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ON A CONJECTURE OF CLEMENS ON RATIONAL CURVES ON HYPERSURFACES

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0. Introduction

In [2], H. Clemens proved the following theorem:

0.1 Theorem. *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d \geq 2n-1$. Then X contains no rational curve.*

In [3],[4] Ein generalized Clemens theorem in two directions; he considered a smooth projective variety M of dimension n, instead of $\mathbb{P}^n$ (which is a mild generalization since any such M can be projected to $\mathbb{P}^n$), and general complete intersections $X \subset M$ of type $(d_1, \dots, d_k)$ and proved:

0.2 Theorem. *If $d_1 + \ldots + d_k \geq 2n - k - l + 1$, any subvariety Y of X of dimension l has a desingularisation $\tilde{Y}$ which has an effective canonical bundle; if the inequality is strict, the sections of $K_{\tilde{Y}}$ separate generic points of $\tilde{Y}$.*

In the case of divisors $Y \subset X$, this result has been improved by Xu [11],[12], who proved:

0.3 Theorem. *Let $Y \subset X$ be a divisor, $\tilde{Y}$ a desingularization of Y, then $p_g(\tilde{Y}) \geq n-1$ if $\sum d_i \geq n+2$.*

In [11], he gave more precise estimates for the minimal genus of a curve in a general surface in $\mathbb{P}^3$.

Now these results are not optimal, excepted in the case of divisors. In fact we will prove in the case of hypersurfaces the following improvement of Clemens and Ein's results:

0.4 Theorem. (See 2.10.) *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d \geq 2n - l - 1$, $1 \leq l \leq n-3$; then any subvariety Y of X of dimension l has a desingularization $\tilde{Y}$ with an effective canonical bundle; if the inequality is strict, the sections of $K_{\tilde{Y}}$ separate generic points of Y.*

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In particular, this proves that general hypersurfaces of degree $d \geq 2n-2$, $n \geq 4$ do not contain rational curves, which was conjectured by Clemens. This result is now optimal since hypersurfaces of degree $\leq 2n-3$ contain lines. Similarly, general hypersurfaces of degree $d \geq 2n-3$ do not contain a surface covered by rational curves, for $n \geq 5$,and this cannot be improved since hypersurfaces of degree $\leq 2n-4$ contain a positive dimensional family of lines. The case $n=4, d=2n-3=5$ is Clemens conjecture on the finiteness of rational curves of fixed degree in a general quintic threefold and is not accessible by our method.

0.5. In the first section, we will prove a very simple proposition (1.1) concerning the global generation of the bundle $T\mathcal{X}(1)_{|X}$, where $\mathcal{X}$ is the universal family of complete intersections, $\mathcal{X} \subset \mathbb{P}^n \times \Pi_i H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))^0$, where the last factor denotes the open set of $\Pi_i H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))$ parametrizing smooth complete intersections, and $X \subset \mathcal{X}$ is a special member of the family. We will show how the theorems of Clemens and Ein are deduced from this. Notice that this is only a formal simplification of the proof of Ein, since the principle of the proof is certainly the same. However, it allows to estimate the codimension of the sublocus of $\Pi_i H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))^0$ where the statement fails to be true. We also give an improvement of Xu's theorem using a refinement of Proposition 1.1. We finally recall from [9], the following kind of applications:

0.6 Theorem. *If $\sum_i d_i > 2n-k+1$, and X is general, no two points of X are rationally equivalent.*

0.7. The second section is devoted to the improvement of these results in the case of hypersurfaces. The main technical point here is Proposition 2.2, which concerns sections of the bundle $\bigwedge^2 T\mathcal{X}(1)_{|X}$. In the above mentioned papers the authors used only sections of $\bigwedge^2 T\mathcal{X}(2)_{|X}$, (which are easily obtained using the wedge products of sections of $T\mathcal{X}(1)_{|X}$), which explains why their results can be improved (by 1).

1. We will begin this section with the proof of the following proposition 1.1; let $S^{d_i} := H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))$, $d_i \geq 2$ and let $\mathcal{X} \subset \mathbb{P}^n \times \Pi_i {S^{d_i}}^0$ be the universal complete intersection; for $t = (t_1, \dots, t_k) \in \Pi_i {S^{d_i}}^0$, let $X_t := {pr_2}^{-1}(t) \subset \mathcal{X}$ be the complete intersection parametrized by t. We assume that $\dim X_t \geq 2$, and that $H^0(T_{X_t}(1)) = \{0\}$, which is certainly true if $K_{X_t} \geq \mathcal{O}_{X_t}(1)$ (with the first assumption), so is not restrictive since this is the only case that we will consider for applications. Then we have:

1.1 Proposition. *The bundle $T\mathcal{X}(1)_{|X_t}$ is generated by global sections.*

Proof. Consider the exact sequence of tangent bundles:

1.1.1. $0 \to T_{X_t}(1) \to T\mathcal{X}(1)_{|X_t} \to (\Pi_i S^{d_i}) \otimes \mathcal{O}_{X_t}(1) \to 0.$

From $h^0(T_{X_t}(1)) = 0$, we deduce:

1.1.2. $H^0(T\mathcal{X}(1)_{|X_t}) = \operatorname{Ker}\mu$, where $\mu : \Pi_i S^{d_i} \otimes S^1 \to H^1(T_{X_t}(1))$ is the coboundary map induced by 1.1.1.

Now $X_t \subset \mathbb{P}^n$ is defined by $t_1 = \ldots = t_k = 0$, so we have the exact sequence:

1.1.3. $0 \to T_{X_t} \to T\mathbb{P}^n_{|X_t} \xrightarrow{\alpha} \Pi_i \mathcal{O}_{X_t}(d_i) \to 0,$

where $\alpha(X_l\partial/\partial X_i) = (X_l\partial t_1/\partial Xj_{|X_t}, \ldots, X_l\partial t_k\partial X_{j_{|X_t}})$. 1.1.3 gives then an isomorphism:

1.1.4.

$$\begin{aligned}\operatorname{Ker}(H^1(T_{X_t}(1)) &\to H^1(T\mathbb{P}^n(1)_{|X_t})) \\ &\cong \Pi_i H^0(\mathcal{O}_{X_t}(d_i+1))/\alpha((H^0(T\mathbb{P}^n_{|X_t}).\end{aligned}$$

Now using the map pr_{1*} between 1.1.1 and 1.1.3:

1.1.5.

$$\begin{array}{ccccccccc} 0 & \to & T_{X_t}(1) & \to & T\mathcal{X}(1)_{|X_t} & \to & (\Pi_i S^{d_i}) \otimes \mathcal{O}_{X_t}(1) & \to & 0 \\ & & Id\downarrow & & pr_{1*}\downarrow & & ev\downarrow & & \\ 0 & \to & T_{X_t}(1) & \to & T\mathbb{P}^n(1)_{|X_t} & \xrightarrow{\alpha} & \Pi_i\mathcal{O}_{X_t}(d_i+1) & \to & 0 \end{array}$$

we see immediately that the map μ of 1.1.2 takes its value in $\operatorname{Ker}(H^1(T_{X_t}(1)) \to H^1(T\mathbb{P}^n(1)_{|X_t}))$, and via the isomorphism of 1.1.4, is simply the map:

1.1.6. $\mu : (\Pi_i S^{d_i}) \otimes S^1 \to \Pi_i H^0(\mathcal{O}_{X_t}(d_i+1))/\alpha(H^0(T\mathbb{P}^n(1)_{|X_t}))$ obtained by composition of the product:$S^{d_i} \otimes S^1 \to S^{d_i+1}$, the restriction to X_t, and the projection modulo $\operatorname{Im}(\alpha)$.

