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COMPACTNESS THEOREMS FOR KÄHLER-EINSTEIN MANIFOLDS OF DIMENSION 3 AND UP

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There has been increasing interest lately in compactness theorems of Riemannian manifolds under various geometric assumptions (see, among others, [3], [10], [1], [7], and [19]). More recently, it has been found that the boundedness condition on the curvature as in [3] and [10] can be replaced by some integral norms of the curvature tensor. One of those often used is the $L^{n/2}$-norm on the curvature tensor, where n is the real dimension of the underlying manifold. For instance, in [1] and [19], the authors show that if $\{(M_i, g_i)\}$ is a sequence of Einstein manifolds of real dimension $2n$ satisfying: (i) $\operatorname{diam}(M_i, g_i) \le \mu$; (ii) $\int_{M_i} \|Rm(g_i)\|_{g_i}^n \, dV_{g_i} \le \mu$; and (iii) $\operatorname{Vol}(M_i, g_i) \ge \frac{1}{\mu}$, where μ is a uniform constant, then the subsequence of $\{(M_i, g_i)\}$ converges to an Einstein orbifold with finitely many isolated singular points. Also see [20] for the case of Kähler-Einstein surfaces. The case that the limit is an orbifold does occur in dimension four (cf. [15], [20]). However, in this paper, we show that it cannot occur for Kähler-Einstein manifolds of higher dimension and nonzero scalar curvature. In order to give our main theorem precisely, we need to introduce some notation first. For any fixed constant $\mu > 0$ and positive integer $n > 0$, denote by $K(\mu, n)$ the set of all Kähler-Einstein manifolds (M, g) of complex dimension n satisfying:

$$\operatorname{diam}(M, g) \le \mu, \tag{0.1}$$

$$\int_M |Rm(g)|_g^n \, dV_g \le \mu, \tag{0.2}$$

$$\operatorname{Vol}_g(M) \ge 1/\mu, \tag{0.3}$$

where $Rm(g)$ denotes the curvature tensor of g. Let $K_+(\mu, n)$ (resp. $K_-(\mu, n)$) be the subset of all (M, g) in $K(\mu, n)$ with $\operatorname{Ric}(g) = \omega_g$ (resp. $\operatorname{Ric}(g) = -\omega_g$), where ω_g is the associated Kähler form of g. We should point out that the diameters of the manifolds in $K_+(\mu, n)$ are

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bounded from above by a constant depending only on n.

Our first main theorem is stated as follows:

Theorem 1. *$K_+(\mu, n)$ (resp. $K_-(\mu, n)$ is compact for $n \geq 3$.*

A related problem is the classification of complete Ricci-flat Kähler manifolds with bounded L^n-norm of the curvature tensor. The examples of such manifolds can be constructed in the following way (cf. [21], [25]). Let $\Gamma \subset \mathrm{SU}(n)$ be a finite group acting on C^n with the origin as its unique fixed point. We further assume that C^n/Γ admits a resolution M such that the push-down of $dz_1 \wedge \cdots \wedge dz_n$ on C^n can be extended nonvanishingly across the exceptional divisor, in other words, the canonical line bundle K_M is trivial. Note that this assumption is automatically true in the case $n \leq 3$. Then M has a complete Ricci-flat Kähler metric with bounded L^n-norm of the curvature. In the case $n = 2$, it was proved before by Hitchin and P. Kronheimer using a different method ([13], [17]).

Theorem 2. *Let (M, g) be a complete Ricci-flat Kähler manifold with the L^n-norm of its curvature tensor bounded. Then M is a resolution of C^n/Γ for some $\Gamma \subset \mathrm{SU}(n)$ with K_M trivial.*

The organization of this paper is as follows. In §1, we recall that for any sequence of Kähler-Einstein manifolds in either $K_+(\mu, n)$ or $K_-(\mu, n)$, a subsequence of it converges to a Kähler-Einstein orbifold in the sense of Cheeger-Gromov (cf. Theorem 1.1). We include an outlined proof of it here following the arguments in §3 of [20]. In §2, we prove the continuity of the dimensions of plurianticanonical or pluricanonical divisors under the convergence of Kähler-Einstein manifolds in Cheeger-Gromov's sense. The basic analytic tool is Hörmander's L^2-estimate for $\bar\partial$-operators. We will also discuss some corollaries of this continuity result. In §3, using Kohn's estimate for $\bar\partial_b$-operators on strongly pseudoconvex CR-manifolds, we study the local structure of the Kähler-Einstein orbifold M_∞ being the limit of Kähler-Einstein manifolds. In particular, we prove that M_∞ is in fact a manifold. §4 contains the proof of Theorem 2. In §5, we complete the proof of Theorem 1 based on the discussions in the previous sections.

