(-;\mathbb{Q})$, while g is A_f or L_f, respectively. In particular, in these notations $\pi_i^{\mathbb{Q}}(g)=0$ for all $i\le k$ if and only if g is rationally k-connected.

Goodwillie proved in [**Goo86**, p. 349] that any 2-connected continuous map $f\colon X\to Y$ gives rise to an isomorphism for all i

$$\pi_i^{\mathbb{Q}}(A_f)\cong H_{i-1}^{S^1}(L_f;\mathbb{Q}). \tag{7.8}$$

Waldhausen proved that if f is k-connected with $k\ge 2$, then so is A_f, see [**Wal78**, Proposition 2.3], and (7.8) gives a rational version of this result:

Corollary 7.9. *If f is 2-connected and rationally k-connected, then so is A_f.*

Proof. If F is a homotopy fiber of f, then LF is a homotopy fiber of L_f, see [**Str11**, Theorem 5.125]. It is easy to see that LF is also the homotopy fiber of the map $LX \times_{S^1} ES^1 \to LY \times_{S^1} ES^1$ induced by L_f. By assumption F is rationally $(k-1)$-connected, so the homotopy exact sequence of the evaluation fibration $\Omega F \to LF \to F$ shows that LF is rationally $(k-2)$-connected. This implies that $H^{S^1}_{i-1}(L_f;\mathbb{Q}) = 0$ for $i \le k$, which proves the lemma thanks to (7.8). q.e.d.

Corollary 7.10. *Any 2-connected rationally k-connected map of simply-connected compact manifolds induces an isomorphism on $\pi_i^{\mathbb{Q}}\mathscr{P}(-)$ for $i < k-2$ and an epimorphism for $i = k-2$.*

Proof. This follows from naturality of (7.1) combined with Corollary 7.9 and the Whitehead theorem mod the Serre class of abelian torsion groups [**Spa66**, Theorem 9.6.22]. q.e.d.

If X is simply-connected, then $X \to *$ is 2-connected, so that (7.8) implies:

Corollary 7.11. *If X is simply-connected, then $\pi_i^{\mathbb{Q}}(A(X), A(*))$ is isomorphic to $H^{S^1}_{i-1}(LX, *;\mathbb{Q})$ for all i.*

Proof. If $f\colon X \to *$, then A_f, L_f are retractions, so (7.7) splits into short exact sequences. In view of (7.2) and (7.6), we get isomorphisms

$$H^{S^1}_{i-1}(LX, *;\mathbb{Q}) \cong H^{S^1}_i(L_f;\mathbb{Q}) \cong \pi^{\mathbb{Q}}_{i+1}(A_f) \cong \pi^{\mathbb{Q}}_i(A(X), A(*)),$$

where the middle isomorphism is given by (7.8). q.e.d.

By (7.1) and Corollary 7.11 the Poincaré series of $\pi_*^{\mathbb{Q}}(\mathscr{P}(M), \mathscr{P}(*))$ equals the difference of the Poincaré series of $HC_{*+1}(M, *;\mathbb{Q})$ and $H_{*+2}(M;\mathbb{Q})$. For future use we record some explicit computations of $HC_*(M)$.

If M is simply-connected and $H^*(M;\mathbb{Q}) \cong \mathbb{Q}[\alpha]/(\alpha^{n+1})$ the Poincaré series for $HC_*(M;\mathbb{Q})$ was found in [**VPB85**, Theorem B] giving the following Poincaré series for $\pi_*^{\mathbb{Q}}(\mathscr{P}(M), \mathscr{P}(*))$:

$$\frac{(1-t^{4n})\,t^{4n+4}}{(1-t^4)\,(1-t^{4n+2})} \qquad \text{if } \alpha \in H^4(M;\mathbb{Q}), \tag{7.12}$$

$$\frac{t^{2n}}{1-t^2} \qquad \text{if } \alpha \in H^2(M;\mathbb{Q}). \tag{7.13}$$

In particular, (7.12) applies to $M = HP^n$, and (7.13) applies when M is CP^n or the total space of any nontrivial S^2-bundle over S^4, see [**GZ00**, Corollary 3.9], which, in fact, is rationally homotopy equivalent to CP^3.

Next we compute $\pi_*^{\mathbb{Q}}(\mathscr{P}(M), \mathscr{P}(*))$ when M is $S^4 \times S^4$ and $S^7 \times S^4$. The Poincaré series of $HC_*(S^4, *;\mathbb{Q})$ equals $t^3(1-t^6)^{-1}$ [**VPB85**, Theorem B], so by dimension reasons $HC_*(S^4;\mathbb{Q})$ is quasifree in the sense of [**BF86**, p.303]. Hence the Künneth formula of [**BF86**, Theorem B(b)] applies and for any connected space X we have

$$HC_*(X \times S^4, *; \mathbb{Q}) \cong HC_*(X, *; \mathbb{Q}) \oplus H_*(LX; \mathbb{Q}) \otimes HC_*(S^4, *; \mathbb{Q}).$$

Recall that taking the Poincaré series converts $\oplus$ to the sum and $\otimes$ to the product of series.