1.1.7. Next let $x \in X_t$ be any point; tensoring everything with $\mathcal{I}_x$ we get similarly isomorphisms:

1.1.8.

$$\begin{aligned}\operatorname{Ker}(H^1(T_{X_t}(1) \otimes \mathcal{I}_x) &\to H^1(T\mathbb{P}^n(1)_{|X_t} \otimes \mathcal{I}_x) \\ &\cong \Pi_i H^0(\mathcal{O}_{X_t}(d_i+1) \otimes \mathcal{I}_x)/\operatorname{Im}(\alpha_x),\end{aligned}$$

where $\alpha_x : H^0(T\mathbb{P}^n(1)_{|X_t} \otimes \mathcal{I}_x) \to \Pi_i H^0(\mathcal{O}_{X_t}(d_i+1) \otimes \mathcal{I}_x)$ is the map induced by α in 1.1.3, and

1.1.9. $H^0(T\mathcal{X}(1)_{|X_t} \otimes \mathcal{I}_x) \cong \operatorname{Ker}\mu_x$,
where $\mu_x : (\Pi_i S^{d_i}) \otimes S^1{}_x \to \Pi_i H^0(\mathcal{O}_{X_t}(d_i+1) \otimes \mathcal{I}_x)/\operatorname{Im}(\alpha_x)$ is the multiplication followed by restriction to X_t and projection mod. $\operatorname{Im}(\alpha_x)$ as in 1.1.6 (Here $S^1{}_x := H^0(\mathcal{O}_{X_t}(1) \otimes \mathcal{I}_x)$).

Now the proof of 1.1 is finished with the obvious observation that μ and μ_x are surjective: indeed, the map given by the inclusion $H^1(T\mathbb{P}^n(1)_{|X_t} \otimes \mathcal{I}_x) \to H^1(T\mathbb{P}^n(1)_{|X_t})$ is injective since $T\mathbb{P}^n(1)_{|X_t}$ is generated by its sections. From $H^0(T_{X_t}(1)) = 0$, we have the exact sequence:

1.1.10. $0 \to H^0(T_{X_t|x}) \to H^1(T_{X_t}(1) \otimes \mathcal{I}_x) \to H^1(T_{X_t}(1)) \to 0,$
which induces an exact sequence:

1.1.11.

$$\begin{aligned} 0 \to H^0(T_{X_t|x}) &\to (\mathrm{Ker}(H^1(T_{X_t}(1) \otimes \mathcal{I}_x) \to H^1(T\mathbb{P}^n(1)_{|X_t} \otimes \mathcal{I}_x)) \\ &\to \mathrm{Ker}(H^1(T_{X_t}(1)) \to H^1(T\mathbb{P}^n(1)_{|X_t})) \to 0, \end{aligned}$$

that is:

1.1.12. $0 \to H^0(T_{X_t|x}) \to \mathrm{Im}(\mu_x) \to \mathrm{Im}(\mu).$

It then follows that $\mathrm{Ker}(\mu_x) \subset \mathrm{Ker}(\mu)$ has codimension equal to: $\dim(\bigoplus_i S^{d_i}) + h^0(TX_{t|x}) = \mathrm{rank}(T\mathcal{X}(1)_{|x})$. By the isomorphisms of 1.1.2, 1.1.6, 1.1.9, we conclude that $H^0(T\mathcal{X}(1)_{|X_t} \otimes \mathcal{I}_x) \subset H^0(T\mathcal{X}(1)_{|X_t})$ has codimension equal to the rank of $T\mathcal{X}$, which means that $T\mathcal{X}(1)_{|X_t}$ is globally generated at x.

Now Proposition 1.1 implies

1.2 Corollary. *For any $l \geq 0$ the bundle $\bigwedge^l T\mathcal{X} \otimes \mathcal{O}_{X_t}(l)$ is generated by global sections, and the bundle $\bigwedge^l T\mathcal{X} \otimes \mathcal{O}_{X_t}(l+1)$ is very ample (in the sense that its global sections restrict surjectively to its sections over any 0-dimensional subscheme of length two of X_t).*

Now $T\mathcal{X}_{|X_t}$ has determinant equal to $K_{X_t} \cong \mathcal{O}_{X_t}(\sum_i d_i - n - 1)$, so we have:

1.2.1. $\bigwedge^l T\mathcal{X} \otimes \mathcal{O}_{X_t}(l) \cong \bigwedge^{N+n-k-l} \Omega\mathcal{X}_{|X_t} \otimes \mathcal{O}_{X_t}(l - \sum_i d_i + n + 1),$
where $N = \dim(\bigoplus_i S^{d_i})$, so $N + n - k = \dim \mathcal{X}$. Thus we conclude:

1.3 Corollary. *$\Omega_{\mathcal{X}}^{N+n-k-l}{}_{|X_t}$ is generated by global sections when $l - \sum_i d_i + n + 1 \leq 0$, and is very ample when this inequality is strict.*

This gives immediately the following refinement 1.4 of Clemens and Ein's results (0.2): Let $\mathcal{M} \subset \Pi_i S^{d_i 0}$ be a subvariety, and let $\tilde{\mathcal{M}} \xrightarrow{\pi} \mathcal{M}$ be an étale map; let $\mathcal{Y} \subset \mathcal{X}_{\tilde{\mathcal{M}}}$ be a subvariety of the family obtained by base change to $\tilde{\mathcal{M}}$; we assume that $pr_2 : \mathcal{Y} \to \tilde{\mathcal{M}}$ is dominant of generic fiber dimension l. Then we have:

1.4 Theorem. *If $\sum_i d_i \geq 2n - k + 1 - l + \mathrm{codim}\,\mathcal{M}$, then any desingularization $\tilde{Y}_t$ of the generic fiber Y_t of $pr_2 : \mathcal{Y} \to \tilde{\mathcal{M}}$ has an effective canonical bundle. If the inequality is strict, then the sections of $K_{\tilde{Y}_t}$ separate generic points of $\tilde{Y}_t$.*

