J. DIFFERENTIAL GEOMETRY
38 (1993) 417-461

CONVERGENCE OF THE ALLEN-CAHN EQUATION TO BRAKKE'S MOTION BY MEAN CURVATURE

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Abstract

The equation $\partial u^\varepsilon/\partial t = \Delta u^\varepsilon - (1/\varepsilon^2)f(u^\varepsilon)$ was introduced by Allen and Cahn to model the evolution of phase boundaries driven by isotropic surface tension. Here $f = F'$ and F is a potential with two equal wells. We prove that the measures $d\mu_t^\varepsilon \equiv ((\varepsilon/2)|Du^\varepsilon|^2 + (1/\varepsilon)F(u^\varepsilon))\,dx$ converge to Brakke's motion of varifolds by mean curvature. In consequence, the limiting interface is a closed set of finite $\mathscr{H}^{n-1}$-measure for each $t \geq 0$ and of finite $\mathscr{H}^n$-measure in spacetime. In particular the limiting interface is a "thin" subset of the level-set flow (which can fatten up) and satisfies the maximum principle when tested against smooth, disjoint surfaces moving by mean curvature. The main tools are Huisken's monotonicity formula, Evans-Spruck's lower density bound and equipartition of energy. In addition, drawing on Brakke's regularity theory, there is almost-everywhere regularity for generic (i.e., nonfattening) initial condition.

Introduction

The equation

$$(*)\qquad \frac{\partial}{\partial t}u^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2}f(u^\varepsilon)$$

was introduced by Allen and Cahn in 1979 to model the motion of phase boundaries by surface tension [2]. Here f is the derivative of a potential F with two wells of equal depth at $u = \pm 1$. The equation is the gradient flow of

$$M^\varepsilon[u] = \int \frac{\varepsilon}{2}|Du|^2 + \frac{1}{\varepsilon}F(u)\,dx,$$

sped up by the factor $1/\varepsilon$.

The effect of $-(1/\varepsilon)^2 f$ is to force u^ε to approximate a characteristic function, with a transition layer of width approximately ε and slope approximately C/ε. Heuristically, the interface should move by mean curvature in the limit.

Received April 14, 1992.

The energy density is

$$d\mu_t^\varepsilon \equiv \left(\frac{\varepsilon}{2}|Du^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon)\right)\, dx.$$

Since μ_t^ε is the quantity whose integral decreases, it must represent surface measure in the limit. Accordingly, we try to pass to limits in the sense of Brakke's varifolds moving by mean curvature. We succeed by combining Brakke's convergence method [5, Chapter 4], an ε-version of Huisken's monotonicity formula [23], and the lower density bound of Evans and Spruck [18, 6.2].

The key step is to prove that the quantities $(\varepsilon/2)|Du^\varepsilon|^2\, dx$ and $(1/\varepsilon)F(u^\varepsilon)\, dx$ become equal in the limit. This is to be expected from the asymptotic form $u^\varepsilon \approx q^\varepsilon(r)$, where q^ε is the one-dimensional standing wave for $(*)$ and r is the signed distance to the front. This result is a kind of "equipartition of energy" for the parabolic problem.

Evans, Soner, and Souganidis [15] have proved that u^ε converges locally uniformly to ± 1 except possibly on the corresponding level-set flow. (For an explanation of the motion of level-sets by mean curvature, see Evans and Spruck [16] or Chen, Giga, and Goto [13].) Bronsard and Kohn [7] proved that

$$u^{\varepsilon_i}(\cdot, t) \to \pm 1 \ \mathscr{L}^n\text{-a.e.} \quad \text{for each } t \geq 0,$$
$$u^\varepsilon(\cdot, \cdot) \to \pm 1 \ \mathscr{L}^{n+1}\text{-a.e.}$$

Thus our results are particularly of interest in the case that the corresponding level-set flow fattens up. In particular, we show that for each $t \geq 0$,

$$u^{\varepsilon_i}(\cdot, t) \to \pm 1 \quad \text{locally uniformly}$$

except on a closed set of $\mathscr{H}^{n-1}$-measure zero, and

$$u^{\varepsilon_i}(\cdot, \cdot) \to \pm 1 \quad \text{locally uniformly}$$

except on a closed set of $\mathscr{H}^n$-measure zero. This closed set is the support of the limiting measure, and lies within the corresponding level-set flow. See Corollary 9.2. It is a "set-theoretic subsolution" of motion by mean curvature as defined in [25], with the distance function property of Soner [34].

Furthermore, our existence theory is also suitable for the regularity theory of Brakke [5] as exploited in Ilmanen [26].

Broader context, including penalized harmonic map problem. This paper addresses questions raised by Bronsard and Kohn [7] and by De Giorgi in his white paper [14]. Particularly interesting is the relation between the

ε-$W^{1,2}$ quantity $(\varepsilon|Du^\varepsilon|^2/2+F(u^\varepsilon)/\varepsilon)\,dx$ and the limiting BV quantity $|Du|\,dx$ where the limit u is the characteristic function of an open set E of finite perimeter.

The paper extends the result of Modica [28] for the elliptic minimization problem to the parabolic (and stationary elliptic) problem. See also Modica and Mortola for the connection with Γ-convergence [29], and Frölich-Struwe [21] for further discussion.

Equation $(*)$ specializes a more general problem where u is vector-valued and F takes its minimum on a submanifold N of $\mathbf{R}^k$. If u^ε takes values near one component of N, then the energy $\int(1/2)|Du|^2+(1/\varepsilon^2)F(u)\,dx$ is asymptotically finite and in the limit we solve the harmonic map flow problem with target N. This was initially proven in Chen [11] for target S^n. Using the monotonicity formula and regularity theory of Struwe [36], the approximation scheme was applied to general targets in Chen-Struwe [12].

If u^ε bridges from one component of N to another, then we must normalize the energy by a factor ε. Each component of N corresponds to a phase of matter. In the limit, phase boundaries should develop that move by mean curvature. In this way it should be possible to model the motion of multiple phases.

In particular, ordinary (two-phase) motion by mean curvature can be seen as harmonic map flow into $S^0=\{-1,+1\}$ with infinite, but normalized energy.

The Allen-Cahn equation has been studied by Barles, Soner and Souganidis [4], Bronsard and Kohn [6], [7], Carr and Pego [8], [9], X. Chen [10], Fife [20], Fusco [22], Rubinstein, Sternberg, and Keller [32], and many others. In particular, de Mottoni and Schatzman established agreement with smooth motion [30], [31], and Evans, Soner, and Souganidis proved compatibility with the level-set flow [15]. Our result implies these earlier results.

Organization. Our first step (§2) is to establish an ε-version of Brakke's inequality

$$\frac{d}{dt}\int \phi\,d\mu_t \le \int -H^2\phi+\vec{H}\cdot S\cdot D\phi\,d\mu_t.$$

(In this formula, S represents orthogonal projection onto the tangent plane of the surface.) However, the resulting ε-version of the formula contains a discrepancy involving the *difference*

$$d\xi_t^\varepsilon \equiv \left(\frac{\varepsilon|Du|^2}{2}-\frac{F(u^\varepsilon)}{\varepsilon}\right)dx.$$

In §3 we derive an ε-version of Huisken's monotonicity formula, which also contains an ξ_t^ε discrepancy. We prove in §4 that $\xi_t^\varepsilon \le 0$, thus establishing the monotonicity formula. This is equivalent to the condition $|Dr^\varepsilon| \le 1$, where r^ε is the ε-approximation to the signed distance to the front.

Next (§5) we prove that the measures $\mu_t^{\varepsilon_i}$ converge subsequentially to a limit μ_t for all $t \ge 0$ at once.

In §6, we establish a version of Brakke's Clearing-Out Lemma [5, 6.3]. This immediately yields local estimates on $\mathscr{H}^{n-1}(\operatorname{spt}\mu_t)$. Using the density argument of Evans and Spruck [18, 6.2], in §7 we get a lower bound on the $(n-1)$-dimensional density of μ_t for $\mathscr{H}^{n-2+\delta}$-a.e. $x \in \operatorname{spt}\mu_t$ and a.e. $t \ge 0$.

Combining the density lower bound with the (negative) ξ_t term in the monotonicity formula, we show $\xi = 0$ in the limit (§8) by a "squeezing" argument. This is what we need to pass to limits in the sense of varifolds (§9) and establish Brakke's inequality for $\{\mu_t\}_{t\ge0}$. As a consequence, $\operatorname{spt}\mu_t$ is $(n-1)$-rectifiable for a.e. $t \ge 0$.

In §10, we estimate $\mathscr{H}^n \lfloor \bigcup_{t\ge0} \operatorname{spt}\mu_t \times \{t\}$ locally.

In §11, we relate the limit measures $\{\mu_t\}_{t\ge0}$ to the characteristic function obtained in Bronsard and Kohn [7] by BV-compactness.

The Allen-Cahn limit yields effectively the same structure that arose as the limit of elliptic regularization in the author's paper [26]. In §12 we briefly indicate how the results of [26] can be applied in the case of the Allen-Cahn equation to show that any initial surface can be perturbed to one whose evolution is smooth $\mathscr{H}^n$-a.e. in space-time.

It seems that approximation by the Allen-Cahn equation yields essentially the full range of Brakke motions (of boundaries), without selecting

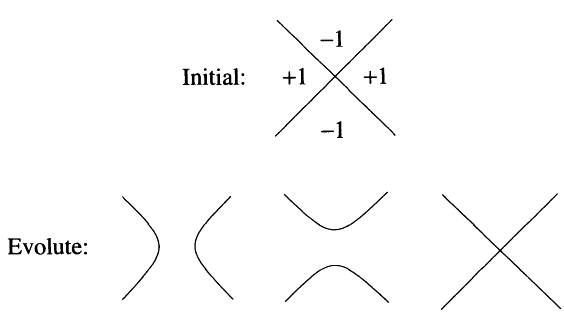

FIGURE 1. THREE CROSS FLOWS

"physically stable" ones. For example, there are three flows for the infinite cross shown in the diagram. The third is extremely unstable under small perturbations, but all three arise as possible limits of the Allen-Cahn equation with suitable initial data. See Figure 1.

1. Preliminaries

In §§1.1–1.4 we set forth the notation and setting of our convergence theorem in as much generality as possible. In §§1.6–1.9 we will define Brakke's varifolds moving by mean curvature (needed in §§2 and 9).

1.1. Equation and measures. Let u^ε be the unique smooth solution of

$$(*)\qquad \begin{aligned} \frac{\partial}{\partial t}u^\varepsilon &= \Delta u^\varepsilon - \frac{1}{\varepsilon^2}f(u^\varepsilon) \quad \text{on } \mathbf{R}^n \times [0, \infty), \\ u^\varepsilon(\cdot, 0) &= u_0^\varepsilon(\cdot) \quad \text{on } \mathbf{R}^n \times \{0\}, \end{aligned}$$

where u_0^ε is as described in §1.4. The potential F is to satisfy

$$f = F', \qquad F = \tfrac{1}{2}g^2,$$

where

$$(1)\qquad \begin{gathered} f(-1) = f(0) = f(1) = 0, \\ f > 0 \quad \text{on } (-1, 0), \qquad f < 0 \quad \text{on } (0, 1), \\ f'(-1), \quad f'(1) > 0, \quad f'(0) < 0, \\ g(-1) = g(1) = 0, \qquad g > 0 \quad \text{on } (-1, 1). \end{gathered}$$

This hypothesis allows the application of X. Chen's result [10] in Lemma 6.1. It can be weakened to a local condition near ± 1.

The model is

$$F = \tfrac{1}{2}(1 - u^2)^2, \quad f = 2u(u^2 - 1), \quad g = 1 - u^2.$$

Measures. Define the Radon measures μ_t^ε, $t \ge 0$, by

$$(2)\qquad d\mu_t^\varepsilon \equiv \left(\frac{\varepsilon}{2}|Du^\varepsilon(\cdot, t)|^2 + \frac{1}{\varepsilon}F(u^\varepsilon(\cdot, t))\right) dx.$$

1.2. Standing wave. Let q^ε be the one-dimensional standing wave for $(*)$, that is

$$(1)\qquad \begin{gathered} q_{rr}^\varepsilon(r) - \frac{1}{\varepsilon^2}f(q(r)) = 0, \qquad r \in \mathbf{R}, \\ q_r^\varepsilon > 0, \quad q^\varepsilon(\pm\infty) = \pm 1, \quad q^\varepsilon(0) = 0, \end{gathered}$$

(see, e.g., [7]), where $(\ )_r$ represents differentiation with respect to r. We can obtain q^ε by solving the first-order equation

$$(2)\qquad q_r^\varepsilon = \frac{1}{\varepsilon}g(q^\varepsilon), \qquad q^\varepsilon(\pm\infty) = \pm 1, \; q^\varepsilon(0) = 0.$$

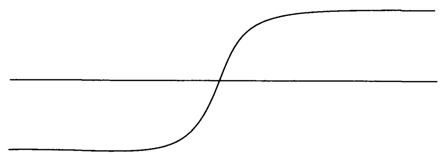

FIGURE 2. STANDING WAVE

To see that (2) implies (1), differentiate and substitute (2) and $f = gg_q$. To ensure that the boundary conditions of (2) are attainable, use the fact that F has two equal wells. See Figure 2.

Note that for any q with $q(\pm\infty) = \pm 1$, we have

$$\begin{aligned} M^\varepsilon[q] &= \int_{-\infty}^{\infty} \frac{\varepsilon}{2} q^2 + \frac{1}{\varepsilon} F(q)\, dr \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\left(\sqrt{\varepsilon} q_r - \frac{1}{\sqrt{\varepsilon}} g(q)\right)^2 + G(q)_r\, dr \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\left(\sqrt{\varepsilon} q_r - \frac{1}{\sqrt{\varepsilon}} g(q)\right)^2 dr + G(1) - G(-1), \end{aligned}$$

where

$$G' = g, \qquad G(-1) + G(1) = 0.$$

Thus q^ε solving (2) is in fact an absolute minimizer subject to its own boundary conditions. Note that $q^\varepsilon(r) = q^1(r/\varepsilon)$. Define

$$\alpha = M^\varepsilon[q^\varepsilon] = (G(1) - G(-1). \tag{3}$$

For the model case $F = (1-u^2)^2/2$, we have

$$q^\varepsilon = \tanh(r/\varepsilon), \quad G = u - u^3/3, \quad \alpha = 4/3.$$

1.3. Initial surface. Since Brakke motions are so general we expect the motion can sustain very rough initial conditions. The most general initial condition is an integral current that is the boundary of a set of finite perimeter, with bounded density ratios. In concrete terms we can start the evolution from any (singular) surface that can be approximated by smooth surfaces. Specifically, let $E_0 \subseteq \mathbf{R}^n$ be an open set, $M_0 = \partial E_0$, and assume

(i) *Density bounds.*

$$\frac{\mathscr{H}^{n-1}(M_0 \cap B_R(x))}{\omega_{n-1} R^{n-1}} \le D \quad \text{for } x \in \mathbf{R}^n, \ R > 0.$$

(ii) *Approximability.* (E_0, M_0) can be approximated strongly by smooth pairs in the sense that there exist pairs $\{(E_0^k, M_0^k)\}_{k\geq 1}$ with E_0^k open, M_0^k a smooth hypersurface, and

$$\chi_{E_0^k} \rightharpoonup \chi_{E_0} \text{ weakly-}* \quad \text{in } BV_{\text{loc}},$$
$$\mathscr{H}^{n-1}\lfloor M_0^k \to \mathscr{H}^{n-1}\lfloor M_0 \quad \text{as Radon measures}.$$

These conditions allow a variety of singularities in M_0. For example, it permits the cross of Figure 1.

1.4. Initial data $\mathbf{u}_0^\varepsilon$. We next show how to make E_0 into suitable initial conditions for $(*)$. Let $\chi_{E_0}: \mathbf{R}^n \to \{0, 1\}$ be the characteristic function of E_0. In order to use the Allen-Cahn equation $(*)$, we must approximate $2\chi_{E_0} - 1$ by smooth functions $u_0^\varepsilon: \mathbf{R}^n \to [-1, +1]$ such that μ_0^ε approximates $\mathscr{H}^{n-1}\lfloor M_0$. To this end we set $u_0^\varepsilon \approx q^\varepsilon(r_0)$, where r_0 is the signed distance to M_0. Specifically we require

(i) $\varepsilon|Du_0^\varepsilon|^2/2 \leq F(u_0^\varepsilon)/\varepsilon$ (initial control of Du),
(ii) $\mu_0^\varepsilon \to \alpha\mathscr{H}^{n-1}\lfloor M_0$ as Radon measures,
(iii) $u_0^\varepsilon \rightharpoonup 2\chi_{E_0} - 1$ in BV_{loc},
(iv) $\mu_0^\varepsilon(B_r(x))/\omega_{n-1}r^{n-1} \leq CD$ for $x \in \mathbf{R}^n$, $r > 0$ (density bounds),
(v) $\|u_0^\varepsilon\|_{C^2} \leq C(\varepsilon)$.