Set $X = S^4$. The Poincaré series for $H_*(LS^4; \mathbb{Q})$ is given in [**VPB85**, Theorem B(2b)] and it simplifies to $1 + (t^3 + t^4)(1 - t^6)^{-1}$. Therefore, the Poincaré series for $HC_*(S^4 \times S^4, *; \mathbb{Q})$ equals

$$2t^3(1-t^6)^{-1} + (t^6 + t^7)(1-t^6)^{-2},$$

and we get the Poincaré series for $\pi_*^{\mathbb{Q}}(\mathscr{P}(S^4 \times S^4), \mathscr{P}(*))$:

$$\frac{2t^2}{1-t^6} + \frac{t^5 + t^6}{(1-t^6)^2} - 2t^2 - t^6. \tag{7.14}$$

Set $X = S^7$. The Poincaré series for $HC_*(S^7, *; \mathbb{Q})$, $H_*(LS^7; \mathbb{Q})$ equal $t^6(1-t^6)^{-1}$, $(1+t^7)(1-t^6)^{-1}$, respectively. Hence the Poincaré series for $HC_*(S^7 \times S^4, *; \mathbb{Q})$ equals $t^6(1-t^6)^{-1} + t^3(1+t^7)(1-t^6)^{-2}$, and therefore, we get the Poincaré series for $\pi_*^{\mathbb{Q}}(\mathscr{P}(S^7 \times S^4), \mathscr{P}(*))$:

$$\frac{t^5}{1-t^6} + \frac{t^2(1+t^7)}{(1-t^6)^2} - t^2 - t^5 - t^9. \tag{7.15}$$

8. Block automorphisms, pseudoisotopies, and surgery

Throughout this section M is a compact manifold with (possibly empty) boundary.

Let $G(M, \partial)$ denote the space of all continuous self-maps $(M, \partial M)$ that are homotopy equivalences of pairs that restrict to the identity on ∂M, and let $\mathrm{Diff}(M, \partial)$ be the group of diffeomorphisms that restrict to the identity of ∂M.

Let $L_j^s(\mathbb{Z}G)$ denote the Wall's L-group of G for surgery up to simple homotopy equivalence. These are abelian groups which are fairly well understood when G is finite. In particular, if G is trivial, then $L_j^s(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ for $j \equiv 0 \pmod 4$ and is finite otherwise.

The following is known to experts but we could not locate a reference.

Theorem 8.1. *If M is a compact orientable manifold and $i \geq 1$, then the dimension of $\pi_i^{\mathbb{Q}} \mathrm{Diff}(M, \partial)$ is bounded above by the dimension of*

$$\mathbb{Q} \otimes \left(\pi_i G(M, \partial) \oplus \pi_i \mathscr{P}(M) \oplus L_{q+1}^s(\mathbb{Z}\pi_1 M) \oplus \Big(\bigoplus_{l \in \mathbb{Z}_+} H_{q-4l}(M) \Big) \right),$$

provided $3i + 9 < \dim M$ and $q = i + 1 + \dim M$.

Proof. Every topological monoid with the identity has abelian fundamental group so tensoring its ith homotopy group with $\mathbb{Q}$ makes sense for $i \geq 1$.

Let $\widetilde{\mathrm{G}}(M, \partial)$ be the topological monoid of block homotopy equivalences of $(M, \partial M)$ that are the identity on the boundary, and let $\widetilde{\mathrm{Diff}}(M, \partial)$ be the subgroup of block diffeomorphisms (see, e.g., [**BM13**] for background on block automorphisms). The inclusion $G(M, \partial) \to \widetilde{G}(M, \partial)$ is a homotopy equivalence, see [**BM13**, p. 21] and there is a fibration

$$\widetilde{G}(M, \partial)/\widetilde{\mathrm{Diff}}(M, \partial) \to B\widetilde{\mathrm{Diff}}(M, \partial) \to B\widetilde{G}(M, \partial),$$

whose homotopy sequence gives

$$\dim \pi_i^{\mathbb{Q}} \widetilde{\mathrm{Diff}}(M, \partial) \leq \dim \left(\pi_i^{\mathbb{Q}} G(M, \partial) \oplus \pi_{i+1}^{\mathbb{Q}} \widetilde{G}(M, \partial)/\widetilde{\mathrm{Diff}}(M, \partial)\right). \tag{8.2}$$

Hatcher [**Hat78**, Chapter 2] constructed a spectral sequence E^n_{pq} converging to $\pi_{p+q+1}\widetilde{\mathrm{Diff}}(M, \partial)/\operatorname{Diff}(M, \partial)$ with

$$E^1_{pq} = \pi_q P(M \times D^p) \quad \text{and} \quad E^2_{pq} = H_p(\mathbb{Z}_2; \pi_q \mathscr{P}(M)),$$

for $q \ll p + \dim(M)$. All elements in $H_{p>0}(\mathbb{Z}_2; -)$ have order 2 [**Bro82**, Proposition III.10.1], so rationally only the terms E^2_{0q} can be nonzero. Hatcher's arguments combined with Igusa's stability theorem [**Igu88**] show that $\pi^{\mathbb{Q}}_{q+1}(\widetilde{\mathrm{Diff}}(M, \partial), \operatorname{Diff}(M, \partial))$ is a quotient of $E^1_{0q} \otimes \mathbb{Q} = \pi^{\mathbb{Q}}_q \mathscr{P}(M)$ provided $\max\{10, 3q + 9\} < \dim(M)$. Thus the homotopy exact sequence of the pair $(\widetilde{\mathrm{Diff}}(M, \partial), \operatorname{Diff}(M, \partial))$ implies for $i \geq 1$ and $3i + 9 < \dim(M)$:

$$\dim \pi_i^{\mathbb{Q}} \operatorname{Diff}(M, \partial) \leq \dim \left(\pi_i^{\mathbb{Q}} \widetilde{\mathrm{Diff}}(M, \partial) \oplus \pi_i^{\mathbb{Q}} \mathscr{P}(M)\right). \tag{8.3}$$