The key idea of this paper occurred to the author during his attendance in Professor J. Kohn's class in Princeton University when he was visiting there. The author would like to express his gratitude to both the institute and Professor Kohn.

1. Convergence to Kähler orbifolds

An n-dimensional complex orbifold M is a topological space satisfying: (1) each point x in M admits an open neighborhood U_x homeomorphic to D^n/Γ_x, where D^n is the unit disc in C^n, and $\Gamma_x \subset U(n)$ is a finite group; and (2) those U_x are patched together by biholomorphic transition functions. Any point x with Γ_x trivial is called a regular point of M. In particular, M is a manifold near such a regular point. Denote by M_{reg} the set of all regular points. All other points are singular points of M, i.e., $\mathrm{Sing}(M) = M\backslash M_{\mathrm{reg}}$. We will confine ourselves to the special case that $\mathrm{Sing}(M)$ consists of isolated points, although it is not necessary for the following discussions. A Kähler metric is just the one on M_{reg} such that for each x in $\mathrm{Sing}(M)$, if $\psi_x : D^n \to U_x$ is the local uniformization, then $\psi_x^* g$ can be extended across the origin.

Now suppose g be a Kähler orbifold metric on M. In the case $\mathrm{Ric}(g) = \lambda\omega_g$ on M for some constant λ, we call (M, g) a Kähler-Einstein orbifold metric.

Theorem 1.1. *Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein manifolds in either $K_+(\mu, n)$ or $K_-(\mu, n)$. By taking a subsequence of it, we may assume that (M_i, g_i) converges to Kähler-Einstein orbifold (M_∞, g_∞) in Cheeger-Gromov's sense, that is, there are finitely many points $x_{i1}, \cdots, x_{iN}$ in M_i, and $x_{\infty 1}, \cdots, x_{\infty N}$ in M_∞, where N is a positive integer depending only on n, μ such that, for any $r > 0$, there are diffeomorphisms ϕ_i from $M_i\backslash \bigcup_{\beta=1}^N B_r(x_{i_\beta}, g_i)$ into M_∞ with $K_r = M_\infty \backslash \bigcup_{\beta=1}^N B_{5r}(x_{\infty\beta}, g_i)$ in the image and satisfying:*

(1) *in the C^5-topology, $(\phi_i^{-1})^* g_i$ converges to g_∞ uniformly on K_r;*

(2) *in the C^5-topology, $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converges to J_∞ uniformly on K_r, where J_i, J_∞ are the almost complex structures of M_i, M_∞, respectively.*

Theorem 1.1 can be derived from the compactness theorem stated in [1] or [19] (see also [20] for the special case of Kähler-Einstein surface). But for the reader's convenience, we outline a proof of it here. For simplicity, we may assume (M_i, g_i) so in $K_+(\mu, n)$ for all i. The key analytic tool is Uhlenbeck's Yang-Mills estimate for curvatures of Yang-Mills connections.

Lemma 1.1. *Let (M_i, g_i) be a Kähler-Einstein manifold given as in Theorem 1.1. Then there are uniform constants C', C'', depending only on the upper bound of n and μ, such that for any f in $C^1(M_i, R)$*

$$
\begin{aligned}
C' \left(\int_{M_i} |f|^{2n/(n-1)} \, dV_{g_i} \right)^{(n-1)/n} &- C'' \int_{M_i} |f|^2 \, dV_{g_i} \\
&\le \int_{M_i} |\nabla f|^2 \, dV_{g_i},
\end{aligned}
\tag{1.1}
$$

where ∇f denotes the gradient of f.

Proof. This follows from a combination of results in C. Croke [5] and P. Li [18].