The technical condition (i) is essential in the argument of §4 and what follows, but probably can be weakened or removed by showing it quickly becomes nearly true. (M. Soner has recently done this [35].)

Proof that this is possible. Let us check that such u_0^ε can be constructed for E_0. Our construction is standard, see, e.g., Modica [28, 2(2)]. First approximate E_0 by E_0^k as in §1.3(ii). We may assume M_0^k is compact by modifying E_0^k off a large ball. Define

$$r^k(x) = \begin{cases} \operatorname{dist}(x, M_0^k), & x \in E_0^k, \\ -\operatorname{dist}(x, M_0^k), & x \notin E_0^k. \end{cases}$$

Note that $|Dr^k| \leq 1$ a.e. and r^k is smooth near M_0^k. Let $\bar{r}^k$ be a smoothing of r^k that agrees with r^k near M_0^k, and which satisfies $|D\bar{r}^k| \leq 1$. Set

$$u_0^{\varepsilon,k}(x) \equiv q^\varepsilon(\bar{r}^k(x)).$$

Since E_0^k is compact we can assume that both (iv) and (v) hold for $u_0^{\varepsilon,k}$ when ε is small enough.

Condition (i) follows from §1.2(2) and $|D\bar{r}^k| \le 1$. Let us consider conditions (ii) and (ii). Fix $\phi \in C_c^0(\mathbf{R}^n)$. For small enough ε (depending on k) we have

$$\begin{aligned}\mu_0^\varepsilon(\phi) &= \int \phi(x)\left(\frac{\varepsilon}{2}|Du_0^\varepsilon|^2 + \frac{1}{\varepsilon}F(u_0^\varepsilon)\right)dx \\ &\approx \int_{M_0}\int_{-\infty}^{\infty}\phi(y,r)\left(\frac{\varepsilon}{2}(q_r^\varepsilon)^2 + \frac{1}{\varepsilon}F(q^\varepsilon)\right)dr\,d\mathscr{H}^{n-1}(y) \\ &\approx \int_{M_0^k}\alpha\phi(y,0)\,d\mathscr{H}^{n-1}(y), \qquad \alpha \text{ as in §1.2(3)}.\end{aligned}$$

Here we use coordinates $x = (y, r)$ near M_0^k, where $y \in M_0^k$ is the closest point to x and $r = r(x)$. Therefore

(ii′) $\mu_0^\varepsilon \approx \alpha\mathscr{H}^{n-1}\lfloor M_0^k$ as Radon measures.

Evidently we also have, by considering the shape of q^ε for small ε,

(iii′) $u_0^\varepsilon \approx 2\chi_{E_0} - 1$ weakly-$*$ in BV_{loc}.

Thus (ii) and (iii) hold with E_0 replaced by E_0^k. By taking a "diagonal" subsequence we see that (i)–(v) are met.

1.5. Crude bounds for $\mathbf{u}^\varepsilon$. For periodic solutions, the maximum principle implies

$$\begin{aligned}&|u^\varepsilon| \le 1, \quad |Du^\varepsilon|, |D^2u^\varepsilon| \le C(\varepsilon, T), \qquad x \in \mathbf{R}^n,\ 0 \le t \le T, \\ &\mu_t^\varepsilon(B_R(x)) \le C(\varepsilon, T)R^n, \qquad x \in \mathbf{R}^n,\ 0 \le t \le T,\ R > 0.\end{aligned}$$

These estimates are independent of the periodicity and hence hold by uniqueness for any smooth solution u^ε of $(*)$.

1.6. Varifold notation. For detailed information about varifolds the reader is referred to [1], [3], [33]. A *general k-varifold* is a Radon measure on $\mathbf{R}^n \times G_k(\mathbf{R}^n)$, where $G_k(\mathbf{R}^n)$ is the Grassman manifold of unoriented k-planes in $\mathbf{R}^n$. We write $\mathbf{V}_k(\mathbf{R}^n)$ for the set of all general k-varifolds.

When S is a k-plane, we also use S to denote the orthogonal projection $\mathbf{R}^n \to S$. We write $A : B$ for the inner product $\sum A_{ij}B_{ij}$ of matrices, and $A \cdot V$ for the application of a matrix to a vector. In this notation we write the first variation formula (see [33])

$$\begin{aligned}\delta V(X) &= \int DX(x) : S\,dV(x, S) \\ \text{(1)} \qquad &\equiv \int \sum_{e_i \| S}\langle D_{e_i}X(x), e_i\rangle\,dV(x, S) \\ &= \int -X(x)\cdot\vec{H}(x)\,d\|V\|(x) \quad \text{if } |\delta V| \ll \|V\|.\end{aligned}$$

Here V is a general varifold, δV is the first variation, $\|V\|$ is the mass measure, $X \in C_c^1(\mathbf{R}^n, \mathbf{R}^n)$, $\{e_1, \cdots, e_k\}$ is an orthonormal basis of S, and $\vec{H} = \vec{H}_V$ is the mean curvature vector if it exists. The quantity $DX : S$ is also written $\operatorname{div}_V X$.

1.7. Rectifiable Radon measures. We call a Radon measure μ *k-rectifiable,* and write $\mu \in \mathscr{M}_k$, if either of the following equivalent conditions is met:

(a) $\mu = \mathscr{H}^k \lfloor X \lfloor \theta$, where X is a locally k-rectifiable, $\mathscr{H}^k$-measurable set, and $\theta \in L^1_{\text{loc}}(\mathscr{H}^k \lfloor X, (0, \infty))$.

(b) The measure-theoretic tangent plane $T_x\mu$ exists μ-a.e., where we define $T_x\mu$ by

$$T_x\mu \equiv \lim_{\lambda \downarrow 0} \mu_{x,\lambda} \quad \text{(in the sense of Radon measures)}$$

provided the limit exists and is a positive multiple of $\mathscr{H}^k$ restricted to some k-plane. Here $\mu_{x,\lambda}(A) \equiv \lambda^{-k}\mu(x + \lambda A)$, for $A \subseteq \mathbf{R}^n$.

In this circumstance there is a corresponding *rectifiable k-varifold* $V = V_\mu$ defined by

$$\int \psi(x, S)\, dV(x, S) = \int \psi(x, T_x\mu)\, d\mu(x) \quad \text{for } \psi \in C_c^0(\mathbf{R}^n \times G_k\mathbf{R}^n, \mathbf{R}).$$

Note that $\mu = \|V\|$. The set of all rectifiable k-varifolds is denoted $\mathbf{RV}_k(\mathbf{R}^n)$, and is by definition in one-to-one correspondence with $\mathscr{M}_k$. We write $\vec{H} = \vec{H}_\mu = \vec{H}_V$.

1.8. Brakke's right-hand side. If $\{M_t\}_{t \geq 0}$ is moving smoothly by mean curvature, we have the following identity for any test function $\phi = \phi(x)$:

$$\frac{d}{dt} \int_{M_t} \phi \, d\mathscr{H}^{n-1} = \int_{M_t} -\phi H^2 + D\phi \cdot \vec{H}\, d\mathscr{H}^{n-1}.$$

We call the first term the *shrinkage* term and the second one the *transport* term. Motivated by this, we define $\mathscr{B}(\mu, \phi)$ for any Radon measure μ as follows.

Singular case. $\mathscr{B}(\mu, \phi) \equiv -\infty$ if either of the following holds:

(i) $\mu\lfloor\{\phi > 0\} \notin \mathscr{M}_k$.

(ii) $|\delta\mu| \lfloor \{\phi < 0\} \not\ll \mu_t\lfloor\{\phi > 0\}$.

(iii) $\int \phi H^2\, d\mu = \infty$.

Nonsingular case. Otherwise,

$$\mathscr{B}(\mu, \phi) \equiv \int -\phi H^2 + D\phi \cdot T_x\mu \cdot \vec{H}\, d\mu.$$

1.9. Brakke motions. A family $\{\mu_t\}_{t\geq 0}$ of Radon measures is called a *Brakke motion* provided

$$\overline{D}_t \mu_t(\phi) \leq \mathscr{B}(\mu_t, \phi) \tag{B}$$

for all $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$ and all $t \geq 0$. Here $\mu_t(\phi) = \int \phi \, d\mu_t$ and $\overline{D}_t f(t)$ is the upper derivate

$$\overline{\lim_{s\to t}}(f(s) - f(t))/(s-t).$$

The inequality (rather than equality) is required for technical reasons in §9. It is a true feature of the flow, as illustrated in [26, §6].

Brakke introduced inequality (B) in his monumental 1978 book [5]. Our definition is slightly stronger than Brakke's since we only look at properties of the mass measure $\mu_t = \|V_t\|$.

2. Brakke's inequality

We wish to derive an ε-version of Brakke's inequality §1.9(b). Let $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$ and derive

$$\begin{aligned}
\frac{\partial}{\partial t}\int \phi \, d\mu_t^\varepsilon &= \int \phi \frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}|Du|^2 + \frac{1}{\varepsilon}F(u)\right) dx \\
&= \int \phi\left(\varepsilon Du \cdot D\frac{\partial}{\partial t}u + \frac{1}{\varepsilon}f(u)\frac{\partial}{\partial t}u\right) dx \\
&= \int \varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)\frac{\partial}{\partial t}u - \varepsilon D\phi \cdot Du \frac{\partial}{\partial t}u \, dx \qquad (1)\\
&= \int -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)^2 \\
&\qquad + \varepsilon D\phi \cdot Du\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right) dx.
\end{aligned}$$

In comparison with (B) this suggests that

$$\begin{aligned}
H^2 \, d\mu_t &\sim \varepsilon\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)^2 dx, \\
\vec{H} \, d\mu_t &\sim \varepsilon Du\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right) dx.
\end{aligned}$$

Recalling $d\mu_t^\varepsilon = ((\varepsilon/2)|Du|^2 + (1/\varepsilon)F(u))\,dx$, we obtain two approximate expressions for H:

$$H \approx \frac{-\Delta u + f(u)/\varepsilon^2}{|Du|} \approx \frac{\varepsilon|Du|(-\Delta u + f(u)/\varepsilon^2)}{(\varepsilon/2)|Du|^2 + (1/\varepsilon)F(u)}.$$

If only we had $(\varepsilon/2)|Du|^2 \approx (1/\varepsilon)F(u)$, these expressions would agree (heuristically).

We next try to interpret μ_t^ε as a varifold. Integrating by parts, we will make the transport term of (1) look like the first variation $-\int S\colon DY(x)\, dV(x, S)$ of the varifold V, where $Y \equiv D\phi$. Let δ denote the identity matrix of $\mathbf{R}^n$.

Define the *stress tensor* T_{ij} by

$$T = \frac{\varepsilon}{2}|Du|^2\delta - \varepsilon Du \otimes Du + \frac{1}{\varepsilon}F(u)\delta .$$

Observe that

$$D_i T_{ij} = \varepsilon D_j u(-\Delta u + f(u)/\varepsilon^2).$$

Then

$$\begin{aligned} \text{Transport term} &= \int \varepsilon\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right) Du \cdot Y\, dx \\ &= \int D_i T_{ij} Y^j\, dx = \int -T\colon DY\, dx \\ &= \int \left(\varepsilon Du \otimes Du - \frac{\varepsilon}{2}|Du|^2\delta - \frac{1}{\varepsilon}F(u)\delta\right) : DY\, dx \\ &= \int -\left(\frac{\varepsilon}{2}|Du|^2 + \frac{1}{\varepsilon}F(u)\right)(\delta - \nu\otimes\nu)\colon DY\, dx \\ &\quad + \int \left(-\frac{1}{\varepsilon}F(u) + \frac{\varepsilon}{2}|Du|^2\right)\nu\otimes\nu\colon DY\, dx \\ &= \int -(\delta - \nu\otimes\nu)\colon DY\, d\mu_t^\varepsilon + \int \nu\otimes\nu\colon DY\, d\xi_t^\varepsilon , \end{aligned} \tag{2}$$

where $\nu = Du/|Du|$, and we define the "discrepancy" Radon measure

$$d\xi_t^\varepsilon \equiv \left(\frac{\varepsilon}{2}|Du|^2 - \frac{1}{\varepsilon}F(u)\right) dx. \tag{3}$$

We are led to define the general varifold

$$V_t^\varepsilon(\psi) \equiv \int \psi(x, Du(x)^\perp)\, d\mu_t^\varepsilon(x), \qquad \psi \in C_c^0(\mathbf{R}^n \times G_{n-1}\mathbf{R}^n, \mathbf{R}). \tag{4}$$

This makes sense since by analyticity the set $\{x\colon Du(x, t) = 0\}$ is either all of $\mathbf{R}^n$ or has Lebesgue measure zero; we exclude the first case. Thus we obtain

$$\begin{aligned} \text{Transport term} &= -\int S\colon DY(x)\, dV_t^\varepsilon(x, S) - \int \nu\otimes\nu\colon DY\, d\xi_t^\varepsilon \\ &= -\delta V_t^\varepsilon(Y) - \int \nu\otimes\nu\colon DY\, d\xi_t^\varepsilon , \qquad Y \equiv D\phi , \end{aligned}$$

and therefore the ε-Brakke formula

$$\frac{d}{dt}\int \phi\, d\mu_t^\varepsilon = \int -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)^2 dx \\ - S\colon D^2\phi\, dV_t^\varepsilon - \nu\otimes\nu\colon D^2\phi\, d\xi_t^\varepsilon. \tag{5}$$

Note that the second term on the right corresponds to the transport term of (B) and is weakly continuous with respect to varifold convergence. The last term is the discrepancy between the transport term of (B) and the transport term of (2). (It cannot be absorbed in the second term without creating negative tangent planes.) Thus a major goal will be to show that $\xi^\varepsilon \to 0$ in the limit. Then the second term (which is a varifold) will match the transport term of (2) (which can be estimated via the equation). This will yield a first variation estimate on the limit varifold.

The vanishing of ξ is heuristically reasonable since we expect $u^\varepsilon \approx q^\varepsilon(r)$ where r is the signed distance to the front, and thus

$$\frac{\varepsilon}{2}|Du|^2 \approx \frac{\varepsilon}{2}(q_r^\varepsilon)^2|Dr|^2 = \frac{1}{\varepsilon}F(u^\varepsilon) \tag{6}$$

by §§1.1, 1.2(2), and the fact that $|Dr| = 1$ a.e.

3. Huisken's monotonicity

Our next task is to derive the ε-version of Huisken's monotonicity formula [24].