Surgery theory allows us to identify $\pi_{i+1}\widetilde{G}(M, \partial)/\widetilde{\mathrm{Diff}}(M, \partial)$ with the relative smooth structure set $\mathcal{S}(M \times D^{i+1}, \partial)$, see [**Qui70**] and [**BM13**, pp. 21–22]. Set $Q = M \times D^{i+1}$ and $q = \dim Q$. If $\dim Q > 5$ and $i \geq 0$, then the surgery exact sequence

$$L^s_{1+\dim Q}(\mathbb{Z}\pi_1 Q) \to \mathcal{S}(Q, \partial) \to [Q/\partial Q, F/O] \to L^s_{\dim Q}(\mathbb{Z}\pi_1 Q) \tag{8.4}$$

is an exact sequence of abelian groups, where F/O is the homotopy fiber of the J-homomorphism $BO \to BF$. Since BF is rationally contractible, the fiber inclusion $F/O \to BO$ is a rational homotopy equivalence, hence rationally F/O is the product of Eilenberg–MacLane spaces $K(\mathbb{Z}, 4l)$, $l \in \mathbb{Z}_+$. It follows that:

$$[Q/\partial Q, F/O] \otimes \mathbb{Q} \cong \bigoplus_{l \in \mathbb{Z}_+} H^{4l}(Q/\partial Q; \mathbb{Q}),$$

where by the Poincaré–Lefschetz duality

$$\widetilde{H}^j(Q/\partial Q; \mathbb{Q}) \cong H^j(Q, \partial Q; \mathbb{Q}) \cong H_{\dim Q - j}(Q; \mathbb{Q}) \cong H_{\dim Q - j}(M; \mathbb{Q}),$$

which completes the proof because of (8.2), (8.3), (8.4). q.e.d.

Corollary 8.5. *Let M be a compact simply-connected manifold and let $i \geq 1$ such that $\pi_i^{\mathbb{Q}} G(M, \partial) = 0$ and $3i + 9 < \dim M$. Let $q = \dim M + i + 1$. If one of the following is true:*

- *q equals 0 or $1 \mod 4$, and $\tilde{H}_*(M; \mathbb{Q}) = H_{2r}(M; \mathbb{Q})$ for some odd r,*
- *q equals 1 or $2 \mod 4$, and $\tilde{H}_*(M; \mathbb{Q}) \cong \bigoplus_{r \in \mathbb{Z}_+} H_{4r}(M; \mathbb{Q})$,*

then $\dim \pi_i^{\mathbb{Q}} \operatorname{Diff}(M, \partial) \leq \dim \pi_i^{\mathbb{Q}} P(M)$.

Proof. The assertion is a consequence of Theorem 8.1 except when $q = 0 \,(\mathrm{mod}\ 4)$. But in this case we can remove $H_{0=q-4l}(M) \cong \mathbb{Z}$ from the right hand side of the inequality in the statement of Theorem 8.1 because in (8.4) the surgery obstruction map $[Q/\partial Q, F/O] \to L_q^s(\mathbb{Z}) \cong \mathbb{Z}$ is nonzero. We could not find this stated in the literature, so here is a proof. Recall that a normal map is a morphism of certain stable vector bundles whose restriction to the zero sections is a degree one map that is a diffeomorphism on the boundary. By plumbing, see [**Bro72**, Theorems II.1.3], for every integer n one can find a compact manifold P and a degree one map $(P, \partial P) \to (D^{q=4l}, \partial D^q)$ that restricts to a homotopy equivalence $\partial P \to \partial D^q$, is covered by a morphism from the stable normal bundle of P to the trivial bundle over D^q, and whose surgery obstruction equals n. The group of homotopy $(q-1)$-spheres is finite, so by taking boundary connected sums of this normal map with itself sufficiently many, say k, times we can arrange that the homotopy sphere ∂P is diffeomorphic to ∂D^q; the surgery obstruction then equals kn. The map $\partial P \to D^q$ preserves the orientation, so identifying ∂P with ∂D^q yields a self-map of ∂D^n that is homotopic to the identity. Attaching the trace of this homotopy to P we can assume that $\partial P \to \partial D^q$ is the identity. Let L be the manifold built by replacing an embedded q-disk in $\operatorname{Int} Q$ with P, so that there is a degree one map $(L, \partial L) \to (Q, \partial Q)$ that equals the identity outside the embedded copy of P. The bundle data match because the restriction of the stable normal bundle of P to ∂P is the stable normal bundle to ∂P, which is trivial. The additivity of the surgery obstruction, see [**Bro72**, II.1.4], shows that the surgery obstruction of the above normal map covering $(L, \partial L) \to (Q, \partial Q)$ equals kn. q.e.d.