Proof. We have $\dim \mathcal{Y} = N + l - \mathrm{codim}\,\mathcal{M}$; by 1.3, if $\sum_i d_i \geq 2n - k + 1 - l + \mathrm{codim}\,\mathcal{M}$, then the bundle $\Omega_{\mathcal{X}_{\tilde{\mathcal{M}}}}^{\dim \mathcal{Y}}{}_{|X_m}$ is generated by the global sections, for all $m \in \tilde{\mathcal{M}}$ such that $\mathcal{M}$ is smooth at $\pi(m)$, since the map $\tilde{\mathcal{M}} \to \mathcal{M}$ is étale. Let $\tilde{\mathcal{Y}}$ be a desingularization of $\mathcal{Y}$, and $j : \tilde{\mathcal{Y}} \to \mathcal{X}_{\tilde{\mathcal{M}}}$ be the natural induced map; then j is generically an immersion. So it follows that $\Omega_{\tilde{\mathcal{Y}}}^{\dim \mathcal{Y}}{}_{|\tilde{Y}_m}$ has a nonzero section, for generic $m \in \tilde{\mathcal{M}}$. Since for a smooth fiber $\tilde{Y}_m$, one has an isomorphism: $\Omega_{\tilde{\mathcal{Y}}}^{\dim \mathcal{Y}}{}_{|\tilde{Y}_m} \cong K_{\tilde{Y}_m}$, we have proved that the canonical bundle $K_{\tilde{Y}_m}$ is effective, for generic $m \in \tilde{\mathcal{M}}$, as we wanted. Similarly, if the inequality is strict, then again by 1.3, the bundle $\Omega_{\mathcal{X}}^{\dim \mathcal{Y}}{}_{|X_m}$ is very ample, for any $m \in \tilde{\mathcal{M}}$, so for a generic point $m \in \tilde{\mathcal{M}}$, satisfying the conditions that j is an immersion generically along $\tilde{Y}_m$ and that $\tilde{Y}_m$ is smooth, we get that the sections of $\Omega_{\tilde{\mathcal{Y}}}^{\dim \mathcal{Y}}{}_{|\tilde{Y}_m} \cong K_{\tilde{Y}_m}$ separate generic points of $\tilde{Y}_m$.

1.5. We explain now how we can obtain the following refinement of Xu's theorem 0.3 in the case of hypersurfaces; of course, only the case where $d = n + 2$ is to be considered, since the case $d > n + 2$ is covered by Ein's theorem.

1.6 Theorem, *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d = n + 2$. Then for any irreducible divisor $Y \subset X$, any desingularization $\tilde{Y}$ of X satisfies that the canonical map of $\tilde{Y}$ is generically finite on its image.*

We consider again $\mathcal{X} \subset \mathbb{P}^n \times S^{d^0}$, the universal hypersurface, and $X_t \subset \mathcal{X}$ a fiber of pr_2; we have shown that $T\mathcal{X}(1)_{|X_t}$ is generated by the global sections, hence gives a map:

1.6.1. $\phi : \mathbb{P}(\Omega_{\mathcal{X}}(-1)_{|X_t}) \to \mathbb{P}^M$.

The proof of the Theorem 1.6 will follow from

1.7 Proposition. *On the set of $GL(n+1)$-invariant hyperplanes of $T\mathcal{X}(1)_{|X_t}$, the positive dimensional fibers of ϕ project onto lines contained in X.*

Here we consider the natural action of $GL(n + 1)$ on

$$\mathcal{X} \subset \mathbb{P}^n \times S^{d^0}.$$

The $GL(n + 1)$-invariant hyperplanes are those which contain the tangent vectors to this action.

1.8. Let us explain how 1.7 implies 1.6: it suffices to show that for any étale map $\mathcal{M} \to S^{d^0}$,with a lifting of the $GL(n+1)$ action, and any $GL(n + 1)$-invariant divisor $\mathcal{Y} \subset \mathcal{X}_{\mathcal{M}}$, ($\mathcal{X}_{\mathcal{M}}$ is the family obtained by base change to $\mathcal{M}$), any desingularization $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ satisfies:

1.8.1. *The sections of $K_{\tilde{\mathcal{Y}}|\tilde{Y}_t} \cong K_{\tilde{Y}_t}$ give a map $\tilde{Y}_t \cdots > \mathbb{P}^{M'}$ generically finite on its image, for generic $t \in \mathcal{M}$.*

Now, at a point y where $\tilde{\mathcal{Y}} \to \mathcal{X}_{\mathcal{M}}$ is an immersion, $T\tilde{\mathcal{Y}}_{|y} \subset T\mathcal{X}_{\mathcal{M}|y}$ is a $GL(n + 1)$-invariant hyperplane. Let $t \in \mathcal{M}$ be generic, and x, y two points of $\tilde{Y}_t$, where $\tilde{\mathcal{Y}} \to \mathcal{X}_{\mathcal{M}}$ is an immersion. If $T\tilde{\mathcal{Y}}_{|x}$, $T\tilde{\mathcal{Y}}_{|y}$ are not in the same fiber of ϕ, then there is a section of $T\mathcal{X}(1)_{|X_t} \cong \Omega_{\mathcal{X}}^{N+n-2}{}_{|X_t}$ (since $d = n + 2$), which vanishes on $T\tilde{\mathcal{Y}}_{|x}$ but not on $T\tilde{\mathcal{Y}}_{|y}$. In other words, the fibers of the map $\psi : \tilde{Y}_t \cdots > \mathbb{P}^{M''}$ given by the image of $H^0(\Omega_{\mathcal{X}}^{N+n-2}{}_{|X_t})$ in $H^0(\Omega_{\tilde{\mathcal{Y}}}^{N+n-2}{}_{|\tilde{Y}_t}) \cong H^0(K_{\tilde{Y}_t})$ are contained over an open set of $\tilde{Y}_t$ in the projection of fibers of ϕ.

So the positive dimensional fibers of ψ, over an open set of $\tilde{Y}_t$ must be lines contained in X_t by 1.7. But if t is generic, the family of lines in X_t has dimension $n-5$, so lines in X_t cannot cover a divisor of X_t, which proves that ψ is generically finite on its image.