3.1. Smooth derivation. Let $\{M_t\}_{t\geq 0}$ be compact manifolds smoothly moving by mean curvature, let $\mu_t = \mathscr{H}^{n-1}\lfloor M_t$, and let $\nu = \nu(x)$ be a unit normal field on M_t. For any smooth function $\phi(x, t)$ we have

$$\begin{aligned}\frac{d}{dt}\int \phi\, d\mu_t &= \int -\phi H^2 + D\phi\cdot\vec{H} + \frac{\partial}{\partial t}\phi\, d\mu_t\\ &= \int -\phi H^2 + 2D\phi\cdot\vec{H} + S\colon D^2\phi + \frac{\partial}{\partial t}\phi\, d\mu_t \quad \text{by §1.6(1)}\\ &= \int -\phi\left(H + \frac{D\phi\cdot\nu}{\phi}\right)^2 + \frac{(D\phi\cdot\nu)^2}{\phi} + S\colon D^2\phi + \frac{\partial}{\partial t}\phi\, d\mu_t.\end{aligned}$$

Now fix a "blowup point" $(y, s)\in\mathbf{R}^n\times(0,\infty)$ and replace ϕ by *Huisken's monotonicity kernel*

$$\rho = \rho_{y,s}(x, t) \equiv \frac{1}{(4\pi(s-t))^{(n-1)/2}}e^{-|x-y|^2/4(s-t)}, \qquad t < s,\ x\in\mathbf{R}^n.$$

Thus

$$D\rho = \frac{(x-y)}{2(s-t)}\rho,$$

$$D^2\rho = \left(\frac{(x-y)\otimes(x-y)}{4(s-t)^2} - \frac{\delta}{2(s-t)}\right)\rho, \qquad \delta \equiv \text{identity matrix},$$

$$\frac{\partial}{\partial t}\rho = \left(\frac{n-1}{2}\frac{1}{(s-t)} - \frac{|x-y|^2}{4(s-t)^2}\right)\rho,$$

$$D\rho\cdot\nu = -\frac{(x-y)\cdot\nu}{2(s-t)}\rho.$$

Miraculously enough, we have

$$\text{(1)} \qquad \frac{(D\rho\cdot\nu)^2}{\rho} + S\colon D^2\rho + \frac{\partial}{\partial t}\rho \equiv 0,$$

so

$$\frac{d}{dt}\int \rho_{y,s}\,d\mu_t = \int -\rho_{y,s}\left(H + \frac{(x-y)\cdot\nu}{2(s-t)}\right)^2 d\mu_t.$$

3.2. Apply to ε-equation. For $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$ we get

$$\begin{aligned}
\frac{d}{dt}\int \phi\,d\mu_t^\varepsilon = \int & -\varepsilon\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)^2 dx \\
& + \varepsilon D\phi\cdot Du\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right) dx \\
& + \frac{\partial}{\partial t}\phi\,d\mu_t^\varepsilon \quad \text{by §2(1)} \\
= \int & -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)^2 dx \\
& + 2\varepsilon D\phi\cdot Du\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right) dx \\
& - \nu\otimes\nu\colon D^2\phi\,d\xi_t^\varepsilon + (\delta - \nu\otimes\nu)\colon D^2\phi\,d\mu_t^\varepsilon + \frac{\partial}{\partial t}\phi\,d\mu_t^\varepsilon \\
& \hfill \text{by §2(2)} \\
= \int & -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u) - \frac{Du\cdot D\phi}{\phi}\right)^2 dx \\
& + \varepsilon\frac{(Du\cdot D\phi)^2}{\phi}dx - \nu\otimes\nu\colon D^2\,d\xi_t^\varepsilon \\
& + (\delta - \nu\otimes -\nu)\colon D^2\phi\,d\mu_t^\varepsilon + \frac{\partial}{\partial t}\phi\,d\mu_t^\varepsilon.
\end{aligned}$$

The second term on the right-hand side of the above equation is equal to

$$\int \frac{(\nu \cdot D\phi)^2}{\phi}(d\mu_t^\varepsilon + d\xi_t^\varepsilon),$$

so we obtain

$$\begin{aligned}\frac{d}{dt}\int \phi \, d\mu_t^\varepsilon = \int -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u) - \frac{Du \cdot D\phi}{\phi}\right)^2 dx \\ + \left(\frac{(\nu \cdot D\phi)^2}{\phi} - (\delta - \nu\otimes\nu) : D^2\phi + \frac{\partial}{\partial t}\phi\right) d\mu_t^\varepsilon \\ + \left(-\nu\otimes\nu : D^2\phi + \frac{(\nu\cdot D\phi)^2}{\phi}\right) d\xi_t^\varepsilon\end{aligned} \tag{2}$$

for $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$. We now wish to insert $\phi = \rho_{y,s}$ into this formula. We set $\phi = \eta\rho$, where η is a smooth cutoff function with

$$|\eta|, |D\eta|, |D^2\eta| \le 1, \qquad \eta = \begin{cases} 0 & \text{off } B_{2R}, \\ 1 & \text{on } B_R. \end{cases}$$

Using the crude bounds of §1.5 and the exponential decay of ρ, $|D\rho|$, $|D^2\rho|$, we can pass to limits as $R \to \infty$ to establish that ρ may be inserted into (2). We calculate

$$-\nu\otimes\nu : D^2\rho + (\nu\cdot D\rho)^2/\rho = \rho/2(s-t).$$

Together with (1) we obtain from (2):

3.3. Monotonicity formula, ε-version. For $y \in \mathbf{R}^n$, $0 \le t < s$,

$$\begin{aligned}\frac{d}{dt}\int \rho_{y,s}\, d\mu_t^\varepsilon = \int -\varepsilon\rho_{y,s}\left(-\Delta u + \frac{1}{\varepsilon^2}f(u) - \frac{Du\cdot D\rho_{y,s}}{\rho_{y,s}}\right)^2 dx \\ + \int \frac{1}{2(s-t)}\rho_{y,s}\, d\xi_t^\varepsilon.\end{aligned}$$

In the next section we will show the negativity of ξ_t^ε.

For later use we record the following lemma, whose proof we leave to the reader. Define

$$\rho_y^r(x) \equiv \frac{1}{(\sqrt{2\pi}r)^{n-1}} e^{-|x-y|^2/(2r^2)},$$

which is related to $\rho_{y,s}(x, t)$ by

$$r^2 = 2(s-t).$$

3.4. Lemma. *Let* μ *be a measure satisfying* §1.4(iv), *namely*

$$\mu(B_R(x))/\omega_{n-1}R^{n-1} \le CD, \qquad R > 0, \; x \in \mathbf{R}^n.$$

Then the following hold:

(i) $\int \rho_x^r \, d\mu \le CD$ *(same* C*)*.

(ii) *For any* $r, \; R > 0, \; x \in \mathbf{R}^n$,

$$\int_{\mathbf{R}^n \setminus B_R(x)} \rho_x^r \, dx \le 2^{n-1} e^{-3R^2/8r^2} D.$$

(iii) *For any* $\delta > 0$ *there is* $\gamma_2 = \gamma_2(\delta) > 0$ *such that for all* $r > 0$ *and all* $x, \; x_1$ *with* $|x - x_1| \le \gamma_2 r$, *we have*

$$\int \rho_{x_1}^r \, d\mu \le (1 + \delta) \int \rho_x^r \, d\mu + \delta D.$$

(iv) *For any* $\delta > 0$ *there is* $\gamma_3 = \gamma_3(\delta) > 0$ *such that for any* $r, \; R > 0$ *with* $1 \le R/r \le 1 + \gamma_3$ *and any* $x \in \mathbf{R}^n$, *we have*

$$\int \rho_x^R \, d\mu \le (1 + \delta) \int \rho_x^r \, d\mu + \delta D.$$

(v) *For any* $r, R > 0$ *with* $R/r \le 1$ *we have* $\int \rho_x^R \, d\mu \le (r/R)^{n-1} \int \rho_x^r \, d\mu$.

(vi) *For any* $\delta > 0$ *there is* $\alpha(\delta) > 0$ *such that for all* $r > 0, \; x \in \mathbf{R}^n$,

$$\int \rho_x^{\alpha(\delta) r} \, d\mu \le \frac{\mu(B_r)}{\omega_{n-1} \alpha(\delta)^{n-1} r^{n-1}} + \delta D.$$

4. Negativity of ξ

Formula 3.3 motivates us to prove that $\xi_t^\varepsilon \le 0$. As discussed following §2(5), we expect $u^\varepsilon \approx q^\varepsilon(r)$, and

$$d\xi_t^\varepsilon = \left(\frac{\varepsilon}{2}|Du^\varepsilon|^2 - \frac{1}{\varepsilon}F(u^\varepsilon)\right) dx \approx 0,$$

since $|Dr| = 1$. Duly motivated, we define $r^\varepsilon(x, t)$ by

$$u^\varepsilon(x, t) = q^\varepsilon(r^\varepsilon(x, t)).$$

Then, as in the argument following §2(5),

$$\frac{(\varepsilon/2)|Du^\varepsilon|^2}{(1/\varepsilon)F(u^\varepsilon)} = |Dr^\varepsilon|^2.$$

Thus it is equivalent to show $|Dr^\varepsilon|^2 \le 1$. By the initial hypothesis §1.4(i), this holds at $t = 0$. We will now use the maximum principle to show it for $t \ge 0$. This argument was used by Modica for the stationary case.

From $(*)$ and $q_r = (1/\varepsilon)g$, $q_{rr} = (1/\varepsilon^2)f = (1/\varepsilon^2)gg_q$, we obtain, writing $r = r^\varepsilon$,

$$u_t^\varepsilon = \Delta u^\varepsilon - f(u^\varepsilon)/\varepsilon^2, \qquad q_r^\varepsilon r_t = q_r^\varepsilon \Delta r + q_{rr}^\varepsilon |Dr|^2 - q_{rr},$$

so

$$r_t = \Delta r + \frac{1}{\varepsilon} g_q(|Dr|^2 - 1)$$

and

$$|Dr|_t^2 = \Delta |Dr|^2 - 2|D^2 r|^2 + \frac{2}{\varepsilon} Dr \cdot Dg_q(|Dr|^2 - 1) + \frac{2}{\varepsilon} g_q Dr \cdot D|Dr|^2.$$

At a maximum of $|Dr^\varepsilon|^2$ that is equal to 1, each term on the right is negative or zero.

Assuming first that the function u^ε is periodic, this implies $|Dr^\varepsilon(\cdot, t)|^2 \le 1$ by the maximum principle. To extend to the nonperiodic case, approximate $u^\varepsilon(\cdot, 0)$ by a periodic function $u_R^\varepsilon(\cdot, 0)$ of large period, and solve $(*)$. Using the crude bounds in §1.5, we pass to limits as the period goes to infinity. Thus we obtain the estimate $|Dr^\varepsilon|^2 \le 1$ for some solution of $(*)$ with the same initial condition $u(\cdot, 0)$. By uniqueness of u^ε, we obtain $|Dr^\varepsilon|^2 \le 1$ for u^ε, and therefore have proven

4.1. Negativity of ξ. For u^ε solving $(*)$ subject to the initial conditions of §1.4, we get $|Dr^\varepsilon|^2 \le 1$ on $\mathbf{R}^n \times [0, \infty)$ or equivalently

$$d\xi_t^\varepsilon \le 0 \quad \text{for all } t \ge 0$$

and therefore, by §3.3,

$$\frac{d}{dt} \int \rho_{y,s} \, d\mu_t^\varepsilon \le 0$$

for $0 \le t < s$, $y \in \mathbf{R}^n$.

5. Passing measures to limits

As a result of §§3.3, 4.1, and 1.4(iv), we immediately have

5.1. Density bounds. There is $c(\varepsilon) \downarrow 0$ with

(i) $\displaystyle\int \rho_{y,s} \, d\mu_t^\varepsilon \le CD$ for $y \in \mathbf{R}^n$, $s \ge c(\varepsilon)$, $0 \le t < s$.

(ii) $\mu_t^\varepsilon(B_R(x)) \le CDR^{n-1}$ for $(x, t) \in \mathbf{R}^n \times [0, \infty)$, $R \ge c(\varepsilon)$.

5.2. Growth bound. For $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$ we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)^2 + \varepsilon D\phi\cdot Du\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)dx \\
&\le \int_{\mathbf{R}^n} -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u) + \frac{D\phi\cdot Du}{2\phi}\right)^2 + \varepsilon|Du|^2\frac{|D\phi|^2}{4\phi}\,dx \\
&\le C_1(\phi)\mu^\varepsilon(\{\phi>0\}), \quad \text{where } C_1(\phi) \equiv \sup_{\{\phi>0\}}\frac{|D\phi|^2}{2\phi} \le \sup|D^2\phi| \\
&\le C_2(\phi)D \quad \text{by §5.1}.
\end{aligned}$$

Then using §2(1) we obtain

5.3. Semidecreasing property. *For $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$, the function $\mu_t^\varepsilon(\phi) - C_2(\phi)Dt$ is nonincreasing.*

5.4. Passing to limits in μ_t^ε. The following argument originates in Brakke [5, §4]. The version which we give is identical to [26, §7].

Choose a countable dense set $B_1 \subset [0, \infty)$. By the weak compactness of Radon measures, the density bound of §5.1, and a diagonal argument, we may select a subsequence $\{\varepsilon_i\}_{i\ge1}$ and measures $\{\mu_t\}_{t\in B-1}$ such that $\mu_t^{\varepsilon_i} \to \mu_t$ for $t \in B_1$. Now let $\{\phi_i\}_{i\ge1}$ be a countable dense set in $C_c^2(\mathbf{R}^n, \mathbf{R}^+)$. By the semidecreasing property 5.3, there is a cocountable set $B_2 \subseteq [0, \infty)$ such that for $t \in B_2$ and all $i \ge 1$, $\mu_s(\phi_i)$ is continuous at t as a function of $s \in B_1$.

For any fixed $t \in B_2$, we can find a further subsequence $\{\mu_t^{\varepsilon_{ij}}\}_{j\ge1}$ and a limit μ_t such that $\mu_t^{\varepsilon_{ij}} \to \mu_t$. Then by Property 5.3, $\{\mu_s(\phi_i)\}_{s\in B_1\cup\{t\}}$ is continuous at t, for each i. Since $\{\phi_i\}_{i\ge1}$ is dense, it follows that μ_t is uniquely determined by $\{\mu_s\}_{s\in B_1}$. Therefore the full subsequence converges: $\mu^{\varepsilon_i} \to \mu_t$. In this way we define μ_t for each $t \in B_2$.

On the countable set $[0, \infty)\backslash B_2$, we perform a further diagonal argument, and thus obtain

5.5. Convergence of measures. *There is a subsequence $\{\varepsilon_i\}_{i\ge1}$ such that $\mu_t^{\varepsilon_i} \to \mu$ for all $t \ge 0$ as Radon measures on $\mathbf{R}^n$.*

5.6. Monotonicity. (*See* §§3.3, 4.1.) *For each $(y, s)' \in \mathbf{R}^n \times (0, \infty)$, the quantity $\int \rho_{y,s}\,d\mu_t$, $0 \le t < s$, is nonincreasing.*

6. Clearing out

To proceed further, we will prove a clearing-out lemma like that of Brakke [5, 6.3] or Evans and Spruck [18, 6.1], but using the monotonicity formula.

6.1. Clearing-Out Lemma. (i) *There is* $\eta > 0$ *depending on* n, F *such that* $\int \rho_{y,s}\, d\mu_t < \eta$ *implies* $(y, s) \notin \overline{\bigcup_{t'\geq 0} \operatorname{spt}\mu_{t'} \times \{t'\}}$.

(ii) *If* $(y, s) \notin \overline{\bigcup_{t'\geq 0} \operatorname{spt}\mu_{t'} \times \{t\}}$, *then there is a neighborhood* U *of* (y, s) *in* $\mathbf{R}^n \times [0, \infty)$ *such that* $u^{\varepsilon_i} \to u$ *uniformly on* U *to either* $+1$ *or* -1.

From the Clearing-Out Lemma come several standard consequences about the support of μ_t.

6.2. Corollary (Clarity of Support). $\operatorname{spt}\mu = \overline{\bigcup_{t'\geq 0} \operatorname{spt}\mu_{t'} \times \{t'\}}$ *where* $d\mu \equiv d\mu_{t'}\, dt'$.

Proof. $\operatorname{spt}\mu \subseteq \overline{\bigcup_{t\geq 0} \operatorname{spt}\mu_t \times \{t\}}$ is immediate. For the converse, let $(y, s) \notin \operatorname{spt}\mu$. Then $(y, s) \in U$, U open, with $U \cap \operatorname{spt}\mu = 0$. Thus $\int \rho_{y,s}\; d\mu_t \to 0$ as $t \uparrow s$, and $(y, s) \notin \overline{\bigcup_{t\geq 0} \operatorname{spt}\mu_t \times \{t\}}$ by Lemma 6.1(i).

Remark. We see that $u \equiv \lim u^{\varepsilon_i}$ is locally constant off $\operatorname{spt}\mu$.

6.3. Corollary (Measure of Support). *Let* $U \subseteq \mathbf{R}^n$ *be open. Then, writing* $(\operatorname{spt}\mu)_t$ *for* $\operatorname{spt}\mu \cap (\mathbf{R}^n \times \{t\})$, *we have*

(i) $\mathscr{H}^{n-1}((\operatorname{spt}\mu)_t \cap U) \leq C(D) \lim_{s\uparrow t} \mu_s(U)$ *for* $t > 0$,

(ii) $\mathscr{H}^{n-1}((\operatorname{spt}\mu)_t \cap B_r) \leq C(D) D R^{n-1}$ *for* $t \geq 0$,

(iii) $(\operatorname{spt}\mu)_0 = M_0$.

We defer the proof to the end of the section.

Remark. C is independent of D if we use Brakke's compactly supported Clearing-Out Lemma. This will only be possible after §9.