9. Manifolds for which ι_N is not injective on rational homotopy

In this section, we derive criteria of when ι_N is not injective on rational homotopy groups and verify the criteria for manifolds in Theorem 1.2. To apply results of Section 8 we need to bound the size of $\pi_i G(M, \partial)$.

Proposition 9.1. *If E is a compact simply-connected manifold with $\pi_l^{\mathbb{Q}}(E) = 0$ for all $l \geq n$, then $\pi_i G(E \times D^m, \partial)$ is finite for all $m \geq \max\{0, n-i\}$.*

Proof. Since E is compact simply-connected, $\pi_l E$ is finitely generated for all l, see [**Spa66**, Corollary 9.6.16], so $\pi_l E$ is finite for $l \geq n$. For any $m \geq \max\{0, n-i\}$

$$\dim(E \times D^m) + i - n \geq \dim E + \max\{0, n-i\} + i - n \geq \dim E,$$

so $H_j(E \times D^m) = 0$ for $j > \dim(E \times D^m) + i - n$ and the claim follows by applying Lemma 9.3 below to $M = E \times D^m$. q.e.d.

Remark 9.2. To apply the above proposition we either fix any n, i and pick m large enough, or assume $i \geq n$ and let m be arbitrary. Note that if M a rationally elliptic manifold, then $\pi_i^{\mathbb{Q}}(M) = 0$ for all $i \geq 2\sup\{l\colon\ H_l(M; \mathbb{Q}) \neq 0\}$, see [**FHT01**, Theorem 32.15].

Lemma 9.3. *Let M be a compact orientable manifold such that for each l the group $\pi_l M$ is finitely generated and $\pi_1 M$ acts trivially on $\pi_l(M)$. If $\pi_l(M)$ is finite for all $l \geq n$ and $H_j(M)$ is finite for all $j > \dim(M) + i - n$, then $\pi_i G(M, \partial)$ is finite.*

Proof. Arguing by contradiction suppose $\pi_i G(M, \partial)$ contains an infinite sequence of elements represented by maps $f_k\colon\ (D^i, \partial D^i) \to G(M, \partial)$. The adjoint $\hat{f}_k\colon\ M \times D^i \to M$ of the map f_k restricts to the identity of $\partial(M \times D^i)$. Adjusting f_k within its homotopy class and passing if necessary to a subsequence we can find $l \geq 1$ such that $\hat{f}_k$ all agree on the $(l-1)$-skeleton and are pairwise non-homotopic on the l-skeleton rel boundary. Denote by 1 the map sending $(D^i, \partial D^i)$ to the identity element of $G(M, \partial)$, and let $\hat{1}$ be its adjoint.

The rest of the proof draws on the obstruction theory as, e.g., in [**MT68**] which applies as $\pi_1(M)$ acts trivially on homotopy groups. The difference cochain $d(\hat{f}_k, \hat{1})$ that occurs in trying to homotope $\hat{f}_k$ to $\hat{1}$ over the l-skeleton relative to the boundary is a cocycle representing a class in the group $H^l(M \times D^i, \partial(M \times D^i); \pi_l M)$, which by Poincaré–Lefschetz duality is isomorphic to $H_{\dim M + i - l}(M \times D^i; \pi_l M) \cong H_{\dim M + i - l}(M; \pi_l M)$.

Let us show that $H_{\dim M + i - l}(M; \pi_l M)$ is finite. If $l \geq n$, this follows from finiteness of $\pi_l M$ and compactness of M. If $l < n$, then $H_{\dim M + i - l}(M)$ is finite by assumption as $\dim M + i - l > \dim M + i - n$. Since $\pi_l M$ is finitely generated for all l, the group $H_{\dim M + i - l}(M; \pi_l M)$ is finite by the universal coefficients theorem.

Hence passing to a subsequence we can assume that $d(\hat{f}_k, \hat{1})$ are all cohomologous, which by additivity of difference cochains implies that $d(\hat{f}_k, \hat{f}_s)$ is a coboundary for all s, k. Thus all $\hat{f}_k$ are homotopic on the l-skeleton rel boundary, which contradicts the assumptions. q.e.d.

The following result, combined with upper bounds on the rational homotopy of the diffeomorphism group obtained in Section 8, yields a lower bound on $\dim\ker\pi_i^{\mathbb{Q}}(\iota_N)$ in terms of rational homotopy groups of stable pseudoisotopy spaces, which in many cases can be computed.