1.9 Proof of Proposition 1.7. Recall from 1.1.2,1.1.6 the isomorphism: $H^0(T\mathcal{X}(1)_{|X_t}) \cong \operatorname{Ker}\mu$, where $\mu : S^d \otimes S^1 \to R_t^{d+1}$ is the multiplication $\mu_0 : S^d \otimes S^1 \to H^0(\mathcal{O}_{X_t}(d+1))$ followed by the projection $H^0(\mathcal{O}_{X_t}(d+1)) \to R_t^{d+1} := S^{d+1}/J_t^{d+1}$, where J_t is the jacobian ideal of the defining equation F_t of X_t. Let now $H \subset \operatorname{Ker}\mu$ be a hyperplane and let $K \subset S^d \otimes S^1$ be a hyperplane such that $K \cap \operatorname{Ker}\mu = H$. A point $x \in X_t$ is in the projection of $\phi^{-1}(H)$ iff the evaluation map $H \to T\mathcal{X}(1)_{|x}$ is not surjective. Let $K_x := K \cap S^d \otimes S_x^1$. Notice that there is at most one point x such that $K_x = S^d \otimes S_x^1$, so we may assume that K_x is a hyperplane of $S^d \otimes S_x^1$, since we are interested in the description of the positive dimensional fibers of ϕ. Using the notation of the proof of 1.1, we have the following exact diagramm:

1.9.2.

$$
\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & 0 & & 0 & & TX_t(1)_x \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(T\mathcal{X}(1)_{|X_t} \otimes \mathcal{I}_x) \cap H & \to & K_x & \stackrel{\mu_x}{\to} & S_x^{d+1}/\alpha(H^0((T\mathbb{P}^n(1) \otimes \mathcal{I}_x)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H & \to & K & \stackrel{\mu}{\to} & R^{d+1}
\end{array},
$$

Under the above assumption, $K_x \subset K$ has codimension equal to $N := \dim S^d$. It is easy to see that the map μ is surjective, so we conclude from 1.9.2 that

$$H^0(T\mathcal{X}(1)_{|X_t} \otimes \mathcal{I}_x) \cap H \subset H$$

has codimension equal to rank$(T\mathcal{X}(1)$ when μ_x is surjective. On the other hand, since K_x is a hyperplane in $S^d \otimes S_x^1$, μ_x will be surjective if K_x does not contain $\operatorname{Ker}(\mu_0^x : S^d \otimes S_x^1 \to H^0(\mathcal{O}_{X_t}(d+1) \otimes \mathcal{I}_x))$. Thus the projection to X_t of the fiber $\phi^{-1}(H)$ is contained in the set $\{x / \operatorname{Ker}\mu_0^x \subset K_x\}$, with one eventual supplementary point where $K_x = S^d \otimes S_x^1$.

Now suppose that H contains $\operatorname{Ker}\mu_0$: Using the exact sequence:

1.9.3. $0 \to T\mathcal{X}_{|X_t} \to T\mathbb{P}^n{}_{|X_t} \oplus S^d \otimes \mathcal{O}_{X_t} \stackrel{dF}{\to} \mathcal{O}_{X_t}(d) \to 0,$
where $dF((u,g))(x) = {}_uF_t(x) + g(x)$, one sees easily that $T\mathcal{X}_{|X_t}$ contains the bundle $M_{d|X_t}$, where M_d is defined by the exact sequence:

1.9.4. $0 \to M_d \to S^d \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0.$

Furthermore one checks readily that $\operatorname{Ker}\mu_0 \subset \operatorname{Ker}\mu$ identifies with the inclusion $H^0(M_d(1)_{X_t}) \subset H^0(T\mathcal{X}(1)_{|X_t})$ and that $M_d(1)$ is generated by global sections. So, if H contains $\operatorname{Ker}\mu_0$, then $\phi^{-1}(H)$ corresponds to hyperplanes V_x in $T\mathcal{X}(1)_x, x \in X_t$ such that $M_{d|x} \subset V_x$. But it is easy to see that $M_{d|x}$ together with the vectors tangent to the infinitesimal action of $GL(n+1)$ generate $T\mathcal{X}(1)_x$, so $\phi^{-1}(H)$ cannot contain a $GL(n+1)$−invariant hyperplane, when H contains $\operatorname{Ker}\mu_0$.

Finally, assume that $\operatorname{Ker}\mu_0 \not\subset H$; then we have:

1.9.5 Lemma. *The set $\{x \in X_t / \operatorname{Ker}\mu_0^x \subset K_x\}$ is contained in a line.*

This is elementary: it suffices to note that if x, y, z are three noncolinear points of X_t, then $\operatorname{Ker}\mu_0^x, \operatorname{Ker}\mu_0^y, \operatorname{Ker}\mu_0^z$ generate $\operatorname{Ker}\mu_0$.

1.10. As in [9], from 1.3 we can also deduce information about the Chow groups $CH_0(X_t)$ for general X_t. In fact, let $\mathcal{M} \subset \Pi_i S^{d_i^0}$ be a subvariety, as in 1.4; then 1.3 gives us:

1.10.1. *For $\sum_i d_i > 2n - k + 1 + \operatorname{codim}\mathcal{M}$, the bundle $\Omega^{\dim \mathcal{M}}_{\mathcal{X}_{\tilde{\mathcal{M}}}\ |X_m}$ is very ample, for any $m \in \mathcal{M}$.*

Now we conclude:

1.11 Theorem. *For $\sum_i d_i > 2n - k + 1 + \operatorname{codim}\mathcal{M}$, no two distinct points of X_m are rationally equivalent, if m is a general point of $\mathcal{M}$.*

We recall from [9] how 1.11 is deduced from 1.10.1: if 1.11 is not true, then there is an étale cover $\tilde{\mathcal{M}}$ of an open set of the smooth part of $\mathcal{M}$, and two distinct sections σ, $\tau : \tilde{\mathcal{M}} \to \mathcal{X}_{\tilde{\mathcal{M}}}$ such that for $m \in \tilde{\mathcal{M}}, \sigma(m)$ is rationally equivalent to $\tau(m)$ in the fiber X_m. The cycle $Z = \sigma(\tilde{\mathcal{M}}) - \tau(\tilde{\mathcal{M}})$ is of codimension $n-k$ in $\mathcal{X}_{\tilde{\mathcal{M}}}$, and the assumption implies that a multiple of it is rationally equivalent to a cycle supported over a proper subset of $\tilde{\mathcal{M}}$. It follows that its class $[Z] \in H^{n-k}(\Omega^{n-k}_{\mathcal{X}_{\tilde{\mathcal{M}}}})$ vanishes in $H^0(R^{n-k}pr_{2*}\Omega^{n-k}_{\mathcal{X}_{\tilde{\mathcal{M}}}})$ over an open set of $\tilde{\mathcal{M}}$. On the other hand, for $m \in \tilde{\mathcal{M}}, H^{n-k}(X_m, \Omega^{n-k}_{\mathcal{X}_{\tilde{\mathcal{M}}}\ |X_m})$ is dual of $H^0(X_m, \Omega^{\dim \tilde{\mathcal{M}}}_{\mathcal{X}_{\tilde{\mathcal{M}}}\ |X_m} \otimes K_{\tilde{\mathcal{M}}}{}^{-1})$ by Serre duality, and one checks the following:(see [9])