6.4. Empty Spot Lemma. *There are* β_1, τ_0, τ_1, $C > 0$ *such that for all* $y \in \mathbf{R}^n$, $R > 0$, *and* $0 < \varepsilon \leq \beta_2 R$, *if* $|u^\varepsilon(\cdot, t)| \geq 1/2$ *on* $B_R(y)$ *then*

(i)
$$|u^\varepsilon| \geq 1 - \varepsilon/R$$

on $B_{R/2}(y) \times [t + \tau_0 \varepsilon^2 |\log(\varepsilon R)|, t + \tau_1 R^2]$,

(ii)
$$\mu_s^\varepsilon(B_{R/2}(y)) \leq C\varepsilon R^{n-2} p$$

for $s \in [t + \tau_0\varepsilon^2 |\log(\varepsilon R)|, t + \tau_1 R^2]$.

To prove Lemma 6.4, we will apply the following lemma of X. Chen who establishes the result by means of a well-chosen subsolution.

6.5. Propagation of interface (adapted from [10, Theorem 3]). Let f satisfy the conditions in §1.1. Let h^ε solve $(*)$. Let $\{N_t\}_{0\leq t\leq T}$ be a smooth solution of the mean curvature flow with bounded curvature, and let $d(x, t)$ be the signed distance to N_t. Suppose there are c, C, $C_1 > 0$ such that

$$|h^\varepsilon(x, 0)| \geq c|d(x, 0)|, \qquad |Dh^\varepsilon(x, 0)| \leq C,$$

whenever $|h^\varepsilon(x, 0)| \le C_1$, where we assume $h^\varepsilon(\cdot, 0)$ takes the same sign as $d(\cdot, 0)$.

Then there exist τ_0, ε_2, $M_2 > 0$ such that for all $0 < \varepsilon \le \varepsilon_2$, $\tau_0\varepsilon^2|\log\varepsilon| \le t \le T$, we have

$$h^\varepsilon(x, t) \ge 1 - \varepsilon \quad \text{for } d(x, t) \ge M_2\varepsilon|\log\varepsilon|,$$
$$h^\varepsilon(x, t) \le -1 + \varepsilon \quad \text{for } d(x, t) \le -M_2\varepsilon|\log\varepsilon|.$$

Proof of Lemma 6.4 *from* §6.5. 1. Let us first prove (i). Since (i) is invariant under scale changes, we may as well assume $R = 1$, $t = 0$. Accordingly, let u^ε be a solution of $(*)$ with $|u^\varepsilon(\cdot, 0)| \ge 1/2$ on B_1; $u^\varepsilon(\cdot, 0) \ge 1/2$ without loss of generality. Let $\{N_t\}_{0\le t\le T}$ be a sphere shrinking by mean curvature with radius

$$r(t) = \sqrt{2(T-t)}, \qquad 0 \le t \le T,$$

where T is chosen so that $r(0) = 5/6$.

2. Let $h(\cdot, 0)$ be a C^2 radial function (independent of ε) such that

$$h(\cdot, 0) = \begin{cases} 1/2 & \text{on } B_{2/3}, \\ -1 & \text{on } \mathbf{R}^n \backslash B_1, \end{cases}$$
$$h(\cdot, 0) > 0 \quad \text{on } B_{5/6}, \qquad h(\cdot, 0) < 0 \quad \text{on } \mathbf{R}^n \backslash B_{5/6},$$
$$|h(x, 0)| \ge |d(x, N_0)| \quad \text{when } |h(x, 0)| \le 1/2,$$
$$|Dh(\cdot, 0)| \le 7 \quad \text{on } \mathbf{R}^n.$$

Then $h(\cdot, 0)$ satisfies the hypotheses of §6.5. Let h^ε be the solution of $(*)$ with initial data $h(\cdot, 0)$, where $0 < \varepsilon \le \varepsilon_2$. By §6.5, we have

$$h^\varepsilon \ge 1 - \varepsilon \quad \text{for } \tau_0\varepsilon^2|\log\varepsilon| \le t \le T, \ x \in B_{r(t) - M_2\varepsilon|\log\varepsilon|}.$$

Now choose $\tau_1 > 0$, $0 < \beta_2 \le \tau_0$ such that

$$r(\tau_1) = 2/3, \quad r(t) - M_2\varepsilon|\log\varepsilon| \ge 1/2 \qquad \text{for } 0 \le t \le \tau_1, \ 0 \le \varepsilon \le \beta_1.$$

Then for $0 < \varepsilon \le \beta_2$ we have

$$h^\varepsilon \ge 1 - \varepsilon \quad \text{on } B_{1/2} \times [\tau_0\varepsilon^2|\log\varepsilon|, \tau_3].$$

3. Now let u^ε satisfy the hypotheses of Lemma 6.4 with $R = 1$, $t = 0$. Then $u^\varepsilon(\cdot, 0) \ge h(\cdot, 0)$ and it follows from the comparison principle that $u^\varepsilon \ge 1 - \varepsilon$ on $B_{1/2} \times [\tau_0\varepsilon^2|\log\varepsilon|, \tau_1]$, as desired for (ii).

4. To prove (ii), we estimate from (i) with general R,

$$\begin{aligned}\mu^\varepsilon(B_{R/2}(y)) &= \int_{B_{R/2}} \frac{\varepsilon}{2}|Du^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon)\,dx \le \int_{B_{R/2}} \frac{2}{\varepsilon}F(u^\varepsilon)\,dx \quad \text{by step 4}\\ &\le \int_{B_{R/2}} \frac{1}{\varepsilon}\left(g\left(1-\frac{\varepsilon}{R}\right)\right)^2 dx \quad \text{by (i)}\\ &\le \int_{B_{R/2}} \frac{1}{\varepsilon}\left(\|g\|_{C^1}\frac{\varepsilon}{R}\right)^2 dx = C\varepsilon R^{n-2}.\end{aligned}$$

Proof of Lemma 6.1. The basic idea is to use Huisken's monotonicity formula to control the mass of μ_t^ε locally, to convert this to pointwise control of u^ε via $(\varepsilon/2)\ |Du^\varepsilon|^2 \le (1/\varepsilon)F$, and then to use X. Chen's work to drive u^ε to ± 1 locally uniformly.

1. Let $y \in \mathbf{R}^n$, $s > t \ge 0$. Let $\eta > 0$ be an arbitrary number to be selected later, and assume $\int \rho_{y,s}\,d\mu_t < \eta$. Observe that by Lemma 3.4(iii), (iv), (v), $\int \rho_{y,s}\,d\mu_t$ varies continuously with (y, s), so there is a small neighborhood U of (y, s) such that $\int \rho_{y',s'}\,d\mu_t \le 2\eta$ for $(y', s') \in U$. By the density estimate 5.1(i) and the exponential decay of ρ, there will be some $i_0 = i_0(\eta, D, \{\mu_t^{\varepsilon_i}\}_{i\ge 1})$ such that for all $i \ge i_0$ and each $(y', s') \in U$, we have $\int \rho_{y',s'}\,d\mu_t^{\varepsilon_i} \le 3\eta$. We may assume that $U \Subset \mathbf{R}^n \times (t, \infty)$. By the ε-monotonicity formula 3.3 and §4.1, we have for $(y', s') \in U$,

$$\int \rho_{y'}^{\varepsilon_i}\,d\mu_{s'-\varepsilon_i^2/2}^{\varepsilon_i} \equiv \int \rho_{y',s'}\,d\mu_{s'-\varepsilon_i^2/2}^{\varepsilon_i} \le \int \rho_{y',s'}\,d\mu_t^{\varepsilon_i} \le 3\eta,$$

provided that ε_i is small enough that $\varepsilon_i^2/2 \le \inf_U(s'-t) > 0$. Then we obtain writing ε for ε_i,

$$\begin{aligned}&\int_{B_\varepsilon(y')\times\{s'-\varepsilon^2/2\}} \frac{\varepsilon}{2}|Du^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon)\,dx\\ &\le \left(\inf_{x\in B_\varepsilon(y')} \rho_{y'}^\varepsilon(x)\right)^{-1} \int \rho_{y'}^\varepsilon\,d\mu_{s'-\varepsilon^2/2}^\varepsilon \qquad (1)\\ &\le \left(\sqrt{2\pi\varepsilon}\right)^{n-1} e^{1/2}3\eta = C\eta\varepsilon^{n-1},\end{aligned}$$

where $C = (n)$.

2. Now let us pretend that

$$|u^\varepsilon(y', s'-\varepsilon^2/2)| \le 1/2. \qquad (2)$$

In the notation of the proof of the negativity of ξ in §4.1, we have

$$u^\varepsilon = q^\varepsilon(r^\varepsilon) = q^1(r^\varepsilon/\varepsilon).$$

Then (2) implies that

$$\left|\frac{r^\varepsilon(y', s' - \varepsilon^2/2)}{\varepsilon}\right| \leq (q^1)^{-1}\left(\frac{1}{2}\right) \equiv C.$$

Also, $|Dr^\varepsilon| \leq 1$ by §4.1. Thus

$$|r^\varepsilon(\cdot, s' - \varepsilon^2/2)| \leq C\varepsilon + 1 \cdot \varepsilon \quad \text{on } B_\varepsilon(y').$$

That is,

$$|u^\varepsilon(\cdot, s' - \varepsilon^2/2)| \leq q^1(C+1) \equiv c < 1 \quad \text{on } B_\varepsilon(y').$$

Then

$$F(u^\varepsilon) \geq c > 0 \quad \text{on } B_\varepsilon(y') \times \{s' - \varepsilon^2/2\},$$

and therefore

$$\text{(3)} \qquad \int_{B_\varepsilon(y')} \frac{\varepsilon}{2}|Du^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon)\,dx \geq \omega_n \varepsilon^n\left(\frac{1}{\varepsilon}c\right) = c\varepsilon^{n-1},$$

where $c = c(n, q^1)$.

3. It follows from (1) and (3) that if $|u^\varepsilon(y', s' - \varepsilon^2/2)| \leq 1/2$, then $c\varepsilon^{n-1} \leq C\eta\varepsilon^{n-1}$. We now fix $\eta = \eta(n, q^1)$ so small that this is impossible. Therefore, we have proven that there is $U \ni (y, s)$ and i_0, independent of ε, such that for $i \geq i_0$ and $(y', s') \in U$, we have

$$|u^{\varepsilon_i}(y', s' - \varepsilon_i^2/2)| \geq 1/2.$$

By decreasing ε_i, we can see that there is a smaller neighborhood V of (y, s) such that $|u^{\varepsilon_i}| \geq 1/2$ on V.

4. Now fix a point $s_0 < s$ and a radius $R > 0$ such that $B_R(y) \times \{s_0\} \subseteq V$ and $(y, s) \in B_{R/2}(y) \times (s_0, s_0 + \tau R^2)$. From Lemma 6.4(iii) it follows that

$$u^{\varepsilon_i} \geq 1 - \frac{\varepsilon_i}{R} \quad \text{on } B_{R/2} \times \left[s_0 + \tau_0\varepsilon_i^2\left|\log\frac{\varepsilon_i}{R}\right|, s_0 + \tau_1 R^2\right],$$

$$\mu_{s'}^{\varepsilon_i}(B_{R/2}(y)) \leq C\varepsilon_i R^{n-1} \quad \text{for } s' \in \left[s_0 + \tau_0\varepsilon_i^2\left|\log\frac{\varepsilon_i}{R}\right|, s_0 + \tau_1 R^2\right],$$

provided $0 < \varepsilon_i \leq \beta_0 R$. Passing to limits, we obtain

$$\begin{aligned} \mu_{s'}(B_{R/2}(y)) &= 0 \quad \text{for } s' \text{ near } s \\ u^{\varepsilon_i} &\to \pm 1 \quad \text{uniformly near } (y, s), \end{aligned}$$

which provides Lemma 6.1(i).

5. To prove Lemma 6.1(ii), let $(y, s) \notin \overline{\bigcup_{t\geq 0} \operatorname{spt} \mu_t \times \{t\}}$. Then $\int \rho_{y,s}\, d\mu_t \to 0$ as $s \to t$, so by the above,

$$u(y', s') = \lim_{i\to\infty} u^{\varepsilon_i}(y', s') = \pm 1 \quad \text{near } (y, s).$$

Proof of Corollary 6.3. (i) It suffices to prove the result for every compact set $K \subseteq U$. Write $X_t \equiv (\operatorname{spt}_\mu)_t$. Let $(x, t) \in X_t \cap K$. Let $\delta > 0$, $\alpha > 0$ be arbitrary. Then by the Clearing-Out Lemma 6.1, for each $r > 0$

$$\eta \leq \int \rho_x^{\alpha r}\, d\mu_{t-\alpha^2 r^2/2}.$$

Choose $\alpha = \alpha(\delta)$ according to Lemma 3.4(vi), and obtain

$$\eta \leq \mu_{t-\alpha^2 r^2/2}(B_r)/\omega_{n-1} r^{n-1} \alpha^{n-1} + \delta D.$$

Choosing $\delta = \delta(D)$ so that $\delta D = \eta/2$, we get

$$\eta \leq \frac{2\mu_{t-\alpha^2 r^2/2}(B_r)}{\omega_{n-1} r^{n-1} \alpha^{n-1}} \tag{1}$$

where $\alpha = \alpha(D)$. Now for a fixed $r > 0$, consider the covering of $X_t \cap K$ by the collection

$$\mathscr{B} = \{B_r(x) \colon x \in X_t\}.$$

By the Besicovitch covering theorem, there are countable subcollections $\mathscr{B}_1, \cdots, \mathscr{B}_{B(n)}$ such that each $\mathscr{B}_i$ is disjoint and

$$X_t \subseteq \bigcup_{i=1}^{B(n)} \bigcup_{B_r(x_j) \in \mathscr{B}_i} B_r(x_j).$$

Now we calculate

$$\begin{aligned}
\mathscr{H}_r^{n-1}(X_t \cap K) &\leq \sum_i \sum_{B_r(x_j)\in\mathscr{B}_i} \omega_{n-1} r^{n-1} \\
&\leq \sum_i \frac{2}{\alpha^{n-1}\eta} \sum_{B_r(x_j)\in\mathscr{B}_i} \mu_{t-\alpha^2 r^2/2}(B_r(x_j)) \quad \text{by (1)} \\
&\leq \sum_i \frac{2}{\alpha^{n-1}\eta} \mu_{t-\alpha^2 r^2/2}(\{x\colon \operatorname{dist}(x, K) \leq r\}) \\
&\leq \frac{2B(n)}{\alpha(D) a^{n-1}\eta} \mu_{t-\alpha^2 r^2/2}(U)
\end{aligned}$$

for r small enough. By sending $r \downarrow 0$, we obtain

$$\mathscr{H}^{n-1}(X_t \cap K) \leq c(D) \lim_{s\uparrow t} \mu_s(U),$$

which is (i). Now (ii) follows from (i) and 5.1(ii).

For (iii), consider any closed ball $B_R(y)$ disjoint from M_0. As $\varepsilon_i \downarrow 0$ we have $|u^{\varepsilon_i}(x, 0)| \to 1$ uniformly for $x \in B_R(y)$. It follows from Lemma 6.4(ii) that $\mu_s(B_{R/2}(y)) = 0$ for $s \in [0, \tau_1 R^2]$. In particular, $(y, 0) \notin X \equiv \overline{\bigcup_{t\geq 0} \operatorname{spt} \mu_t \times \{t\}}$; i.e., $y \notin X_0$. Thus $X_0 \subseteq M_0$. Now $M_0 \subseteq X_0$ because $M_0 = \operatorname{spt} \mu_0$ by §1.4(ii). Hence $X_0 = M_0$, proving (iii).

7. Density lower bound

We now apply the technique of Evans and Spruck [18, Lemma 6.2] to prove two lower density estimates. The Hausdorff measure $n - 2 + \varepsilon$ is a slight improvement on their work.

7.1. Density lower bound. *Let $\{\mu_t\}_{t\geq 0}$ satisfy the Clearing-Out Lemma* 6.1. *Define*

$$Z^0 \equiv \left\{(x, t) \in \operatorname{spt} \mu \colon \varlimsup_{r\downarrow 0} \int \rho_x^r \, d\mu_t < \eta\right\},$$
$$Z_t^0 \equiv Z^0 \cap (\mathbf{R}^n \times \{t\}),$$

where η is as in Lemma 6.1. *Then for $\delta > 0$, $\mathscr{H}^{n-2+\delta}(Z_t^0) = 0$ for a.e. $t \geq 0$.*

The second density lemma involves the *forward* heat kernel.

7.2. Forward density lower bound. *Let $\{\mu_t\}_{t\geq 0}$ satisfy the Clearing-Out Lemma. Define*

$$Z^- \equiv \{(x, t) \in \operatorname{spt} \mu \colon \varlimsup_{s\downarrow t} \int \rho_{y,s}(x, t) \, d\mu_s(y) < \eta\},$$
$$Z_t^- \equiv Z^- \cap (\mathbf{R}^n \times \{t\}).$$

Then for $\varepsilon > 0$, $\mathscr{H}^{n-2+\varepsilon}(Z_t^-) = 0$ for a.e. $t \geq 0$.