Theorem 9.4. *If E is a compact manifold, and k, i are integers such that $k \geq 0$, $i \geq 1$ and $\max\{2i+9, 3i+7\} < k + \dim\partial E$, then there is $\varepsilon = \varepsilon(E,i,k) \in \{0,1\}$ such that*

$$\dim\ker\pi_i^{\mathbb{Q}}\big(\iota_{E\times S^{k+\varepsilon}}\big) \geq \frac{\dim\pi_i^{\mathbb{Q}}\mathscr{P}(\partial E)}{2} - \dim\pi_i^{\mathbb{Q}}\operatorname{Diff}(E\times D^{k+\varepsilon},\partial).$$

Proof. Set $d_i = \dim\pi_i^{\mathbb{Q}}\mathscr{P}(\partial E)$. Lemma 5.2(ii) applied to the manifold $D^k\times\partial E$ shows the existence of $\varepsilon\in\{0,1\}$ such that the image of $\pi_i^{\mathbb{Q}}$-homomorphism induced by the inclusion

$$P_\partial(D^{k+\varepsilon}\times\partial E)\to P(D^{k+\varepsilon}\times\partial E)$$

has dimension $\geq \frac{d_i}{2}$.

Set $m = k+\varepsilon$ and $N = E\times S^m$. Let D^m denote the upper hemisphere of S^m, and set $D = E\times D^m$ with the corners smoothed. Let $\operatorname{Diff}^J(D,\partial)$ be the subgroup of $\operatorname{Diff}(D,\partial)$ consisting of diffeomorphisms whose ∞-jet at $E\times\partial D^m$ equals the ∞-jet of the identity map. Following [**Igu88**, Chapter 1, Proposition 1.3] one can show that the inclusion $\operatorname{Diff}^J(D,\partial)\to\operatorname{Diff}(D,\partial)$ is a weak homotopy equivalence. Consider the following commutative diagram of continuous maps:

$$\begin{tikzcd}
P(D^m\times\partial E) \arrow[dr, "\tau"'] & P_\partial(D^m\times\partial E) \arrow[l, "\sigma"'] \arrow[r, "\iota"'] & \operatorname{Diff}^J(D^m\times E,\partial) \arrow[d, "\rho"] \\
& P(S^m\times\partial E) \arrow[r, "\iota_N"] & \operatorname{Diff}(S^m\times E)
\end{tikzcd} \tag{9.5}$$

in which σ is the inclusion, the maps τ, ρ extend diffeomorphisms by the identity, and ι is the restriction of ι_N. The reason we have to deal with ∞-jets is that the extension of a diffeomorphism in $\operatorname{Diff}(D,\partial)$ by the identity of N is not a diffeomorphism.

The inclusions $\partial E\to D^m\times\partial E\to S^m\times\partial E$ induce $\pi_i^{\mathbb{Q}}$-monomorphisms of stable pseudoisotopy spaces as $S^m\times\partial E$ retracts onto $\partial E\to D^m$. The same is true unstably since i is in Igusa's stable range. Thus there is a subspace W of $\pi_i^{\mathbb{Q}}P_\partial(D^m\times\partial E)$ of dimension $\geq\frac{d_i}{2}$ that is mapped isomorphically to a subspace U of $\pi_i^{\mathbb{Q}}P(S^m\times\partial E)$ by $\tau\circ\sigma$. Hence the kernel of $\pi_i^{\mathbb{Q}}(\iota)|_W$ embeds into the kernel $\pi_i^{\mathbb{Q}}(\iota_N)|_U$, and the kernel of $\pi_i^{\mathbb{Q}}(\iota)|_W$ clearly satisfies the claimed inequality. q.e.d.

Remark 9.6. Sadly, there is not a single example of E, i, k with indecomposable $\operatorname{Int}E$ for which we know the value of ε.

Proposition 9.7. *Let E be the total space of a linear disk bundle over a closed manifold such that E and ∂E are simply-connected, the algebra $H^*(E;\mathbb{Q})$ has a single generator, and the algebra $H^*(\partial E;\mathbb{Q})$ does not have a single generator. Then there are sequences i_l, m_l such that the sequence* $\dim\ker \pi_{i_l}^{\mathbb{Q}}(\iota_{E\times S^{m_l}})$ *is unbounded.*

Proof. By [**VPB85**, Corollary 2] the sequence $\dim HC_i(E;\mathbb{Q})$ is bounded while $\dim HC_i(\partial E;\mathbb{Q})$ is unbounded. Since $0 \le \dim \pi_i^{\mathbb{Q}}\mathscr{P}(*) \le 1$, we conclude (see Section 7) that the sequence $\dim \pi_i^{\mathbb{Q}}\mathscr{P}(E)$ is bounded and $\dim \pi_i^{\mathbb{Q}}\mathscr{P}(\partial E)$ is unbounded. The class of rationally elliptic spaces contains all closed manifolds whose rational cohomology algebra has ≤ 2 generators, and is closed under fibrations, see [**FHT93**], so E is rationally elliptic. Hence Proposition 9.1 applies for all sufficiently large i and any m, and we have $\pi_i^{\mathbb{Q}}G(E\times D^m,\partial) = 0$, which by Theorem 8.1 gives a uniform upper bound on $\dim \pi_i^{\mathbb{Q}}\,\mathrm{Diff}(E\times D^m,\partial)$, and the result follows from Theorem 9.4. q.e.d.