1.11.1. *The class $(\alpha_Z)_m \in \operatorname{Hom}(H^0(X_m, \Omega^{\dim \tilde{\mathcal{M}}}_{\mathcal{X}_{\tilde{\mathcal{M}}}\ |X_m}), K_{\tilde{\mathcal{M}},m})$ obtained as the image of $[Z]$ by the composite:*

$$\begin{aligned} H^{n-k}(\Omega^{n-k}_{\mathcal{X}_{\tilde{\mathcal{M}}}}) \to H^{n-k}(X_m, \Omega^{n-k}_{\mathcal{X}_{\tilde{\mathcal{M}}}\ |X_m}) &\cong (H^0(X_m, \Omega^{\dim \tilde{\mathcal{M}}}_{\mathcal{X}_{\tilde{\mathcal{M}}}\ |X_m} \otimes K_{\tilde{\mathcal{M}}}{}^{-1}))^* \\ &\cong \operatorname{Hom}(H^0(X_m, \Omega^{\dim \tilde{\mathcal{M}}}_{\mathcal{X}_{\tilde{\mathcal{M}}}\ |X_m}), K_{\tilde{\mathcal{M}},m}) \end{aligned}$$

is equal to $\sigma^* - \tau^*$.

Here σ^*, τ^* are the pull-back maps of holomorphic forms by the sections σ, $\tau : \tilde{\mathcal{M}} \to \mathcal{X}_{\tilde{\mathcal{M}}}$. Now this is finished since by 1.10.1, $\Omega^{\dim \mathcal{M}}_{\mathcal{X}_{\tilde{\mathcal{M}}}\ |X_m}$

is very ample, when $\sum_i d_i > 2n - k + 1 + \operatorname{codim} \mathcal{M}$, which implies immediately that for $\sigma(m) \neq \tau(m)$, the map $\sigma^* - \tau^*$ cannot be zero at m, in contradiction with $(\alpha_Z)_m = 0$.

2. In this section we will consider the case where $k = 1$, that is hypersurfaces of degree d in $\mathbb{P}^n$. Let $\mathcal{X} \subset \mathbb{P}^n \times (S^d)^0$ be the universal hypersurface; the main point in the previous section was to get the global generation of $\bigwedge^l T\mathcal{X}(l)_{|X_t}$, using global sections of $T\mathcal{X}(1)_{|X_t}$. I do not know the answer to the following question:

2.1 Question. *When is $\bigwedge^2 T\mathcal{X}(1)_{|X_t}$ generated by global sections, at least for generic t?*

(This should be true when K_X is ample.)

However, for our applications, the following proposition will suffice to improve the results of Section 1: view $H^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t})$ as a space of sections of a certain line bundle over the grassmannian of codimension-two subspaces of $T\mathcal{X}(1)_{|X_t}$. Assume $n \geq 4$ and $K_X \geq \mathcal{O}_X(1)$; then we have:

2.2 Proposition. *For generic t, $H^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t})$ has no base point on the set of $GL(n+1)$-invariant codimension-two subspaces of $T\mathcal{X}_{|X_t}$.*

Here we are considering the natural action of $GL(n+1)$ on

$$\mathcal{X} \subset \mathbb{P}^n \times S^d : g(x, F) = (g(x), (g^{-1})^*(F));$$

by invariant subspace, we mean subspaces containing the vectors tangent to the orbits of $GL(n+1)$.

Proof. Consider the inclusion $j : \mathcal{X} \hookrightarrow \mathbb{P}^n \times S^d$; it gives the exact sequence:

2.2.1. $0 \to T\mathcal{X}_{|X_t} \to T\mathbb{P}^n{}_{|X_t} \oplus S^d \otimes \mathcal{O}_{X_t} \xrightarrow{df} \mathcal{O}_{X_t}(d) \to 0,$
where $dF((u,H))_{(x)} = dF_{t(x)}(u) + H(x)$ if F_t is the equation of X_t in $\mathbb{P}^n$. Let M_d be the bundle on $\mathbb{P}^n$ defined by the exact sequence:

2.2.2. $0 \to M_d \to S^d \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0.$

From 2.2.2, we get an inclusion $M_{d|X_t} \subset T\mathcal{X}_{|X_t}$ and an exact sequence:

2.2.3. $0 \to M_{d|X_t} \to T\mathcal{X}_{|X_t} \to T\mathbb{P}^n{}_{|X_t} \to 0.$

In particular, we obtain an inclusion:

2.2.4. $H^0(\bigwedge^2 M_d(1)_{|X_t}) \subset H^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t}).$

Now we have the following lemma:

2.3 Lemma. *$H^0(\bigwedge^2 M_d(1))$, viewed as a set of sections of a certain line bundle on the grassmannian of codimension-two subspaces of the bundle M_d, has for base points the set $\{(x,T), x \in \mathbb{P}^n, T \subset M_{d(x)}$, such that T contains the ideal of a line Δ through $x\}$.*

Proof. The exact sequence defining M_d gives an isomorphism: $H^0(\bigwedge^2 M_d(1)) \cong \operatorname{Ker} \mu'$, where $\mu' : \bigwedge^2 S^d \otimes S^1 \to S^d \otimes S^{d+1}$ is the Koszul map defined by: $\mu'((P \wedge Q) \otimes A) = P \otimes AQ - Q \otimes AP$. Now $\operatorname{Ker} \mu'$

contains the elements: $PA\wedge PB\otimes C-PA\wedge PC\otimes B+PB\wedge PC\otimes A$, for $P\in S^{d-1}, A, B, C\in S^1$. It follows that the image of the restriction map: $H^0(\bigwedge^2 M_d(1))\to\bigwedge^2 M_d(1)_{|x}\subset\bigwedge^2 S^d$ contains the elements $PA\wedge PB$, for $P\in S^{d-1}, A, B\in S^1_x$, where $S^1_x := H^0(\mathcal{O}_{\mathbb{P}^n}(1)\otimes\mathcal{I}_x)$. Let $T\subset M_{d,x} := H^0(\mathcal{O}_{\mathbb{P}^n}(d)\otimes\mathcal{I}_x)$ be of codimension two, and suppose $H^0(\bigwedge^2 M_d(1))$ vanishes on it. Then for any $P\in S^{d-1}$, $[T:P]_x := \{A\in S^1_x \,/\, PA\in T\}$ must be an hyperplane, that is, the map $m_P : S^1_x\to S^d_x/T$ of multiplication by P is not surjective. If $[T:P]_x = S^1_x$ for generic P, then $T = S^d_x$, which is not true; otherwise m_P has generic rank one. Differentiating this condition at a generic point $P\in S^{d-1}$, we find $[T:P]_x\cdot S^{d-1}\subset T$, so 2.3 is proved since $[T:P]_x$ is the component of degree 1 of the ideal of a line Δ containing x. The converse follows from the fact that if T contains the ideal of a line Δ containing x, the composite map:

2.3.1. $H^0(\bigwedge^2 M_d(1))\to\bigwedge^2 M_d(1)_{|x}\to\bigwedge^2(M_{d|x}/T)$

factors through the restriction map:

2.3.2. $H^0(\bigwedge^2 M_d(1))\to H^0(\bigwedge^2 M_d^{\Delta}(1))$,

where M_d^{Δ} is defined by the exact sequence:

2.3.3. $0\to M_d^{\Delta}\to H^0(\mathcal{O}_{\Delta}(d))\to\mathcal{O}_{\Delta}(d)\to 0.$

Now it is easy to see that $H^0(\bigwedge^2 M_d^{\Delta}(1)) = \{0\}$.