We will prove the second density Lemma 7.2 and leave the first, whose proof is simpler, to the reader.

Proof of the bound 7.2.1. We have

$$Z^- = \bigcup_{\substack{\tau>0\\ \eta_2<\eta}} Z^{\eta_2, \tau},$$

where

$$Z^{\eta_2,\tau} \equiv \left\{(x, t) \in \operatorname{spt} \mu \colon \int \rho_{y,s}(x, t) \, d\mu_s(y) \leq \eta_2 \text{ for all } s \in (t, t + \tau]\right\}.$$

It suffices to prove $\mathscr{H}^{n-2+\varepsilon}(Z_t^{\eta_2,\tau}) = 0$ for each fixed $\eta_2 < \eta$, $\tau > 0$.

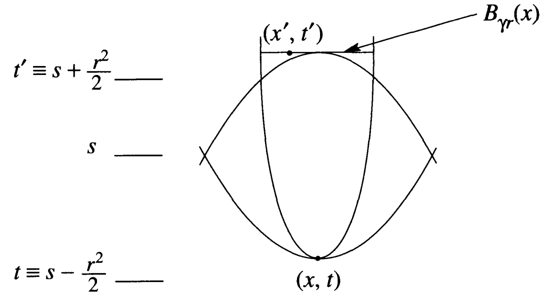

FIGURE 3. CONTROLLING FUTURE DENSITY

Let $\delta > 0$ be a constant to be chosen momentarily. Let $\gamma = \gamma(\delta)$ be the constant γ_2 in Lemma 3.4(iii). Let $(x, t) \in Z^{\eta_2, \tau}$, $s \in [t, t+\tau]$, and define r by $r^2/2 \equiv s - t$. Then for $x' \in B_{\gamma r}(x)$, we have

$$\begin{aligned}\int \rho_{x', s+r^2/2}(y, s)\, d\mu_s(y) &\equiv \int \rho^r_{x'}(y)\, d\mu_s(y)\\ &\leq (1+\delta)\int \rho^r_x(y)\, d\mu_s(y) + \delta D \quad \text{by Lemma 3.4(iii)}\\ &= (1+\delta)\int \rho_{y,s}(x, t)\, d\mu_s(y) + \delta D\\ &\leq (1+\delta)\eta_2 + \delta D \quad \text{since } (x, t) \in Z^{\eta_2, \tau}, \ s \in (t, t+\tau].\end{aligned}$$

We now choose $\delta = \delta(D, \eta - \eta_2)$ so that $(1+\delta)\eta_2 + \delta D < \eta$. Then $\int \rho_{x', s+r^2/2}(y, s)\, d\mu_s(y) < \eta$ for $x' \in B_{\gamma r}(x)$. Note that $\gamma = \gamma(D, \eta - \eta_2)$. See Figure 3.

Then by the Clearing-Out Lemma, we have $(x', t') \notin \overline{\bigcup_{t'' \geq 0} \operatorname{spt} \mu_{t''}}$ and in particular $(x', t') \notin Z^{\eta_2, \tau}$, where $t' \equiv s + r^2/2 = t + r^2$. We have shown that the relation

$$|t' - t| \leq 2\tau, \qquad |x' - x| \leq \gamma r, \quad \text{where } r^2 = t' - t$$

forces either $(x, t) \notin Z^{\eta_2, \tau}$ or $(x', t') \notin Z^{\eta_2, \tau}$. In consequence, for $(x, t) \in Z^{\eta_2, \tau}$ we have proven that $P_{2\tau}(x, t) \cap Z^{\eta_2, \tau} = (x, t)$, where $P_{2\tau}(x, t)$ is the truncated double solid paraboloid defined by

$$2\tau \geq |t' - t| \geq \frac{|x' - x|^2}{\gamma^2},$$

as shown in the figure.

2. We further subdivide $Z^{\eta_2, \tau}$ into sets of the form

$$Z' \equiv Z^{\eta_2, \tau, x_0, t_0} \equiv Z^{\eta_2, \tau} \cap (B_1(x) \times [t_0 - \tau, t_0 + \tau]), \qquad x \in \mathbf{R}^n, \ t > 0.$$

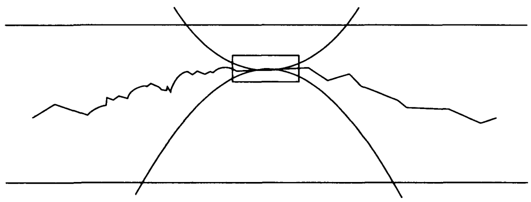

FIGURE 4. PICTURE OF Z'

Then $Z^{\eta_2,\tau}$ is a countable union of such sets Z', so it suffices to prove $\mathscr{H}^{n-2+\varepsilon}(Z'_t) = 0$ for a.e. $t \geq 0$, where $Z'_t \equiv Z' \cap (\mathbf{R}^n \times \{t\})$. Because $P_{2\tau}(x, t)$ is taller than Z' for $(x, t) \in Z'$, the set Z' meets each vertical line $\{x\} \times \mathbf{R}$ in at most one point. See Figure 4.

Fix $\delta_1 > 0$ and cover the projection $\pi_{\mathbf{R}^n}(Z') \subseteq B_1(x) \subseteq \mathbf{R}^n \times \{0\}$ by a collection of balls $\{B_{r_i}(x_i)\}_{i\geq 1}$, where $x_i \in \pi_{\mathbf{R}^n}(Z')$, $r_i \leq \delta_1$, and

$$\sum_{i=1}^{\infty} \omega_n r_i^n \leq 2\mathscr{L}^n(B_1(x_0)).$$

Let (x_i, t_i) be the point in Z' corresponding to x_i. By step 1, the cylinders $B_{r_i}(x_i) \times [t_i - r_i^2/\gamma^2, t_i + r_i^2/\gamma^2]$ collectively cover Z'. We calculate with approximate Hausdorff measure

$$\begin{aligned}
\int_{t_0-\tau}^{t_0+\tau} \mathscr{H}_{\delta_1}^{n-2+\varepsilon}(Z'_t)\, dt &\leq \int_{t_0-\tau}^{t_0+\tau} \sum_{\{i\,:\, t\in[t_i-r_i^2/\gamma^2, t_i+r_i^2/\gamma^2]\}} \omega_{n-2+\varepsilon} r_i^{n-2+\varepsilon}\, dt \\
&= \sum_{i=1}^{\infty} \int_{t_i-r_i^2/\gamma^2}^{t_i+r_i^2/\gamma^2} \omega_{n-2+\varepsilon} r_i^{n-2+\varepsilon}\, dt \\
&= \sum_{i=1}^{\infty} \frac{2\omega_{n-2+\varepsilon}}{\gamma^2} r^{n+\varepsilon} \\
&\leq C(\eta - \eta_2, D)\delta_1^{\varepsilon} 2\mathscr{L}^n(B_1(x_0)).
\end{aligned}$$

Let $\delta_1 \downarrow 0$ and obtain by the monotone convergence theorem

$$\int_{t_0-\tau}^{t_0+\tau} \mathscr{H}^{n-2+\varepsilon}(Z'_t)\, dt = 0.$$

Taking countable unions, we find

$$\int_0^{\infty} \mathscr{H}^{n-2+\varepsilon}(Z_t^-)\, dt = 0,$$

yielding the result.

8. Vanishing of ξ (equipartition of energy)

Define $d\xi^\varepsilon \equiv d\xi_t^\varepsilon\, dt$, $d\mu^\varepsilon \equiv d\mu_t^\varepsilon\, dt$ and recall $\xi^\varepsilon \le 0$, $|\xi^\varepsilon| \le \mu^\varepsilon$. Assume (via a further subsequence) that $\xi^{\varepsilon_i} \to \xi$, $\mu^{\varepsilon_i} \to \mu \equiv d\mu_t\, dt$ as Radon measures on $\mathbf{R}^n \times [0, \infty)$. We now are in a position to prove $\xi = 0$. Recall formula 3.3 and §4.1:

$$(1) \qquad \frac{d}{dt}\int \rho_{y,s}(x, t)\, d\mu_t^\varepsilon(x) \le -\int \frac{\rho_{y,s}(x, t)}{2(s-t)}\, d|\xi_t^\varepsilon|(x) \le 0.$$

Heuristically, if $\xi < 0$, then $(\varepsilon/2)|Du^\varepsilon|^2 < (1/\varepsilon)F(u^\varepsilon)$, that is, the interface is spread too much (relative to q^ε) and has a greater than $(n-1)$-dimensional character. Then the $(n-1)$-dimensional monotonicity integral decreases rapidly toward zero as it focuses. This is expressed by the above formula. On the other hand, we have the density lower bound 7.2. These two will contradict one another, yielding the following theorem.

8.1. Vanishing of ξ. *Let u^ε, $\{\mu_t\}_{t\ge 0}$ be as in* §§1, 5. *Then* $\xi = 0$.

Note that we cannot control the measures $\lim_{\varepsilon_i \downarrow 0} \xi_t^\varepsilon$ for every t.

Proof of 8.1. Integrating (1) and passing to limits, we find that for any $(y, s) \in \mathbf{R}^n \times (0, \infty)$ and any $\sigma > 0$,

$$\begin{aligned}\iint_{\mathbf{R}^n \times [0, s-\sigma]} \frac{\rho_{y,s}(x, t)}{2(s-t)}\, d|\xi|(x, t) &\le \int_{\mathbf{R}^n} \rho_{y,s}(x, 0)\, d\mu_0(x) \\ &\le CD \quad \text{by Lemma 3.4(i)}.\end{aligned}$$

Fix $R, T > 0$ and integrate against $d\mu_s\, ds$ to obtain

$$\begin{aligned}&\int_0^{T+1}\int_{B_R(0)}\iint_{\mathbf{R}^n \times [0, s-\sigma]} \frac{\rho_{y,s}(x, t)}{2(s-t)}\, d|\xi|(x, t)\, d\mu_s(y)\, ds \\ &\quad \le \int_0^{T+1}\int_{B_R(0)} CD\, d\mu_s(y)\, ds \\ &\quad \le CD^2(T+1)R^{n-1} \quad \text{by §5.1(iii)} \\ &\quad < \infty.\end{aligned}$$

See Figure 5.

Then since $|\xi|$ and $d\mu_s\, ds$ are Radon measures and the integrand is continuous and bounded on its domain, by Fubini's theorem we obtain

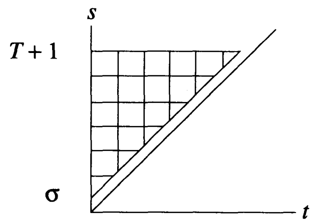

FIGURE 5. REGION OF INTEGRATION

$$\iint_{\mathbf{R}^n\times[0,T+1]}\int_{t+\sigma}^{T+1}\frac{1}{2(s-t)}\int_{B_R(0)}\rho_{y,s}(x,t)\,d\mu_s(y)\,ds\,d|\xi|(x,t)$$
$$\le CD^2(T+1)R^{n-1}.$$

Passing $\sigma\downarrow 0$ we obtain the same inequality with $\sigma=0$, by the monotone convergence theorem. It follows that

$$\int_t^{t+1}\frac{1}{2(s-t)}\int_{B_R(0)}\rho_{y,s}(x,t)\,d\mu_s(y)\,ds\le C(x,t)<\infty \tag{2}$$

for $|\xi|$-a.e. $(x,t)\in\mathbf{R}^n\times[0,T]$.

2. Now for any $x\in B_{R/2}(0)$, $s>t>0$, we have

$$\begin{aligned}\int_{\mathbf{R}^n}\rho_{y,s}(x,t)\,d\mu_s(y)&=\int_{B_R(0)}\rho_{y,s}(x,t)\,d\mu_s(y)+\int_{\mathbf{R}^n\setminus B_R(0)}\rho_x^{\sqrt{2(s-t)}}\,d\mu_s\\&\le\int_{B_R(0)}\rho_{y,s}(x,t)\,d\mu_s(y)\\&\quad+2^{n-1}e^{-\frac{3}{8}\frac{(R/2)^2}{2(s-t)}}D\quad\text{by 3.5(ii)}.\end{aligned}$$

Thus for $|\xi|$-a.e. $(x,t)\in B_{R/2}(0)\times[0,T]$,

$$\begin{aligned}&\int_t^{t+1}\frac{1}{2(s-t)}\int_{\mathbf{R}^n}\rho_{y,s}(x,t)\,d\mu_s(y)\,ds\\&\le C(x,t)+\int_t^{t+1}\frac{1}{2(s-t)}2^{n-1}e^{-\frac{3}{32}\frac{R^2}{s-t}}D\,ds\\&<\infty.\end{aligned}\tag{3}$$

By taking R, T sufficiently large, the above holds for $|\xi|$-a.e. $(x,t)\in\mathbf{R}^n\times[0,\infty)$.

3. Fix (x,t) satisfying (3). We aim to prove that

$$\lim_{s\downarrow t}\int_{\mathbf{R}^n}\rho_{y,s}(x,t)\,d\mu_s(y)=0.$$

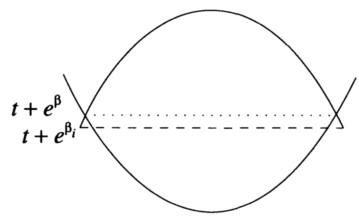

FIGURE 6. CONTROLLING $f(t+e^{\beta})$

Define $\beta = \log(s-t)$ and

$$h(s) \equiv \int_{\mathbf{R}^n} \rho_{y,s}(x,t)\, d\mu_s(y),$$

so (3) becomes

$$\int_{-\infty}^{0} h(t+e^{\beta})\, d\beta < \infty. \tag{4}$$

This shows h is frequently small, and we will next use monotonicity to control h in the other places.

4. Let $\gamma \in (0, 1]$ be any number. By (4), there will be a sequence $\beta_i \to -\infty$ such that

$$\beta_1 > \beta_2 > \cdots, \quad |\beta_i - \beta_{i+1}| \le \gamma, \quad h(t+e^{\beta_i}) \le \gamma. \tag{5}$$

Now let $\beta \in (-\infty, \beta_1]$ and suppose $\beta \in [\beta_i, \beta_{i-1})$ (see Figure 6). Then

$$\begin{aligned} h(t+e^{\beta}) &= \int \rho_{y,t+e^{\beta}}(x,t)\, d\mu_{t+e^{\beta}}(y) \\ &= \int \rho_{x,t+2e^{\beta}}(y, t+e^{\beta})\, d\mu_{t+e^{\beta}}(y) \\ &\le \int \rho_{x,t+2e^{\beta}}(y, t+e^{\beta_i})\, d\mu_{t+e^{\beta_i}}(y) \\ &\qquad\qquad \text{by monotonicity, since } \beta_i \le \beta \\ &= \int \rho_x^R\, d\mu_{t+e_i^{\beta}} \end{aligned}$$

where $R^2/2 \equiv 2e^{\beta} - e^{\beta_i}$. On the other hand, by (5)

$$\begin{aligned} \gamma \ge h(t+e^{\beta_i}) &= \int \rho_{y,t+e^{\beta_i}}(x,t)\, d\mu_{t+e^{\beta_i}}(y) \\ &= \int \rho_x^r\, d\mu_{t+e^{\beta_i}}, \end{aligned}$$

where $r^2/2 \equiv e^{\beta_i}$. Note that

$$1 \le R/r = \sqrt{2e^{\beta-\beta_i} - 1} \le 1 + C\gamma.$$

5. Now let $\delta \in (0, 1]$ and set $\gamma = \min(\delta, \gamma_3(\delta)/C)$ where $\gamma_3(\delta)$ is as in Lemma 3.4(iv). Then applying the inequalities of step 4 for this choice of γ, we find that for all $\beta \le \beta_1(\gamma(\delta))$,

$$\begin{aligned} h(t + e^{\beta}) &\le \int \rho_x^R \, d\mu_{t+e^{\beta_i}} \le (1 + \delta) \int \rho_x^r \, d\mu_{t+e^{\beta_i}} + \delta D \\ &\le 2\gamma + \delta D. \end{aligned}$$

This is true for all $\beta \le \beta_1(\gamma)$. As $\delta \downarrow 0$ we obtain

$$\lim_{s \downarrow t} h(s) = 0 \quad \text{for } |\xi|\text{-a.e. } (x, t). \tag{6}$$

6. On the other hand, by Theorem 7.2,

$$\overline{\lim_{s\downarrow t}}\, h(s) \ge \eta > 0 \qquad d(\mathscr{H}^{n-2+\varepsilon} \lfloor \operatorname{spt} \mu_t) \, dt\text{-a.e. } (x, t). \tag{7}$$

The fact that $\mu_t(B_r(x)) \le CDr^{n-1}$ for all $(x, t) \in \mathbf{R}^n \times [0, \infty)$, $r > 0$ shows that

$$d|\xi| \le d\mu = d\mu_t \, dt \ll d(\mathscr{H}^{n-2+\varepsilon} \lfloor \operatorname{spt} \mu_t) \, dt. \tag{8}$$

Combining (6), (7), and (8) gives

$$0 \ge \overline{\lim_{s\downarrow t}}\, h(s) \ge \eta \qquad |\xi|\text{-a.e.}$$

from which we conclude $\xi = 0$.