Remark 9.8. If in Proposition 9.7 the algebras $H^*(\partial E;\mathbb{Q})$, $H^*(E;\mathbb{Q})$ are singly generated, we can still compute the dimensions of $\pi_i^{\mathbb{Q}}\mathscr{P}(\partial E)$, $\pi_i^{\mathbb{Q}}\mathscr{P}(E)$ using [**VPB85**, Theorem B]. In view of Section 8 and Theorem 9.4 this gives a computable lower bound on $\dim\ker \pi_i^{\mathbb{Q}}(\iota_{E\times S^m})$; of course the bound might be zero.

Let us investigate when Proposition 9.7 does not apply.

Lemma 9.9. *Let $p\colon T\to B$ be a linear S^k-bundle over a closed manifold B with $\dim B>0$ such that T, B are simply-connected and $H^*(T;\mathbb{Q})$ is singly generated, and let e be the rational Euler class of p. Then $k<\dim B$ and the following holds:*

(1) *If $B=S^d$, then either $e=0$ and $\frac{d}{2}=k$ is even,*
or $e\neq 0$ and $d=k+1$ is even.

(2) *If $B=CP^d$ with $d\ge 2$, then $k=1$ and $e\neq 0$.*

(3) *If $B=HP^d$ with $d\ge 2$, then either $k=2$, or $k=3$ and $e\neq 0$.*

Proof. This is a straightforward application of the Gysin sequence

$$\textbf{(G)}\qquad H^{j-k-1}(B;\mathbb{Q}) \xrightarrow[\cup e]{} H^j(B;\mathbb{Q}) \xrightarrow[p^*]{} H^j(T;\mathbb{Q}) \to \longrightarrow H^{j-k}(B;\mathbb{Q}) \xrightarrow[\cup e]{} H^{j+1}(B;\mathbb{Q}).$$

If $\dim B\le k$, then $e=0$ for dimension reasons, so p^* is injective and $H^k(T;\mathbb{Q})$ surjects onto $H^0(B;\mathbb{Q})\cong\mathbb{Q}$. If $\dim B = k$, then $\dim H^k(T;\mathbb{Q})=2$ contradicting that $H^*(T;\mathbb{Q})$ is singly generated. If $\dim B<k$ and $H^*(T;\mathbb{Q})=\langle a\rangle$, then a has degree $\le \dim B<k$ and hence $a\in \mathrm{Im}\,p^*$ so that p^* is a surjection of $H^k(B;\mathbb{Q})=0$ onto $H^k(T;\mathbb{Q})\cong\mathbb{Q}$, which is a contradiction. Thus $k<\dim B$.

Let $B=S^d$. Then (**G**) implies $H^j(T;\mathbb{Q})=0$ except for $j=0$, $k+d$, and possibly for $j=k,d$. If $e\neq 0$, then $d=k+1$ is even, and (**G**) gives $H^k(T;\mathbb{Q})=0=H^d(T;\mathbb{Q})$. If $e=0$, then (**G**) shows that $H^j(T;\mathbb{Q})$ are nonzero for $j=k,d$. Since $H^*(T;\mathbb{Q})$ is singly generated, k, d must be even because an odd degree class is not a power of an even degree class, and any odd degree class has zero square. As $k<d$, we have $d=2k$ completing the proof of (1).

To prove (2) let $B=CP^d$ and note that simple connectedness of T shows that if $k=1$, then $e\neq 0$. To rule out $k\geq 2$ use (**G**) to conclude that $p^*\colon H^2(B)\to H^2(T)$ is injective, hence as $H^*(T;\mathbb{Q})$ is singly generated, the generator must come from B and hence its $(n+1)$th power is zero, but then it cannot generate the top dimensional class in degree $\dim T=k+\dim B\geq 2+2n$.

To prove (3) let $B=HP^d$. Similarly to (2) if $k\geq 4$, then $H^*(T;\mathbb{Q})$ is not singly generated. The same holds for $k=1$ as then $T\to B$ is the trivial S^1-bundle because HP^d is 2-connected. Thus k must equal 2 or 3. Finally, if e were zero for $k=3$, then (**G**) gives that $H^3(T;\mathbb{Q})$ and $H^4(T;\mathbb{Q})$ are nonzero, so $H^*(T;\mathbb{Q})$ could not be singly generated. q.e.d.

Remark 9.10. (a) The exceptional cases above do happen. Examples are the unit tangent bundle to S^d with d even (whose total space is a rational homology sphere), the Hopf bundles $S^1\to S^{2d+1}\to CP^d$ and $S^3\to S^{4d+3}\to HP^d$, and the canonical S^1 quotient $S^2\to CP^{2d+1}\to HP^d$ of the latter bundle. All nontrivial S^2-bundles over S^4 have singly generated total space, see [**GZ00**, Corollary 3.9]. Each of these total spaces appears as ∂E where $\operatorname{Int} E$ admits a complete metric of $K\geq 0$.

(b) The assumption that $B=S^n, CP^n$ or HP^n is there only to simplify notations by excluding some cases not relevant to our geometric applications. The proof of Lemma 9.9 applies to some other bases, e.g., the Cayley plane or biquotients with singly generated cohomology, which are classified in [**KZ04**]. In particular, the unit tangent bundle to the Cayley plane does not have singly generated cohomology.

(c) One can use results of [**Hal78**] to give a rational characterization of fiber bundles $T\to B$ such that T, B are simply-connected manifolds and $H^*(T;\mathbb{Q})$ is singly generated. We will not pursue this matter because with the exception mentioned in (b) it is unclear if such bundles arise in the context of nonnegative curvature.