From 2.3 and 2.2.3, 2.2.4, we conclude immediately:

2.4 fact. *Let $V\subset T\mathcal{X}_{|x}$ be a codimension-two subspace which is a base point of $H^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t})$. Then $V\cap M_{d|x}$ must be a hyperplane of $M_{d|x}$ or must contain the ideal of a line Δ containing x.*

To deal with the first case, we show:

2.5 Lemma. *Let P be the quotient $\bigwedge^2 T\mathcal{X}(1)_{|X_t})/\bigwedge^2 M_d(1)_{|X_t}$. Then the map $H^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t}))\to H^0(P)$ is surjective, and P is generated by global sections.*

Proof. The first assertion comes from the vanishing:(see[6])

2.5.1. $H^1(\bigwedge^2 M_d(1)_{|X_t}) = \{0\}$.

In fact consider the exact sequence:

2.5.2. $0\to\bigwedge^2 M_d(1)_{|X_t}\to\bigwedge^2 S^d\otimes\mathcal{O}_{X_t}(1)\to M_d\otimes\mathcal{O}_{X_t}(d+1)\to 0.$

It follows that:

2.5.3.

$$H^1(\bigwedge^2 M_d(1)_{|X_t}) = H^0(M_d\otimes\mathcal{O}_{X_t}(d+1))/\operatorname{Im}(\bigwedge^2 S^d\otimes S^1),$$

and this is equal to

$$\operatorname{Ker}(S^d\otimes H^0(\mathcal{O}_{X_t}(d+1))\to H^0(\mathcal{O}_{X_t}(2d+1)))/\operatorname{Im}(\bigwedge^2 S^d\otimes S^1).$$

But it is shown by M. Green in [6] that the following sequence is exact at the middle:

2.5.4. $\bigwedge^2 S^d \otimes S^1 \to S^d \otimes S^{d+1} \to S^{2d+1}$,
where the first map is the Koszul map μ' of 2.3. Since $\mathrm{Ker}(S^d \otimes S^{d+1} \to S^{2d+1})$ surjects onto $\mathrm{Ker}(S^d \otimes H^0(\mathcal{O}_{X_t}(d+1)) \to H^0(\mathcal{O}_{X_t}(2d+1)))$, we conclude immediately, as in [5], that 2.5.4 remains exact after restriction to X_t, that is, by 2.5.3, that $H^1(\bigwedge^2 M_d(1)_{|X_t}) = \{0\}$.

As for the first statement, we have an exact sequence:

2.5.5. $0 \to M_d \otimes T\mathbb{P}^n(1)_{|X_t} \to P \to \bigwedge^2 T\mathbb{P}^n(1)_{|X_t} \to 0$.

Again $H^1(M_d \otimes T\mathbb{P}^n(1)_{|X_t}) = \{0\}$ by the exact sequence:

2.5.6.

$$0 \to M_d \otimes T\mathbb{P}^n(1)_{|X_t} \to S^d \otimes T\mathbb{P}^n(1)_{|X_t} \to T\mathbb{P}^n(d+1)_{|X_t} \to 0,$$

the equality $H^1(T\mathbb{P}^n(1)_{|X_t}) = \{0\}$ $(n \geq 4)$, and the fact that $H^0(T\mathbb{P}^n(d+1)_{|X_t})$ is generated by $H^0(T\mathbb{P}^n(1)_{|X_t})$.

Finally $\bigwedge^2 T\mathbb{P}^n(1)_{|X_t}$ is generated by global sections, as is $M_d \otimes T\mathbb{P}^n(1)_{|X_t}$, which follows from the Euler sequence and the fact that $M_d(2)$ is generated by global sections. This last fact is seen as follows: we have $H^0(M_d(2)) = \mathrm{Ker}(S^d \otimes S^2 \overset{mult.}{\to} S^{d+2})$; this contains the elements $PA \otimes B - PB \otimes A$, for $P \in S^{d-2}, A, B \in S^2$. Evaluating these elements in $M_d(2)_{|x}$, we get for$A(x) = 0, B(x) \neq 0$ the elements $PA, A(x) = 0, P \in S^{d-2}$, of $M_d(2)_x = H^0(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_x)$. Clearly, they generate $H^0(\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_x)$.

Now 2.4 and 2.5 show:

2.6 Corollary. *If $V \subset T\mathcal{X}_{|x}$ is a codimension-two subspace which is a base point of $H^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t})$, then $V \cap M_{d|x}$ must contain the ideal of a line Δ containing x.*

Indeed, if $V \cap M_{d|x}$ is a hyperplane of $M_{d|x}$, the map

$$H^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t}) \to \bigwedge^2 (T\mathcal{X}_{|x}/V)$$

factors through the map: $H^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t}) \to P_x$ which is surjective by 2.5.

2.7. To finish the proof of Proposition 2.2, we now specialize to the case of the Fermat variety X defined by the equation $F = \sum_i X_i^d = 0$. We may do it because of the following lemma:

2.7.1 Lemma. *$h^0(\bigwedge^2 T\mathcal{X}(1)_{|X_t})$ is independant of $t \in S^{d^0}$.*

Proof. Using the exact sequence (see 2.5) defining P:

$$0 \to \bigwedge^2 M_d(1)_{|X_t} \to \bigwedge^2 T\mathcal{X}(1)_{|X_t} \to P \to 0,$$

and 2.5.1, it suffices to prove that $h^0(\bigwedge^2 M_d(1)_{|X_t})$ and $h^0(P)$ are independant of $t \in S^{d^0}$. For the first one, this comes from the exact sequence (see 2.5.2, 2.5.4)

2.7.2.

$$0 \to H^0(\bigwedge^2 M_d(1)_{|X_t}) \to \bigwedge^2 S^d \otimes H^0(\mathcal{O}_{X_t}(1)) \\ \to S^d \otimes H^0(\mathcal{O}_{X_t}(d+1)) \to H^0(\mathcal{O}_{X_t}(2d+1)) \to 0,$$

where all spaces, starting from the second one have constant rank. For the second one, this follows from the exact sequence 2.5.4, with $H^1(M_d \otimes T\mathbb{P}^n(1)_{|X_t}) = \{0\}$. So it suffices to know that $H^0(M_d \otimes T\mathbb{P}^n(1)_{|X_t})$ and $H^0(\bigwedge^2 T\mathbb{P}^n(1)_{|X_t})$ have ranks independant of t. But this is immediate for the second one by Bott vanishing theorem, and for the first one by the exact sequence:

2.7.3.

$$0 \to H^0(M_d \otimes T\mathbb{P}^n(1)_{|X_t}) \to S^d \otimes h^0(T\mathbb{P}^n(1)_{|X_t}) \\ \to H^0(T\mathbb{P}^n(d+1)_{|X_t}) \to 0,$$

where all terms starting from the second one have constant rank by Bott vanishing theorem.