9. Passing varifolds to limits

9.1. Brakke's inequality for the limit. *Let $\{\mu_t\}_{t\ge0}$ be a limit for the Allen-Cahn equation as in* §§1.1 *and* 5.4. *Then $\{\mu_t\}_{t\ge0}$ satisfies Brakke's inequality*

$$\overline{D}_t \int \phi \, d\mu_t \le \int -\phi H^2 + D\phi \cdot (T_x\mu_t)^{\perp} \cdot \vec{H} \, d\mu_t$$

for each $t \ge 0$ and $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$, where the right-hand side is understood to be $\mathscr{B}(\mu_t, \phi)$.

Remarks. (i) Using results 5.1 and 7.1 it is now easy to see that $(\operatorname{spt} \mu)_t$ is $(n-1)$-rectifiable for a.e. $t \ge 0$. In fact, Brakke's work shows the following more precise result. By the second half of the proof of [5, 6.12] together with Lemma 6.1, it is possible to show that whenever $\overline{D}_t^- \mu_t(\phi) > -\infty$ then

$$\Theta^{*n-1}(\mu_t, \cdot) \geq c\eta \qquad \mathscr{H}^{n-1} \llcorner (\operatorname{spt}\mu)_t\text{-a.e.},$$

and passing to limits $t_i \uparrow t$ via the Allard compactness theorem [1], $(\operatorname{spt}\mu)_t$ is $(n-1)$-rectifiable whenever $\overline{D}_t^- \mu_t(\phi) > -\infty$.

(ii) The proof of Brakke's Inequality 9.1 also establishes $T_x\mu_t \cdot \vec{H}(x) = 0$ for $\mu_t \llcorner \{\phi > 0\}$-a.e. x (perpendicularity of mean curvature) whenever $\overline{D}_t \int \phi \, d\mu_t > -\infty$.

9.2. Corollary (Inclusion). *Let* $\{\Gamma_t\}_{t\geq 0}$ *be the level-set flow of Evans and Spruck and Chen, Giga, and Goto, with* $\Gamma_0 = \overline{M_0}$ M_0 *as in* §1.3. *Then* $(\operatorname{spt}\mu)_t \subseteq \Gamma_t$ *for each* $t \geq 0$, *and* $u^{\varepsilon_i} \to \pm 1$ *locally uniformly in the complement of* $\operatorname{spt}\mu$.

Proof. By [27, 10.7] any Brakke motion remains within the corresponding level-set flow. The second statement repeats Lemma 6.1(ii).

This implies, in particular, the result of Evans, Soner and Souganidis [15] that $u^{\varepsilon_i} \to \pm 1$ locally uniformly in the complement of Γ_t.

9.3. Lower Semicontinuity and Rectifiability Lemma. *Let* $\{u^i(\cdot)\}_{t\geq 0}$ *be a sequence of smooth functions on* $\mathbf{R}^n$ *with* $\mathscr{L}^n(\{|Du^i| = 0\}) = 0$ *for each* $i \geq 1$. *Let* $\{\varepsilon_i\}_{i\geq 1}$ *be a sequence converging to zero. Define* μ^i, ξ^i, V^i *as in* §§1, 2, *namely*

$$d\mu^i = \left(\frac{\varepsilon_i |Du^i|^2}{1} + \frac{F(u^i)}{\varepsilon_i}\right) dx,$$

$$d\xi^i = \left(\frac{\varepsilon_i |Du^i|^2}{2} - \frac{F(u^i)}{\varepsilon^i}\right) dx,$$

$V^i \in \mathbf{V}_k(\mathbf{R}^n)$, $\|V^i\| = \mu^i$, $(V^i)^{(x)}$ *supported at* $(Du^i(x))^\perp$ *for each* $x \in \mathbf{R}^n$. *Let* $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$ *and define*

$$\mathscr{B}^{\varepsilon_i}(u_i, \phi) \equiv \int -\varepsilon_i \phi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2$$
$$+ \varepsilon_i D\phi \cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx.$$

Assume

(i) $\mu^i \to \mu$ *as Radon measures on* $\mathbf{R}^n$,
(ii) $|\xi^i|(\{\phi > 0\}) \to 0$,

(iii) $\xi^i \le 0$, $i \ge 1$,

(iv) $\mathscr{B}^{\varepsilon_i}(u^i, \phi) \ge -C_4$ *for* $i \ge 1$,

(v) $\mathscr{H}^{n-1}(\operatorname{spt}\mu \cap \{\phi > 0\}) < \infty$.

Then the following hold:

(vi) $\mu\lfloor\{\phi > 0\}$ *is real* $(n-1)$*-rectifiable.*

(vii) *There is* $V \in \mathbf{RV}_{n-1}(\mathbf{R}^n$ *such that* $V^i\lfloor\{\phi > 0\} \to V$ *and* $\|V\| = \mu\lfloor\{\phi > 0\}$.

(viii) *For all* $Y \in C_c^1(\{\phi > 0\}, \mathbf{R}^n)$,

$$\delta V(Y) = \lim_{i\to\infty} \int -\varepsilon Y \cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx.$$

(ix) $\mathscr{B}(\mu, \phi) \ge \overline{\lim}_{i\to\infty} \mathscr{B}^{\varepsilon_i}(u^i, \phi)$.

9.4. Standard estimates. *For* $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$, μ *a Radon measure,* $C_1(\phi) \equiv \sup|D^2\phi|$, *we have the following*:

$$\text{(i)} \quad \int D\phi \cdot (T_x\mu)^\perp \cdot \vec{H}\, d\mu \le \int \frac{1}{2}\phi H^2\, d\mu + C_1(\phi)\mu(\{\phi > 0\}),$$

when these are defined.

$$\text{(ii)} \qquad \int \phi H^2\, d\mu \le -2\mathscr{B}(\phi, \mu) + 2C_1(\phi)\mu(\{\phi > 0\}),$$

when these are defined. Let $u\colon \mathbf{R}^n \to \mathbf{R}$ *be smooth,* $\varepsilon > 0$, *then*

$$\text{(iii)} \qquad \begin{aligned} &\int \varepsilon D\phi \cdot Du \left(-\Delta u + \frac{1}{\varepsilon^2} f(u)\right) dx \\ &\le \int \frac{1}{2}\varepsilon\phi \left(-\Delta u + \frac{1}{\varepsilon^2} f(u)\right)^2 dx \\ &\quad + 2C_1(\phi) \int_{\{\phi > 0\}} \frac{\varepsilon}{2}|Du|^2\, dx. \end{aligned}$$

$$\text{(iv)} \qquad \begin{aligned} &\int \varepsilon\phi \left(-\Delta u + \frac{1}{\varepsilon^2} f(u)\right)^2 dx \\ &\le -2\mathscr{B}^\varepsilon(\mu, \phi) + 4C_1(\phi) \int_{\{\phi > 0\}} \frac{\varepsilon}{2}|Du|^2\, dx. \end{aligned}$$

Proof. (i) follows from Cauchy's inequality, with $C_2(\phi) \equiv \sup|D^2\phi| \ge \sup|D\phi|^2/\phi$. (ii) follows from (i). (iii) and (iv) are proven entirely analogously for u.

Proof of Lemma 9.3. 1. By the compactness theorem for Radon measures, there is a subsequence $\{V^{i_j}\}_{j\geq 1}$ and a limit $\widetilde{V} \in V_k(\mathbf{R}^n)$ such that $V_{i_j} \to \widetilde{V}$ as varifolds.

2. Now let $\{V^{i_j}\}_{j\geq 1}$ be any such subsequence. Fix $U \Subset \{\phi > 0\}$. Write $\nu^i = Du^i/|Du^i|$, $\mathscr{L}^n$-a.e. Then for $Y \in C_c^1(U, \mathbf{R}^n)$, by §2(2) we obtain

(1)

$$\begin{aligned} \delta V^i(Y) &= \int DY : S\, dV^i(x, S) \\ &= \int DY : (\delta - \nu^i \otimes \nu^i)\, d\mu^i \\ &= \int -\varepsilon_i \cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx + \int -\nu^i \otimes \nu^i : DY\, d\xi^i. \end{aligned}$$

Passing to limits and using $|\xi^i|(\{\phi > 0\}) \to 0$, we find

$$\begin{aligned} \delta\widetilde{V}(Y) &= \int DY(x) : S\, d\widetilde{V}(x, S) \\ &= \lim_{j\to\infty} - \int \varepsilon_{i_j} Y \cdot Du^{i_j} \left(-\Delta u^{i_j} + \frac{1}{\varepsilon_{i_j}^2} f(u^{i_j})\right) dx \end{aligned} \tag{2}$$

and

$$\begin{aligned} |\delta\widetilde{V}(Y)| &\leq |Y| \varlimsup_{i\to\infty} \int_U \varepsilon_i |Du^i| \left|-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right| dx \\ &\leq |Y| \varlimsup_{i\to\infty} \int_U \frac{\varepsilon_i}{2} \frac{|Du^i|^2}{\phi} + \varepsilon_i \phi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx \\ &\leq |Y| \varlimsup_{i\to\infty} (2C(\phi, U)\mu^i(\{\phi > 0\}) + 2C_4 + 4C_1(\phi)\mu^i(\{\phi > 0\})) \\ &\qquad\qquad \text{by §9.4(iv), Lemma 9.3(iv)} \\ &= |Y|(C(\phi, U)\mu(\{\phi > 0\}) + 2C_4). \end{aligned}$$

3. Thus

$$|\delta\widetilde{V}(Y)| \leq C(\phi, U, \mu, C_4)|Y|,$$

which shows $|\delta\widetilde{V}\lfloor\{\phi > 0\}|$ is a Radon measure on $\{\phi > 0\}$. In consequence, by Allard's Rectifiability Theorem [1, 5.5(2)] and hypothesis (v), we have $\widetilde{V}\lfloor\{\phi > 0\} \in \mathbf{RV}_{n-1}(\{\phi > 0\})$. Since $\|\widetilde{V}\| = \mu$, (vi) is established.

4. In fact, by rectifiability the varifold $V = \widetilde{V}\lfloor\{\phi > 0\}$ is uniquely determined by μ, independent of the subsequence, and so $V^i\lfloor\{\phi > 0\} \to V$ as varifolds. This establishes (vii). Now (viii) follows from (2).

5. We will now prove (ix). We adapt Brakke's upper semicontinuity proof [5, 4.28]. First we do the $\int \phi H^2 \, d\mu$ term. Let $\psi \in C_c^2(\{\phi > 0\}, \mathbf{R}^+)$ with $\psi^{1/2} \in C^1$. Due to the rectifiability of μ, we can approximate by smooth functions (see, e.g., [27, 7.4]) to get

$$\left(\int \psi H^2 \, d\mu\right)^{1/2} = \sup\left\{\int \psi^{1/2}\vec{H}\cdot Y \, d\mu \colon Y \in C_c^\infty(\mathbf{R}^n, \mathbf{R}^n), \|Y\|_{L^2(\mu)} \le 1\right\}.$$

Now, using $\xi^i \to 0$ (twice) yields

$$\begin{aligned}
\int \psi^{1/2} Y \cdot \vec{H} \, d\mu &= -\delta V(\psi^{1/2} Y) = -\lim_{i\to\infty} \delta V_i(\psi^{1/2} Y) \\
&= \lim_{i\to\infty} \int \varepsilon_i \psi^{1/2} Y \cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i} f(u^i)\right) dx \\
&\quad + \lim_{i\to\infty} \int \nu^i \otimes \nu^i \colon D(\psi^{1/2} Y) \, d\xi^i \quad \text{by (1)} \\
&\le \varliminf_{i\to\infty} \left(\int \varepsilon_i |Du^i|^2 |Y|^2 \, dx\right)^{1/2} \\
&\quad \cdot \left(\int \varepsilon_i \psi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx\right)^{1/2} \\
&\le \lim_{i\to\infty} \left(\int |Y|^2 \, d\mu^i\right)^{1/2} \\
&\quad \cdot \varliminf_{i\to\infty} \left(\int \varepsilon_i \psi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx\right)^{1/2} \\
&= \|Y\|_{L^2(\mu)} \varliminf_{i\to\infty} \left(\int \varepsilon_i \psi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx\right)^{1/2},
\end{aligned}$$

which shows

$$\int \psi H^2 \, d\mu \le \varliminf_{i\to\infty} \int \varepsilon_i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx.$$

Passing ψ to ϕ by the monotone convergence theorem, we obtain

$$\begin{aligned}
(3) \qquad \int \phi H^2 \, d\mu &\le \varliminf_{i\to\infty} \int \varepsilon_i \phi \left(\Delta u^i + \frac{1}{\varepsilon_i^3} f(u^i)\right)^2 dx \\
&\le C(\phi, \mu, C_4) \quad \text{by §9.4(iv), Lemma 9.3(iv)}.
\end{aligned}$$

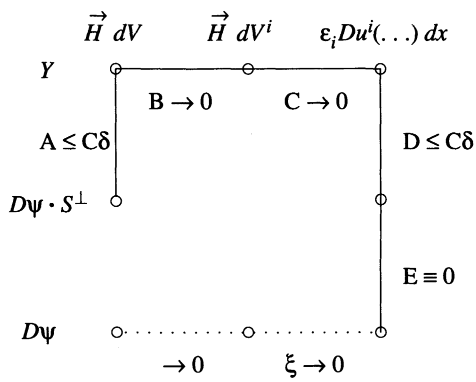

FIGURE 7. SEVEN EPSILON PROOF

6. Next we do the transport term. We will first show that, for $\psi \in C_c^2(\{\phi > 0\}, \mathbf{R}^+)$,

$$\lim_{i\to\infty} \int \varepsilon_i D\psi \cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx = \int D\psi \cdot S^\perp \cdot \vec{H}\, d\mu, \tag{4}$$

where $S = S(x)$ denotes $T_x\mu$. Note that $D\psi(x)\cdot S^\perp(x)$ is a μ-integrable vector field. Since μ is rectifiable, it is possible to choose a vector field $Y \in C_c^1(\{\phi > 0\}\mathbf{R}^n)$ such that

$$\int |Y(x) - D\psi(x)\cdot S^\perp(x)|^2\, d\mu(x) \le \delta^2. \tag{5}$$

Then (see Figure 7)

$$\begin{aligned}
\int D\psi \cdot S^\perp \cdot \vec{H}\, d\mu &= \int (D\psi \cdot S \perp - Y)\cdot \vec{H}\, d\mu && \text{(A)}\\
&\quad - \delta V(Y) + \delta V^i(Y) && \text{(B)}\\
&\quad - \delta V^i(Y) - \int \varepsilon_i Y \cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx && \text{(C)}\\
&\quad + \int \varepsilon_i (Y - (D\psi\cdot\nu^i)\nu^i)\cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx && \text{(D)}\\
&\quad + \int \varepsilon_i ((D\psi\cdot\nu^i)\nu^i - D\psi)\cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx && \text{(E)}\\
&\quad + \int \varepsilon_i D\psi \cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx, && \text{(F)}
\end{aligned}$$

where $\nu^i = Du^i/|Du^i|$ $\mathscr{L}^n$-a.e. By (3) and (5),

$$|\mathrm{A}| \le \delta \left(\int_{\operatorname{spt}\psi} H^2 \, d\mu \right)^{1/2} \le C(\psi, \phi)\delta .$$

By continuity of first variation and (vii), $\lim_{i\to\infty} \mathrm{B} = 0$. By (iii) and (1),

$$\lim_{i\to\infty} \mathrm{C} = \lim_{i\to\infty} \int \nu \otimes \nu : DY \, d\xi^i = 0 .$$

Next

$$\begin{aligned} |\mathrm{D}| &\le \left(\int \varepsilon_i |Du^i|^2 |Y - (D\psi, \nu^i)\nu^i|^2 \, dx \right)^{1/2} \\ &\quad \cdot \left(\sup_{\{\psi>0\}} \left(\frac{1}{\phi}\right) \int_{\operatorname{spt}\psi} \varepsilon_i \phi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i) \right)^2 dx \right)^{1/2} \\ &\le \left(2 \int |Y - D\psi \cdot S^{\perp}| \, dV^i \right)^{1/2} C(\operatorname{spt}\psi, \phi)^{1/2} \\ &\quad \cdot C(\phi, \mu_i(\{\phi > 0\}), C_4)^{1/2} \quad \text{by (iii), §9.4(iv)}. \end{aligned}$$

Thus

$$\lim_{i\to\infty} |\mathrm{D}| \le C(\phi, \psi)\delta .$$

Evidently $\mathrm{E} \equiv 0$ and thus

$$\begin{aligned} \int D\psi \cdot S^{\perp} \cdot \vec{H} \, d\mu &= \mathrm{F} \\ &= \lim_{i\to\infty} \int \varepsilon_i D\psi \cdot Du^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i) \right) dx \pm 2C(\psi, \phi)\delta . \end{aligned}$$

Let $\delta \downarrow 0$ to obtain the desired result (4).