Theorem 9.11. *Let $N=S^m\times E$ and $i\leq m-3$ where E and i satisfy one of the following:*

1. *E is the total space of a linear D^{2d}-bundle over S^{2d}, $d\geq 2$, with nonzero Euler class, and $i=8d-5+j(4d-2)$ for some odd $j\geq 1$.*
2. *E is the total space of a linear D^4-bundle over HP^d, $d\geq 1$, with nonzero Euler class, and $i=8d+3+j(4d+2)$ for some odd $j\geq 1$.*

3. *E is the total space a linear D^3-bundle over HP^d, $d \geq 1$ with nonzero first Pontryagin class, and $i = 4d+2+j(2d+1)$ for some even $j \geq 0$.*
4. *E is the product of S^4 and the total space of a D^4-bundle over S^4 with nonzero Euler class, and $i = 6j+3$ for some odd $j \geq 3$.*

Then $\pi_i^{\mathbb{Q}}\mathscr{P}(N) = 0$, and furthermore, $\dim \pi_i^{\mathbb{Q}}\mathscr{P}(\partial N) = 1$ in cases (1), (2), (3) and $\dim \pi_i^{\mathbb{Q}}\mathscr{P}(\partial N) = j$ in case (4).

Proof. The inclusions $E \to N$ and $\partial E \to \partial N$ are $(m-1)$-connected, so they induce isomorphisms on $\pi_i^{\mathbb{Q}}\mathscr{P}(-)$ for $i \leq m-3$.

Case 1. Here ∂E is the total space of S^{2d-1}-bundle over S^{2d}, so the homotopy sequence of the bundle shows that ∂E is 2-connected while the Gysin sequence implies $H^*(\partial E;\mathbb{Q}) \cong H^*(S^{4d-1};\mathbb{Q})$. So any degree one map $\partial E \to S^{4d-1}$ is a rational homology isomorphism, and hence a rational homotopy equivalence. The map is 2-connected, so by Corollary 7.10 it induces an isomorphism on $\pi_i^{\mathbb{Q}}\mathscr{P}(-)$. Now (7.3), (7.4), (7.5) give the Poincaré polynomials

$$\frac{t^3}{1-t^4} + \frac{t^{6d-4}}{1-t^{4d-2}} \quad \text{for } \pi_*^{\mathbb{Q}}\mathscr{P}(S^{2d}) \quad \text{and}$$
$$\frac{t^{8d-5}}{1-t^{4d-2}} \quad \text{for } \pi_*^{\mathbb{Q}}(\mathscr{P}(S^{4d-1}), \mathscr{P}(*)).$$

Reducing the exponents mod 4 yields the desired conclusion.

Case 2. Here ∂E is a simply-connected rational homology S^{4d+3}. Then (7.3), (7.12), (7.5) give the Poincaré polynomials

$$\frac{t^3}{1-t^4} + \frac{(1-t^{4d})\,t^{4d+4}}{(1-t^4)\,(1-t^{4d+2})} \quad \text{for } \pi_*^{\mathbb{Q}}\mathscr{P}(HP^d),$$
$$\frac{t^{8d+3}}{1-t^{4d+2}} \quad \text{for } \pi_*^{\mathbb{Q}}(\mathscr{P}(S^{4d+3}), \mathscr{P}(*)).$$

Reducing the exponents mod 4 implies the claim.

Case 3. Nontriviality of the first Pontryagin class implies, see [**Mas58**, pp. 273–274], that the algebra $H^*(\partial E;\mathbb{Q})$ is isomorphic to $\mathbb{Q}[\alpha]/\alpha^{2d+2}$ for some $\alpha \in H^2(\partial E;\mathbb{Q})$. Then (7.3), (7.12), (7.13) give the Poincaré polynomials

$$\frac{t^3}{1-t^4} + \frac{(1-t^{4d})\,t^{4d+4}}{(1-t^4)\,(1-t^{4d+2})} \quad \text{for } \pi_*^{\mathbb{Q}}\mathscr{P}(HP^d), \tag{9.12}$$

$$\frac{t^{2(2d+1)}}{1-t^2} \quad \text{for } \pi_*^{\mathbb{Q}}(\mathscr{P}(\partial E), \mathscr{P}(*)). \tag{9.13}$$

The monomials with exponent i appear in (9.13) and do not occur in the first summand of (9.12). The second summand can be written as $\sum_{s=1}^{d} t^{4d+4s} \sum_{r\geq 0} t^{(4d+2)r}$, hence the exponents of its monomials are $4d+4s+(4d+2)r$ which all lie in the union of the disjoint intervals

$[4d+4+(4d+2)r,\, 8d+(4d+2)r]$. Each number $4d+2+(4d+2)r$ lies in the gap between the intervals, so letting $j=2r$ completes the proof. In fact, many more values of i are allowed because the exponents $4d+4s+(4d+2)r$ of distinct pairs (s,r) differ by 4 or 6 while (9.13) contains every even exponent $\geq 4d+2$.