2.8. So let X be the Fermat variety, $x \in X$ and $V \subset T\mathcal{X}_{|x}$ be a codimension-two subspace, which is a base point of $H^0(\bigwedge^2 T\mathcal{X}(1)_{|X})$, and is invariant under the infinitesimal action of $GL(n+1)$, which means that it contains:

2.8.1. $J_x := \{(u(x), -_{\tilde{u}}F)\} \subset T\mathcal{X}_{|x} \subset T\mathbb{P}^n{}_{|x} \times S^d$,
where $u \in H^0(T\mathbb{P}^n)$, and $\tilde{u}$ is a lifting of u in the Lie algebra of $GL(n+1)$, so $\tilde{u} = \sum_i A_i \partial/\partial X_i$, $A_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ and $_{\tilde{u}}F = \sum_i A_i \partial F/\partial X_i$.

We know by 2.6 that V contains the ideal of a line Δ containing x: $I_\Delta(d) \subset M_{d|x} \subset T\mathcal{X}_{|x}$. Let $T\mathcal{X}^\Delta_{|x} := T\mathcal{X}_{|x}/I_\Delta(d)$, and let J^Δ_x be the image of J_x in $T\mathcal{X}^\Delta_{|x}$. Since V contains J_x and $I_\Delta(d)$, the map:

$$H^0(\bigwedge^2 T\mathcal{X}(1)_{|X}) \to H^0(\bigwedge^2 T\mathcal{X}(1)_{|x}) \to \bigwedge^2 (T\mathcal{X}/V)$$

factors through the map:

2.8.2. $\beta : H^0(\bigwedge^2 T\mathcal{X}(1)_{|X}) \to \bigwedge^2 (T\mathcal{X}^\Delta_{|x}/J^\Delta_x)$,
and it suffices to show that β is surjective, to conclude that V cannot be a base point of $H^0(\bigwedge^2 T\mathcal{X}(1)_{|X})$.

Now we do the following: We can choose two coordinates X_i, X_j, which give independant coordinates on Δ; also, we may assume that not all coordinates $X_k, k \neq i, j$ vanish at x, because there are at least two nonvanishing coordinates at any $x \in X$. Let $A_\lambda := X_i - \lambda X_j$, for

$\lambda \in \mathbb{C}$ and let $P_\lambda := (X_i^{d-1} - \lambda^{d-1} X_j^{d-1})/(X_i - \lambda X_j) \in S^{d-2}$. Recall from 1.1.2, 1.1.6 the isomorphism:

2.8.3. $H^0(T\mathcal{X}(1)_{|X}) \cong \operatorname{Ker}(\mu : S^d \otimes S^1 \to R^{d+1})$;
it follows that for any $T \in S^2$:

2.8.4. $TP_\lambda \otimes A_\lambda \in H^0(T\mathcal{X}(1)_{|X})$, since

$$TP_\lambda \cdot A_\lambda = T(X_i^{d-1} - \lambda^{d-1} X_j^{d-1}) \in J(F).$$

Now we have:

2.8.5. *$TP_\lambda \otimes A_\lambda \wedge SP_\lambda \otimes A_\lambda \in H^0(\bigwedge^2 T\mathcal{X}(2)_{|X})$ vanishes on $\{A_\lambda = 0\}$ for any $T, S \in S^2$.*

To see this, note that along $\{A_\lambda = 0\}, TP_\lambda \otimes A_\lambda$ gives a vertical vector, that is an element of $TX \subset T\mathcal{X}$, since in the exact sequence:

2.8.6. $0 \to TX_{|y} \to T\mathcal{X}_{|y} \xrightarrow{\pi} S^d \to 0$,
one has $\pi(TP_\lambda \otimes A_\lambda) = TP_\lambda \cdot A_\lambda(y)$, which vanishes when $A_\lambda(y) = 0$. This vertical vector is easy to compute, retracing through the construction of the isomorphism: $H^0(T\mathcal{X}(1)_{|X}) \cong \operatorname{Ker}(\mu)$; in fact we have $TP_\lambda \cdot A_\lambda = T(X_i^{d-1} - \lambda^{d-1} X_j^{d-1})$ in S^{d+1}, and this is equal to

$$(1/d)T(\partial F/\partial X_i - \lambda^{d-1} \partial F/\partial X_j).$$

Then we have the following:

2.8.7. *For $A_\lambda(y) = 0$, one has*

$$\begin{aligned}(TP_\lambda \otimes A_\lambda)_y &= (1/d)T(y)(\partial/\partial X_i - \lambda^{d-1}\partial/\partial X_j)\\ &\in TX(1)_{|y} \subset T\mathbb{P}^n(1)_{|y}.\end{aligned}$$

So clearly $TP_\lambda \otimes A_\lambda$ and $SP_\lambda \otimes A_\lambda$ are proportional along $\{A_\lambda = 0\}$, which proves 2.8.5.

It follows that, after dividing by A_λ, we get a section $(TP_\lambda \otimes A_\lambda \wedge SP_\lambda \otimes A_\lambda)/A_\lambda$ of $\bigwedge^2 T\mathcal{X}(1)_{|X}$. Clearly, if $W \subset T\mathcal{X}_{|x}$ is the subspace generated by the $TP_\lambda \otimes A_\lambda$, when T and λ vary, the sections $(TP_\lambda \otimes A_\lambda \wedge SP_\lambda \otimes A_\lambda)/A_\lambda$ generate the subspace $\bigwedge^2 W(1) \subset \bigwedge^2 T\mathcal{X}(1)_{|x}$ since for generic $\lambda, A_\lambda(x) \neq 0$ (we have assumed that X_i, X_j are independant on Δ).