7. Now we will pass (4) to limits to obtain the analogous result for ϕ. Let $\|\phi - \psi\|_{C^2} \le \delta^2$. The error on the left-hand side of (4) is bounded by

$$\begin{aligned} &\varlimsup_{i\to\infty} \left(\int \frac{|D(\phi - \psi)|^2}{\phi - \psi} \varepsilon_i |Du^i|^2 \, dx \right)^{\frac{1}{2}} \left(\int \varepsilon_i (\phi - \psi) \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i) \right)^2 dx \right)^{\frac{1}{2}} \\ &\quad \le (2\mu(\{\phi > 0\}) \sup |D^2(\phi - \psi)|)^{1/2} C(\phi, \mu, C_4)^{1/2} \quad \text{by (3)} \\ &\quad \le C(\phi, \mu, C_4)\delta . \end{aligned}$$

The error on the right-hand side is bounded by a similar quantity. Thus we have proven (4) with ψ replaced by ϕ. This gives (ix), and completes the proof of Lemma 9.3.

9.5. Density Lemma. *Let $\{\mu^i\}_{i\geq 1}$ be a sequence of real $(n-1)$-rectifiable Radon measures. Let $\phi \in C^2_c(\mathbf{R}^n, \mathbf{R}^+)$. Suppose*

(i) $\mu^i \lfloor\{\phi > 0\} \to \mu\lfloor\{\phi > 0\}$ *as Radon measures on* $\{\phi > 0\}$,

(ii) $-C_4 \leq \int -\phi H^2 + D\phi \cdot \vec{H}\, d\mu$ *for* $i \geq 1$,

(iii) $\Theta^{*n-1}(\mu^i, x) \geq \eta$ μ^i*-a.e.*

Then

(iv) $\mu\lfloor\{\phi > 0\}$ *is real* $(n-1)$*-rectifiable,*

(v) $\Theta^{*n-1}(\mu) \geq \eta$ μ*-a.e.*

Proof. By §9.4(i), (ii) we obtain local first variation bounds on the varifolds V^i associated with μ^i. Then Allard's lemma [1, 5.4] on passing density to limits yields a subsequence and a varifold V such that $V^{i_j}\lfloor\{\phi > 0\} \to V$ in $\mathbf{V}_{n-1}(\{\phi > 0\})$, $\|V\| = \mu\lfloor\{\phi > 0\}$, $\|\delta V\|$ is Radon on $\{\phi > 0\}$, and $\Theta^{*n-1}(\mu, \cdot) \geq \eta$ $\mu\lfloor\{\phi > 0\}$-a.e.

Thus by [1, 5.5], V is real $(n-1)$-rectifiable.

The proof of 9.1 is now nearly identical to [26, 7.1], which was derived in turn from [5, Chapter 4].

Proof of Brakke's inequality 9.1. Let $t_0 \geq 0$, $\phi \in C^2_c(\mathbf{R}^n, \mathbf{R}^+)$, and assume without loss of generality

$$-\infty < D_0 \equiv \overline{D}_{t_0}\mu_t(\phi). \tag{1}$$

Then there is a sequence $\delta_q \downarrow 0$ and $t_q \to t_0$ such that

$$D_0 - \delta_q \leq (\mu_{t_q}(\phi) - \mu_{t_0}(\phi))/t_q - t_0.$$

We may assume that $t_q > t$ for all q. (The other case is similar.)

By the convergence $\mu^{\varepsilon_i}_t \to \mu_t$, there is a sequence $r_q \to \infty$ such that

$$D_0 - 2\delta_q \leq \frac{\mu_t^{\varepsilon_{r_q}}(\phi) - \mu_t^{\varepsilon_{r_q}}(\phi)}{t_q - t_0} = \frac{1}{t_q - t_0}\int_{t_0}^{t_q} \frac{d}{dt}\mu_t^{\varepsilon_{r_q}}(\phi)\, dt.$$

Because $|\xi^\varepsilon| \to 0$ on $\mathbf{R}^n \times [0, \infty)$, we can increase r_q if need be to ensure

$$\iint_{\{\phi>0\}\times[t_0, t_q]} d|\xi^{\varepsilon_{r_q}}| \leq \delta_q^2(t_q - t_0). \tag{2}$$

2. Now by Property 5.3, there is $D_1 = D_1(\phi)$ such that

$$\frac{d}{dt}\mu_t^{\varepsilon_i}(\phi) \leq D_1 \quad \text{for } i \geq 1,\ t \geq 0.$$

If

$$Z = \left\{ t \in [t_0, t_q] \colon \frac{d}{dt}\mu_t^{\varepsilon_{r_q}}(\phi) \geq D_0 - 3\delta_q \right\},$$

then we have

$$D_0 - 2\delta_q \le \frac{1}{t_q - t_0} \int_{[t_0, t_q]\setminus Z} (D_0 - 3\delta_q)\, dt + \frac{1}{t_q - t_0} \int_Z D_1\, dt,$$

from which we obtain

$$|Z| \ge \frac{\delta_q}{D_1 - D_0 + 3\delta_q}(t_q - t_0) \ge \frac{\delta_q}{2(D_1 - D_0)}(t_q - t_0)$$

for q large. Thus by (2)

$$|Z| \inf_{t\in Z} \xi_t^{\varepsilon_{r_q}}(\{\phi > 0\}) \le \delta_q^2 (t_q - t_0)$$

and therefore there exists $s_q \in Z \subseteq [t_0, t_q]$ such that

$$D_0 - 3\delta_q \le \left.\frac{d}{dt}\right|_{s_q} \mu_t^{\varepsilon_{r_q}}(\phi) \le \mathscr{B}^{\varepsilon_{r_q}}(u^{\varepsilon_{r_q}}(\cdot, s_q), \phi), \tag{3}$$

and also that

$$\left|\xi_{s_q}^{\varepsilon_{r_q}}\right|(\{\phi > 0\}) \le 2(D_1 - D_0)\delta_q. \tag{4}$$

3. We now pass a subsequence of $\{\mu_{s_q}^{\varepsilon_{r_q}}\}_{fq\ge 1}$ to a limit μ. By applying the semimonotone property 5.3 and the growth bound (1), it is possible to prove (see [26, 7.1]) that

$$\mu \llcorner \{\phi > 0\} = \mu_{t_0} \llcorner \{\phi > 0\}, \tag{5}$$

so that $\mathscr{B}(\mu, \phi) = \mathscr{B}(\mu_{t_0}, \phi)$.

4. On the other hand, by the support lemma 6.3(ii),

$$\mathscr{H}^{n-1}(\operatorname{spt}\mu_{t_0} \cap \{\phi > 0\}) < \infty.$$

Together with (3), (4), (5) we have verified the hypotheses of Lemma 9.3 for the sequences $\{u^{\varepsilon_{r_q}}(\cdot, s_q)\}_{q\ge 1}$ and $\{\mu_{s_q}^{\varepsilon_{r_q}}\}_{q\ge 1}$ on $\{\phi > 0\}$. Therefore from (1), (3), (4), (5), as $q \to \infty$, we obtain

$$\overline{D}_{t_0}\mu_t(\phi) \equiv D_0 \le \mathscr{B}(\mu_{t_0}, \phi).$$

Hence we have proven Brakke's Inequality 9.1.

10. Space-time measure

Let $\{\mu_t\}_{t\ge 0}$ be a Brakke motion. Define

$$d\mu = d\mu_t\, dt, \qquad d\nu = H^2\, d\mu_t\, dt, \quad \nu \ll \mu.$$

We will estimate $\mathscr{H}^n \lfloor \operatorname{spt}\mu$. The estimate is a mild generalization of results in [26, §12] to the current situation.

10.1. Long-Term Clearing-Out Lemma [26, 12.5]. *There is* $\eta_2 = \eta_2(\eta, D) > 0$ *and* $\alpha > 0$ *such that for all* $x \in \mathbf{R}^n$, $\beta \geq \alpha$, $t \geq \beta r^2$, *if*

$$\mu(B_r(x) \times [t - \beta r^2, t]) < \eta_2 \beta r^{n+1},$$
$$\nu(B_r(x) \times [t - \beta r^2, t]) < \eta_2 r^{n-1}/\beta$$

then $(x, t) \notin \operatorname{spt}\mu$.

This proposition implies that if the surface reaches the center of a sphere, then it has had a certain amount either of mass on average or of mass which has been lost in the crossing.

Proof. Fix $\beta \geq \alpha > 0$, $\eta_2 > 0$, $r > 0$, $x \in \mathbf{R}^n$, $t \geq \beta r^2$. By the hypothesis,

$$\int_{t-\beta r^2}^{t} \mu_s(B_r(x))\, ds < \eta_2 \beta r^{n+1}.$$

Thus there is $t_1 \in [t - \beta r^2, t - \beta r^2/2]$ with $\mu_{t_1}(B_r(x)) < 2\eta_2 r^{n-1}$. Fix $\phi \in C_c^1(B_r(x), \mathbf{R}^+)$ with $\phi = 1$ on $B_{r/2}(x)$, $|D\phi| \leq 3/r$. Then for all $s \in [t - \beta r^2/2, t]$,

$$\begin{aligned}
\mu_s(\phi) &\leq \mu_{t_1}(\phi) + \int_{t_1}^{s} \mathscr{B}(\mu_{s'}, \phi)\, ds' \\
&\leq 2\eta_2 r^{n-1} + \int_{t-\beta r^2}^{t} \int_{B_r(x)} |D\phi|\, |\vec{H}|\, d\mu_{s'}\, ds' \\
&\leq 2\eta_2 r^{n-1} + \frac{3}{r} \left(\int_{r-\beta r^2}^{t} \int_{B_r(x)} H^2\, d\mu_{s'}\, ds' \right)^{\frac{1}{2}} \left(\int_{t-\beta r^2}^{t} \mu_{s'}(B_r(x))\, ds' \right)^{\frac{1}{2}} \\
&\leq 2\eta_2 r^{n-1} + \frac{3}{r} \left(\eta_2 \frac{r^{n-1}}{\beta} \right)^{1/2} (\eta_2 \beta r^{n+1})^{1/2} \text{ by hypothesis} \\
&= 5\eta_2 r^{n-1}.
\end{aligned}$$

Now take $\alpha = 1/4$, so $\beta \geq 1/4$; define s by $2(t - s) \equiv (r/2)^2$, so $s \in [t - \beta r^2/2, t]$. By Lemma 3.4(vi) there is $\delta(\gamma)$ with $\lim_{\gamma \downarrow 0} \delta(\gamma) = 0$ such that

$$\begin{aligned}
\int \rho_{x,t}(y, s)\, d\mu_s(y) &= \int \rho_x^{r/2}\, d\mu_s \\
&\leq \frac{1}{\gamma^{n-1}} \frac{\mu_s(B_{r/2})}{\omega_{n-1}(r/2)^{n-1}} + \delta(\gamma) D \leq \frac{5C\eta_2}{\gamma^{n-1}} + \delta(\gamma) D.
\end{aligned}$$

Choose γ; then η_2 small enough so that

$$\int \rho_{x,t}(y, s)\, d\mu_s(y) < \eta .$$

Thus $(x, t) \notin \operatorname{spt}\mu$ by the Clearing-Out Lemma.

10.4. Corollary. $\Theta_*^{n-1}(\mu + \nu, (x, t)) \geq c\eta_2(\eta, D)$ *on* $\operatorname{spt}\mu$.

Proof. By Lemma 10.1, for $\beta \geq \alpha$ and $r > 0$,

$$(\mu + \nu)(B_r(x) \times [t - \beta r^2, t]) \geq \eta_2 \min(\beta r^{n+1}, \beta^{-1} r^{n-1}).$$

Now take $r \leq 1/\alpha$ and $\beta = 1/r \geq \alpha$. q.e.d.

10.5. Corollary. (i) $\mathscr{H}^n \lfloor \operatorname{spt}\mu$ *is locally finite.*

(ii) u^{ε_i} *converges locally uniformly on* $\mathbf{R}^n \times [0, \infty) \setminus \operatorname{spt}\mu$ *to* ± 1.

Proof. Statement (ii) repeats Lemma 6.1(ii). Let us prove (i). Since $\nu \ll \mu$, $\operatorname{spt}\mu = \operatorname{spt}(\mu + \nu)$. Then by Corollary 10.4 and Simon [33, 3.2(1)],

$$\mathscr{H}^n(\operatorname{spt}(\mu + \nu) \cap U) \leq \frac{(\mu + \nu)(U)}{c\eta_2(n, D)}, \tag{1}$$

for $U \subseteq \mathbf{R}^n \times [0, \infty)$ open.

Let us estimate $\nu(B_R \times [T_1, T_2])$. Fix $\phi \in C_c^2(B_{2R})$ with $0 \leq \phi \leq 1$, $\phi = 1$ on B_R, $|D^2\phi| \leq C/R^2$. Then by Lemma 9.3(ii),

$$\begin{aligned}
\nu(B_R \times [T_1, T_2]) &\leq \int_{T_1}^{T_2} \int \phi H^2 \, d\mu_t \, dt \\
&\leq \int_{T_1}^{T_2} -2\mathscr{B}(\mu_t, \phi) + 2 \sup |D^2\phi| \mu_t(\{\phi > 0\}) \, dt \\
&\leq 2\mu_{T_1}(B_{2R}) + \frac{C}{R^2} \int_{T_1}^{T_2} \mu_t(B_{2R}) \, dt .
\end{aligned}$$

Thus by (1),

$$\begin{aligned}
&\mathscr{H}^n(\operatorname{spt}\mu \cap B_R \times [T_1, T_1]) \\
&\quad \leq \frac{1}{c\eta_2(n, D)} \left(2\mu_{T_1}(B_{2R}) + \frac{C}{R^2} \int_{T_1}^{T_2} \mu_t(B_{2R}) \, dt + \int_{T_1}^{T_2} \mu_t(B_R) \, dt \right) \\
&\quad < \infty .
\end{aligned}$$

11. Convergence in BV

We will relate the measures $\{\mu_t\}_{t \geq 0}$ to the limit of the functions u^{ε_i} in BV_{loc}. This was first established by Bronsard and Kohn [7]. We begin by reviewing their argument.

Let us write $\nabla \equiv \nabla^{\mathbf{R}^n \times \mathbf{R}}$ and recall from §1.2

$$G' = g, \quad G(-1) + G(+1) = 0, \quad \alpha = G(1) - G(-1).$$

Then by $(*)$ and Cauchy-Schwartz,

$$\begin{aligned}\iint_{B_R \times [T_1, T_2]} |\nabla G(u^\varepsilon)|\, dx\, dt &= \iint_{B_R \times [T_1, T_2]} g(u^\varepsilon)|\nabla u^\varepsilon|\, dx\, dt \\ &\le \left(\iint \frac{2}{\varepsilon} F(u^\varepsilon)\, dx\, dt\right)^{1/2} \\ &\quad \cdot \left(\iint \varepsilon \left(-\Delta u^\varepsilon + \frac{1}{\varepsilon^2} f(u^\varepsilon)\right)^2 + \varepsilon |Du^\varepsilon|^2\, dx\, dt\right)^{1/2}.\end{aligned}$$

Thus using $\xi^{\varepsilon_i} \to 0$, we get

$$\begin{aligned}&\overline{\lim_{i\to\infty}} \iint_{B_R \times [T_1, T_2]} |\nabla G(u^{\varepsilon_i})|\, dx\, dt \\ &\qquad \le \mu(B_R \times [T_1, T_2])^{1/2} (\mu + \nu)(B_R \times [T_1, T_2])^{1/2},\end{aligned}$$

which was estimated in §10. By a similar argument, for $t \ge 0$

$$\begin{aligned}&\overline{\lim_{i\to\infty}} \int_{B_R} |DG(u^{\varepsilon_i}(x, t))|\, dx \\ &\qquad \le \overline{\lim_{i\to\infty}} \left(\int \frac{2}{\varepsilon_i} F(u^{\varepsilon_i})\, dx\right)^{1/2} \cdot \left(\int \varepsilon_i |Du^{\varepsilon_i}|^2\, dx\right)^{1/2} \\ &\qquad \le 2\mu_t(B_R) < \infty.\end{aligned}$$

These are uniform bounds in $BV_{\mathrm{loc}}(\mathbf{R}^n \times [0, \infty))$, $BV_{\mathrm{loc}}(\mathbf{R}^n \times \{t\})$, respectively.