Case 4. Here ∂E is 2-connected rational homology $S^4\times S^7$. Then (7.3), (7.14), (7.15) give the Poincaré polynomials

$$\frac{t^3}{1-t^4}+\frac{2t^2}{1-t^6}+\frac{t^5+t^6}{(1-t^6)^2}-2t^2-t^6 \quad \text{for } \pi_*^{\mathbb{Q}}\mathscr{P}(S^4\times S^4), \tag{9.14}$$

$$\frac{t^5}{1-t^6}+\frac{t^2+t^9}{(1-t^6)^2}-t^2-t^5-t^9 \quad \text{for } \pi_*^{\mathbb{Q}}(\mathscr{P}(\partial E),\mathscr{P}(*)). \tag{9.15}$$

The term

$$\frac{t^9}{(1-t^6)^2}-t^9=\sum_{j\geq 2} j\, t^{6j+3},$$

in (9.15) has exponents that reduce to 3 mod 6, so it has some common exponents only with the term $t^3(1-t^4)^{-1}$ in (9.14). The exponents corresponding to odd j reduce to 1 mod 4, so do not appear in (9.14). For the same reasons the exponents do not appear elsewhere in (9.15), which completes the proof. q.e.d.

Remark 9.16. Case 4 illustrates that the following proposition is not optimal.

Proposition 9.17. *Let M be a compact manifold with nonempty boundary and let B be a closed b-dimensional manifold of nonzero Euler characteristic. If* $\max\{2i+7, 3i+4\}<\dim\partial M$*, then* $\dim\ker\pi_i^{\mathbb{Q}}(\iota_{M\times B})\geq \dim\ker\pi_i^{\mathbb{Q}}(\iota_M)$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathscr{P}(\partial M) \arrow[d, "\delta_\infty"'] & P(\partial M\times I^b) \arrow[l] \arrow[d, "\delta_b"'] & P(\partial M) \arrow[l, "\Sigma^b"'] \arrow[r, "\iota_M"'] \arrow[d, "\times\,\mathbf{id}_B"] & \mathrm{Diff}(M) \arrow[d, "\times\,\mathbf{id}_B"] \\
\mathscr{P}(\partial M\times B) & P(\partial M\times B) \arrow[l] \arrow[r, "\times\chi(B)"] & P(\partial M\times B) \arrow[r, "\iota_{M\times B}"] & \mathrm{Diff}(M\times B)
\end{tikzcd}$$

where I^b is identified with an embedded disk in B and δ_b is the extension by the identity. The middle bottom arrow is the $\chi(B)$-power map with respect to the group composition. The unlabeled arrows are the canonical maps into the direct limit, and δ_∞ is the stabilization of δ_b.

The rightmost square commutes, while the middle one homotopy commutes [**Hat78**, Appendix I]. Since δ_b homotopy commutes with Σ, the leftmost square also homotopy commutes.

It suffices to show that the map $\times\,\mathbf{id}_B$ of pseudoisotopy spaces is $\pi_i^{\mathbb{Q}}$-injective. Since we are in the pseudoisotopy stable range, Σ^b and the unlabeled arrows are $\pi_i^{\mathbb{Q}}$-isomorphisms. The $\chi(B)$-power map induces the multiplication by $\chi(B)$ on the rational homotopy group, see [**Spa66**, Corollary 1.6.10]. Hence the power map is also $\pi_i^{\mathbb{Q}}$-isomorphism as $\chi(B) \neq 0$. Finally, $\pi_i^{\mathbb{Q}}$-injectivity of δ_∞ follows because $\mathscr{P}(-)$ is a homotopy functor and δ_∞ has a left homotopy inverse induced by the coordinate projection $\partial M \times B \to \partial M$. q.e.d.

Proof of Theorem 1.2. By Theorem 1.1 it suffices to check that the group $\ker \pi_{k-1}^{\mathbb{Q}}(\iota_{U\times S^m})$ is nonzero. A lower bound on $\dim \ker \pi_{k-1}^{\mathbb{Q}}(\iota_{U\times S^m})$ is given by Theorem 9.4 and we wish to find cases when the bound is positive.

If the sphere bundle associated with the vector bundle with total space U does not have singly generated rational cohomology, then the lower bound in Theorem 9.4 can be made arbitrary large by Proposition 9.7 and Theorem 8.1. This applies when U is the tangent bundle to CP^d, HP^d, $d \geq 2$, and the Cayley plane.

If U is the total space of a vector bundle over S^{2d}, $d \geq 2$, with nonzero Euler class, then a positive lower bound in Theorem 9.4 comes from Corollary 8.5 and the part 1 of Theorem 9.11. The same argument works to the Hopf $\mathbb{R}^4$ bundle over HP^d because it has nonzero Euler class, so the part 2 of Theorem 9.11 applies.

A nontrivial $\mathbb{R}^3$ over HP^d, $d \geq 1$, cannot have a nowhere zero section, so it must have nonzero Pontryagin class, see [**Mas58**, Theorem V, p.281]. Then a positive lower bound in Theorem 9.4 comes from Corollary 8.5 and the part 3 of Theorem 9.11. This applies to the Hopf $\mathbb{R}^3$ bundle and the bundles in (4). Finally, (5) follows from Proposition 9.17. q.e.d.

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