So, to show that β (2.8.2) is surjective, it suffices to show:

2.8.8. *The composite map: $W \hookrightarrow T\mathcal{X}_{|x} \to T\mathcal{X}_{|x}^\Delta / J_x^\Delta$ is surjective*, or equivalently:

2.8.9. $W_\Delta + J_x^\Delta = T\mathcal{X}_{|x}^\Delta$, *where W_Δ is the projection of W in $T\mathcal{X}_{|x}^\Delta$.*

But $W(1)$, viewed as a subspace of $T\mathbb{P}^n(1)_{|x} \oplus S^d \otimes \mathcal{O}_x(1)$ is generated by the elements $(-(1/d)T(x)(\partial/\partial X_i - \lambda^{d-1}\partial/\partial X_j, TP_\lambda \cdot A_\lambda(x))$, for $\lambda \in \mathbb{C}, T \in S^2$, with $P_\lambda := (X_i^{d-1} - \lambda^{d-1} X_j^{d-1})/(X_i - \lambda X_j)$. Clearly, when λ, T move, the restrictions to Δ of the elements $TP_\lambda \cdot A_\lambda(x))$ generate

$H^0(\mathcal{O}_\Delta(d))$, since X_i, X_j are independant on Δ. Finally the kernel of the projection $W_\Delta \to H^0(\mathcal{O}_\Delta(d))$ is generated by the vertical vector $(1/d)T(x)(\partial/\partial X_i - \lambda^{d-1}\partial/\partial X_j)$ for $T(x) \neq 0$ and $A_\lambda(x) = 0$. It follows that, as a subspace of $T\mathbb{P}^n(1)_{|x} \oplus H^0(\mathcal{O}_\Delta(d)) \otimes \mathcal{O}_x(1)$, W_Δ is equal to:

2.8.10. $\{(u, g),\ u \in < \partial/\partial X_i, \partial/\partial X_j > \otimes \mathcal{O}_x(2)/\ dF(u) + g(x) = 0\}$.

So W_Δ is of codimension $n-2$ in $T\mathcal{X}(1)_{|x}$, since $\partial/\partial X_i$, $\partial/\partial X_j$ are independant in $T\mathbb{P}^n(-1)_{|x}$. To prove that $W_\Delta + J_x^\Delta = T\mathcal{X}_{|x}^\Delta$, it suffices to verify that $J_x^\Delta \cap W_\Delta$ is of codimension $n-2$ in J_x^Δ.

But by 2.8.1 and 2.8.10, we find:

2.8.11.

$$J_x^\Delta \cap W_\Delta = \{(u(x), -_{\tilde{u}}F)/u(x) \in < \partial/\partial X_i, \partial/\partial X_j > \otimes \mathcal{O}_x(2)\},$$

where the equality holds in $T\mathcal{X}(1)_{|x} \subset T\mathbb{P}^n(1)_{|x} \oplus H^0(\mathcal{O}_\Delta(d)) \otimes \mathcal{O}_x(1)$, and this is clearly of codimension $n-2$ in J_x^Δ, since the projection $J_x^\Delta \to T\mathbb{P}^n_{|x}$ is surjective, and $\partial/\partial X_i$, $\partial/\partial X_j$ are independant in $T\mathbb{P}^n(-1)_{|x}$ (this follows from the assumption that not all coordinates X_k, $k \neq i, j$ vanish at x). So the proof of Proposition 2.2 is finished.

2.9. Although it should be clear from the reasoning in the proof of Theorem 1.4, we repeat the argument which gives the next result:

2.10 Theorem. *Let $d \geq 2n - l - 1$, $1 \leq l \leq n-3$; then for $X \subset \mathbb{P}^n$ general of degree d and $Y \subset X$ a subvariety of dimension l, $K_{\tilde{Y}}$ is effective, where $\tilde{Y}$ is any desingularization of Y. If the inequality is strict, the canonical map of $\tilde{Y}$ is of degree one on its image.*

Proof. It suffices to show that for any étale map $\mathcal{M} \to (S^d)^0$, and for any $GL(n+1)$-invariant subvariety $\mathcal{Y} \subset \mathcal{X}_\mathcal{M}$ dominating $\mathcal{M}$, with generic fiber dimension l, if $\tilde{\mathcal{Y}}$ is a desingularization of $\mathcal{Y}$, $H^0(K_{\tilde{\mathcal{Y}}|\tilde{Y}_t}) \neq 0$, (resp. $H^0(K_{\tilde{\mathcal{Y}}|\tilde{Y}_t})$ separates the points of an open set of $\tilde{Y}_t$ when the inequality is strict), for t generic in $\mathcal{M}$.

But for t generic in $\mathcal{M}$ and y generic in Y_t, $\mathcal{Y}$ is smooth at y and $T\mathcal{Y}_{|y} \subset T\mathcal{X}_{\mathcal{M}|y}$ is a space of codimension $n-1-l$, invariant under $GL(n+1)$. Now we have by Proposition 1.1 that $T\mathcal{X}_\mathcal{M}(1)_{|X_t}$ is generated by global sections, and by Proposition 2.2 that $H^0(\bigwedge^2 T\mathcal{X}_\mathcal{M}(1)_{|X_t})$ has no base point on the set of $GL(n+1)$-invariant codimension two subspaces of $T\mathcal{X}_\mathcal{M}(1)_{|X_t}$ for t generic in $\mathcal{M}$. Let y be generic in Y_t as above and let $\sigma_{l+1}, \ldots, \sigma_{n-3}$ be sections of $T\mathcal{X}_\mathcal{M}(1)_{|X_t}$, such that $< T\mathcal{Y}_{|y}, (\sigma_i)_{1=l,\ldots,n-3} >$ is a codimension two $GL(n+1)$-invariant subspace V of $T\mathcal{X}_\mathcal{M}(1)_{|y}$; there exists $\omega \in H^0(\bigwedge^2 T\mathcal{X}_\mathcal{M}(1)_{|X_t})$ which does not vanish on V; now

$$\omega(V) = \omega \wedge \sigma_l \wedge \ldots \wedge \sigma_{n-3}(T\mathcal{Y}_{|y}),$$

and $\omega \wedge \sigma_l \wedge \ldots \wedge \sigma_{n-3}$ is a section of

$$\bigwedge^{n-1-l} T\mathcal{X}_{\mathcal{M}}(n-2-l)_{|X_t}) \cong \Omega^{N+l}_{\mathcal{X}_{\mathcal{M}}\,|X_t}(n-2-l-K_{X_t}).$$

So if $K_{X_t} \geq \mathcal{O}_{X_t}(n-2-l)$, that is, when $d \geq 2n-l-1$, there is a section of $\Omega^{N+l}_{\mathcal{X}_{\mathcal{M}}\,|X_t}$ which does not vanish in $\Omega^{N+l}_{\tilde{\mathcal{Y}}\;|\tilde{Y}_t} \cong K_{\tilde{\mathcal{Y}}|\tilde{Y}_t}$. Similarly, if the inequality is strict, there is a section of $\Omega^{N+l}_{\mathcal{X}_{\mathcal{M}}\,|X_t}(-1)$ which does not vanish in $\Omega^{N+l}_{\tilde{\mathcal{Y}}\;|\tilde{Y}_t}(-1) \cong K_{\tilde{\mathcal{Y}}|\tilde{Y}_t}(-1)$; hence the canonical map of $\tilde{Y}_t$ is of degree one on its image in this case.

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