Now $u^{\varepsilon_i} \to u = \pm 1$ locally uniformly on $\mathbf{R}^n \times [0, \infty) \setminus \operatorname{spt} \mu$, so $G(u^{\varepsilon_i}) \to G(u) = \alpha u/2$ locally uniformly on $\mathbf{R}^n \times [0, \infty) \setminus \operatorname{spt} \mu$. Since $\mathscr{H}^{n+1} \lfloor \mu = 0$, $\mathscr{H}^n \lfloor \operatorname{spt} \mu_t \times \{t\} = 0$ for each t, we have

11.1. ***BV* convergence.**

(i) $G(u^{\varepsilon_i}) \rightharpoonup G(u) = \alpha u/2$ weakly-$*$ *in* $BV_{\mathrm{loc}}(\mathbf{R}^n \times [0, \infty))$.

(ii) $G(u^{\varepsilon_i}(\cdot, t)) \rightharpoonup G(u(\cdot, t)) = \alpha u(\cdot, t)/2$ weakly-$*$ *in* $BV_{\mathrm{loc}}(\mathbf{R}^n \times \{t\})$ *for* $t \ge 0$.

Then for any $\phi \in C_c^2(\mathbf{R}^n, \mathbf{R}^+)$, we calculate

$$\begin{aligned}\frac{\alpha}{2}\int \phi|Du|\,dx &= \int \phi|DG(u)|\,dx \\ &\le \underline{\lim}_{\varepsilon_i\downarrow 0}\int \phi|Dg(u^{\varepsilon_i})|\,dx \\ &\le \underline{\lim}_{\varepsilon_i\downarrow 0}\int \phi\left(\frac{\varepsilon_i^2}{2}|Du^{\varepsilon_i}|^2 + \frac{1}{\varepsilon_i}F(u^{\varepsilon_i})\right)dx \\ &= \int \phi\,d\mu_t.\end{aligned}$$

Thus we obtain

11.2. Relation of μ_t to $|Du(\cdot, t)|$. *For the limit u, $\{\mu_t\}_{t\ge 0}$ defined above, we have $u \in BV_{\mathrm{loc}}(\mathbf{R}^n \times [0, \infty))$, $u(\cdot, t) \in BV_{\mathrm{loc}}(\mathbf{R}^n \times \{t\})$ for all $t \ge 0$, and*

$$(\alpha/2)|Du(\cdot, t)|\,dx \le d\mu_t \qquad \textit{for all } t \ge 0,$$

where $\alpha = \int_{-1}^{1} g(u)\,du$.

The discrepancy is due to weak lower semicontinuity.

We might reasonably hope that the density of μ_t is an integral multiple of α μ_t-a.e. (for a.e. $t \ge 0$), because Brakke proved integrality for the original construction [5, §4]. One-dimensional examples are illuminating here.

12. Remarks on regularity

In his 1978 book [5], Brakke was able to develop a remarkable almost-everywhere regularity theory for k-varifolds $\{V_t\}_{t\ge 0}$ moving weakly by mean curvature. We will discuss informally the application of Brakke's work, via results in [26], to the singular limit of $(*)$. The result is the "generic" regularity theorem 12.2.

In this section we restrict the initial data to be compact, since there is not yet an avoidance principle for noncompact sets moving by mean curvature.

Let us first briefly describe the result of [26]. Starting from the same kind of initial surface as in §1.3 the argument in [26] has two steps.

A. Establish a structure $(T, \{\mu_t\}_{t\ge 0})$ (effectively $(Du, \{\mu_t\}_{t\ge 0})$ of §11.2) moving by mean curvature in Brakke's sense.

B. Under the *Nonfattening Hypothesis* (described below), modify $\{\mu_t\}_{t\ge 0}$ so that it satisfies the *unit density* hypothesis of Brakke's regularity theory.

Step A of [17] uses the approximation scheme of *elliptic regularization.* The present paper, in effect, shows that the Allen-Cahn equation yields an alternate existence scheme for step A. To be precise, we refer to §§9.1, 11.1, 11.2, Corollary 10.5, and by defining $E \equiv \{(x, t) : u(x, t) = 1\} \setminus \operatorname{spt} \mu$ we deduce the following.

12.1. Existence of enhanced pair. *Given an initial surface* $M_0 = \partial E_0$ *strongly approximated by smooth surfaces and with the density bounds of* §1.3, *there exists an enhanced pair moving by mean curvature with initial condition* M_0, *that is, a pair* $(E, \{\mu_t\}_{t\geq 0})$ *satisfying the following*:

(i) $E \subseteq \mathbf{R}^n \times [0, \infty)$ *is an open set of locally finite perimeter, such that*
 (a) $E_0 = E \cap (\mathbf{R}^n \times \{0\})$,
 (b) $E_t \equiv E \cap (\mathbf{R}^n \times \{t\})$ *is of locally finite perimeter for each* $t \geq 0$,
 (c) $\mathscr{H}^n(\partial^* E \cap (B_R \times [t, t+\tau])) \leq C R^{n-1}(\tau^{1/2} + \tau)$ ($C^{1/2}$ *continuity*).

(ii) $\{\mu_t\}_{t\geq 0}$ *is a Brakke motion with* $\mu_0 = \alpha \mathscr{H}^{n-1} \lfloor M_0$.

(iii) $\mu_t \geq \alpha \mathscr{H}^{n-1} \lfloor \partial^* E_t$ *as Radon measures for* $t \geq 0$.

We may then apply step B of [26] to obtain the regularity theorem 12.2. For the convenience of the reader we will briefly sketch the argument of [26].

Brakke's theory of regularity [5, 6.12] relies on the hypothesis of *unit density*, that is, $\Theta^k(\mu_t, \cdot) = 1$, μ_t-a.e. for a.e. $t \geq 0$, where $\mu_t = \|V_t\|$. The measures $\{\mu_t\}_{t\geq 0}$ constructed above are not known to satisfy this hypothesis. To get around this, we proceed as follows.

Given a (compact) initial surface Γ_0, possibly singular, we fill a neighborhood of Γ_0 with disjoint surfaces $\{\Gamma_0^\gamma\}_{\gamma\in(-\delta, \delta)}$ homologous to Γ_0. Let each Γ_0^γ evolve according to the level-set flow of Evans and Spruck and Chen, Giga, and Goto to form families $\{\Gamma_t^\gamma\}_{t\geq 0}$.

By [16, CGG] these families remain disjoint for positive time. Therefore all but countably many of them satisfy the *Nonfattening Hypothesis*:

$$(\dagger) \qquad \mathscr{L}^{n+1}\left(\bigcup_{t\geq 0} \Gamma_t \times \{t\}\right) = 0.$$

Thus Corollary 9.2 implies that for these γ, $\mathscr{L}^{n+1}(\operatorname{spt} \mu) = 0$.

From this, the Constancy Theorem [33, 26.27] and (i), it follows that E is uniquely determined by E_0, independent of the choice of u_0^ε and subsequence $\{\varepsilon_i\}_{i\geq 1}$.

According to the results in [26, 9.1], it is possible to modify $\{\mu_t\}_{t\geq 0}$ so that $\mu_t = \alpha \mathscr{H}^{n-1} \lfloor \partial^* E_t$ for each t and $\{\mu_t\}_{t\geq 0}$ still solves Brakke's inequality (B). Hence by De Giorgi's theorem on sets of finite perimeter, μ_t has density α μ_t-a.e. for $t \geq 0$. Moreover, by [26, Chapter 12] Brakke's theory applies to show:

Any initial surface can be perturbed to one whose evolution is smooth $\mathscr{H}^n$-almost everywhere in spacetime.

The steps of the preceding argument are carried out in [26]. We now state the result precisely.

12.2. Generic-data almost-everywhere regularity theorem for Allen-Cahn limit. Let $h: \mathbf{R}^n \to \mathbf{R}$ be a Lipschitz function with compact level-sets $\Gamma_0^\gamma \equiv h^{-1}(\gamma)$. For each $\gamma \in \mathbf{R}$, let $\{\Gamma_t^\gamma\}_{t\geq 0}$ be the corresponding level-set flow.

Let $I \subseteq \mathbf{R}$ be the set of γ such that $\{\Gamma_t^\gamma\}_{t\geq 0}$ satisfies the initial surface hypotheses §1.3(i), (ii). (We may assume that I is full measure). For $\gamma \in I$, $t \geq 0$ let $\mu_t^\gamma \equiv \lim \mu_t^{\gamma, \varepsilon_i}$ as in §§5.4 and 9.1, and let $u^\gamma = \lim u^{\gamma, \varepsilon_i}$ as in §§11.1 and 11.2.

Define

$$E \equiv \{u = 1\} \setminus \operatorname{spt} \mu, \qquad E_t \equiv E \cap (\mathbf{R}^n \times \{t\}).$$

Then E, E_t are open sets of finite perimeter and $u + 1 = 2\chi_E$, and the following statements are true:

A. *Inclusion* (originally proven by Evans, Soner, and Souganidis [15]). For $\gamma \in I$, $\operatorname{spt} \mu_t^\gamma \subseteq \Gamma_t^\gamma$ for $t \geq 0$.

B. *Generic nonfattening.* For all but countably many γ, $\{\Gamma_t^\gamma\}_{t\geq 0}$ satisfies (†).

C. *Uniqueness.* For $\gamma \in I$ such that (†) holds, the sets E^γ, E_t^γ are uniquely determined independent of all approximations.

D. *Matching.* For $\gamma \in I$ such that (†) holds, define the reduced flow

$$\overline{\mu}_t^\gamma \equiv \alpha \mathscr{H}^{n-1} \lfloor \partial^* E_t^\gamma .$$

Then $\{\overline{\mu}_t^\gamma\}_{t\geq 0}$ satisfies Brakke's inequality (B), and has density α for $\overline{\mu}_t^\gamma$-a.e. x, and all $t \geq 0$.

E. *Almost-everywhere regularity.* For $\gamma \in I$ such that (†) holds, define $d\overline{\mu}^\gamma \equiv d\overline{\mu}_t^\gamma \, dt$. Then

$$\operatorname{spt} \overline{\mu}^\gamma = \overline{\partial^* E^\gamma} = \partial^* E^\gamma \quad \mathscr{H}^n\text{-a.e.},$$

and $\operatorname{spt} \overline{\mu}^\gamma$ is a smooth n-manifold $\mathscr{H}^n$-a.e.

These results follow from Brakke's work via [26, 10.7, 11.4, 12.9] (with [26, 7.1] extended to real varifolds with a lower density bound) together with Brakke's Inequality 9.1 and §11.2. What is new is that they apply to the singular limit of $(*)$.

13. Questions

1. *Sharp density lower bound*: Improve the constant η in Lemma 6.1 to $\eta = \alpha$.

2. *Integral density*: Following Brakke [5, 4] prove that the density of μ_t is an integral multiple of α μ_t-a.e. (for a.e. $t \geq 0$).

3. *Unit density*: Devise hypotheses that imply that the density of μ is $\alpha\mu_t$-a.e.

4. *Strong convergence* for $H > 0$: Following Evans and Spruck [19], show that if $H > 0$ initially on M_0, there is no cancellation in $Du^{\varepsilon_i} \rightharpoonup Du$.

5. *Systems*: Extend to systems $u\colon \mathbf{R}^n \to \mathbf{R}^m$ (multiple phase interfaces, triple junctions).

6. *Vanishing of* ξ: Is there further significance of the equipartition of energy for motion by mean curvature?

Acknowledgments

This work was done at the Regional Geometry Institute at Mt. Holyoke, the Center for Nonlinear Analysis at Carnegie-Mellon, and the Institute for Advanced Study with partial support of NSF Grant DMS-9100383.

I would like to thank J. Taylor, P. D. Smith, and H. M. Soner for illumination, and to express my deepest gratitude to my advisor, L. C. Evans, for much education and advice, including discussion of "broader context."

References

[1] W. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972) 417–491.

[2] S. Allen & J. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metall. **27** (1979) 1084–1095.

[3] F. Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure*, Ann. of Math. (2) **87** (1968) 321–391.

[4] G. Barles, H. M. Soner, & P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim., to appear.

[5] K. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, NJ, 1978.

[6] L. Bronsard & R. V. Kohn, *On the slowness of phase boundary motion in one space dimension*, preprint #13, Lefschetz Center, Brown University, 1989.

[7] ——, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations **90** (1991) 211–237.

[8] J. Carr & R. Pego, *Very slow phase separation in one dimension*, Proc. Conf. on Phase Transitions (M. Rascle, ed.), 1987.

[9] ——, *Metastable patterns in solutions of* $u_t = \varepsilon^2 u_{xx} - f(u)$, Comm. Pure Appl. Math. **42** (1989) 523–576.

[10] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, IMA preprint #637, University of Minnesota, 1990.

[11] Y. Chen, *The weak solutions to the evolution problem of harmonic maps*, Math. Z. **201** (1989) 69–74.

[12] Y. Chen & M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. **201** (1989) 83–103.

[13] Y.-G. Chen, Y. Giga, & S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geometry **33** (1991) 749–786.

[14] E. De Giorgi, *Some conjectures on flow by mean curvature*, white paper, 1990.

[15] L. C. Evans, H. M. Soner, & P. E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math., to appear.

[16] L. C. Evans & J. Spruck, *Motion of level sets by mean curvature*. I, J. Differential Geometry **33** (1991) 635–681.

[17] ——, *Motion of level sets by mean curvature*. II, Trans. Amer. Math. Soc., to appear.

[18] ——, *Motion of level sets by mean curvature*. III, J. Geometry Analysis, to appear.

[19] ——, *Motion of level sets by mean curvature*, IV, preprint.

[20] P. C. Fife, *Dynamics of internal layers and diffusive interfaces*, SIAM, CBMS, 1988.

[21] J. Frölich & M. Struwe, *Variation problems on vector bundles*, preprint, 1990.

[22] G. Fusco, *A geometric approach to the dynamics of* $u_t = \varepsilon^2 u_{xx} + f(u)$ *for small* ε, Proc. Conf. Stuttgart (K. Kirchgassner, ed.), Lecture Notes in Physics #359, 1988.

[23] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geometry **31** (1990) 285–299.

[24] T. Ilmanen, *Generalized motion of sets by mean curvature on a manifold*, Indiana University J. Math. **41** (1992) 671–705.

[25] ——, *The level-set flow on a manifold*, Proc. 1990 Summer Inst. in Differential Geometry (R. E. Greene and S. T. Yau, ed.), Amer. Math. Soc., 1992.

[26] ——, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. No. 1304, 1993.

[27] S. Luckhaus & L. Modica, *Gradient theory of phase transitions with Gibbs-Thompson curvature effects*, Arch. Rational Mech. Anal. **107** (1989) 71–83.

[28] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1986) 123–142.

[29] L. Modica & S. Mortola, *Un esempio di* Γ*-convergenza*, Boll. Un. Mat. Ital. (5) 14-B (1977) 285–299.

[30] P. de Mottoni & M. Schatzman, *Evolution geometrique d'interfaces*, C. R. Acad. Sci. Paris **309** (1989) 453–458.

[31] ——, *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc., to appear.

[32] J. Rubinstein, P. Sternberg & J. B. Keller, *Fast reaction, slow diffusion and curve-shortening*, SIAM J. Appl. Math. **49** (1989) 116–133.

[33] L. Simon, *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[34] H. M. Soner, *Motion of a set by the mean curvature of its boundary*, J. Differential Equations, to appear.

[35] ——, *Ginzburg-Landau equation and motion by mean curvature*, I: *Convergence*, preprint.

[36] M. Struwe, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geometry **28** (1988) 485–502.

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