Lemma 1.2. *Let N be the integer $[\mu/(C')^n]+1$, where C' is the Sobolev constant given in (1.1), and $[a]$ denotes the integer part of the real number a. Then there is a universal constant $C \geq 0$, such that for any $r \in (0, 1)$ and any Kähler-Einstein manifold (M_i, g_i) as in Theorem 1.1, there are finitely many points $x_{i1}^r, \cdots, x_{in}^r$ in M_i such that for any $x \in M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}^r, g_i)$,*

$$(1.2) \qquad \|R(i)\|_{g_i}(x) \leq \frac{C}{r^n}\left(\int_{B_{r/4}(x, g_i)} \|R(i)\|_{g_i}^2(x)\, dV_{g_i}\right)^{1/2},$$

where $B_r(x_{i\beta}^r, g_i)$ is the geodesic ball with radius r and center at $x_{i\beta}^r$, and $\|R(i)\|_{g_i}$ is the norm of $R(i)$ with respect to g_i.

Proof. A straightforward computation shows

$$(1.3) \qquad -\Delta_{g_i}(\|R(i)\|_{g_i}) \leq \|R(i)\|_{g_i} + C(n)(\|R(i)\|_{g_i})^2,$$

where Δ_{g_i} is the laplacian of g_i, and $C(n)$ is a positive constant depending only on n, whose actual value is not important to us. Define

$$(1.4) \qquad \mathrm{E}_i = \left\{ x \in M_i \,\middle|\, \int_{B_{r/4}(x, g_i)} \|R(i)\|_{g_i}^2\, dV_{g_i} \geq \varepsilon \right\}.$$

Then by the well-known covering lemma, E_i can be covered by N geodesic balls of radius $\frac{r}{2}$. Take $x_{i1}^r, \cdots, x_{iN}^r$ to be the centers of these balls. Then for any $x \in M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}^r, g_i)$,

$$(1.5) \qquad \int_{B_{r/4}(x, g_i)} \|R(i)\|_{g_i}^n\, dV_{g_i} \leq \varepsilon.$$

Let $\eta\colon R_+^1 \to R_+^1 = \{t \in R^1 | t \geq 0\}$ be a cut-off function satisfying $\eta \equiv 1$ for $t \leq 1$, and $\eta \equiv 0$ for $t \geq 2$ and $|\eta'(t)| \leq 1$.

For any $x \in M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}^r, g_i)$, denote by $\rho_x(\cdot)$ the distance function on M_i from x.

Put $f = \|R(i)\|_{g_i}$. Multiplying $\eta^2(8\rho_x/r)f$ on both sides of (1.3) and then integrating by parts, one obtains

$$(1.6) \qquad \begin{aligned} &\int_{M_{-i}} |\nabla(\eta f)|^2\, dV_{g_i} \\ &\quad \leq \int_{M_i} \eta^2 f^2\, dV_{g_i} + \int_{M_i} |\nabla\eta|^2 f^2\, dV_{g_i} + \int_{M_i} \eta^2 f^3\, dV_{g_i}. \end{aligned}$$

By Lemma 1.1 and Hölder's inequality,

$$
\begin{aligned}
& C'\left(\int_{M_i}|\eta f|^{2n/(n-1)}\,dV_{g_i}\right)^{(n-1)/n}-C''\int_{M_i}|\eta f|^2\,dV_{g_i} \\
(1.7)\qquad & \le \int_{M_i}\left(\eta^2+\frac{64|\eta'|^2}{r^2}\right)|f|^2\,dV_{g_i} \\
& +\left(\int_{M_i}|\eta f|^n\,dV_{g_i}\right)^{1/n}\left(\int_{B_{r/4}(x\,,\,g_i)}|f|^{2n/(n-1)}\,dV_{g_i}\right)^{(n-1)/n}.
\end{aligned}
$$

Therefore, for some constant $C \ge 0$ depending only on n, we have
(1.8)

$$
\left(\int_{B_{r/8}(x\,,\,g_i)}|f|^{2n/(n-1)}\,dV_{g_i}\right)^{(n-1)/n} \le \frac{C}{r^2(C'-\sqrt{\varepsilon})}\int_{B_{r/4}(x\,,\,g_i)}|f|^2\,dV_{g_i}.
$$

Similarly, by multiplying $\eta^2 f^{(n+1)/(n-1)}$ on both sides of (1.3) and processing as above, we have

$$
\begin{aligned}
(1.9)\qquad & \left(\int_{B_{r/16}(x\,,\,g_i)}|f|^{2(n/(n-1))^2}\,dV_{g_i}\right)^{(n-1)/n} \\
& \le \frac{C}{r^2(\frac{n-1}{2n}C'-\sqrt{\varepsilon})}\int_{B_{r/8}(x\,,\,g_i)}|f|^{2n/(n-1)}\,dV_{g_i}.
\end{aligned}
$$

Let $\varepsilon \le ((n-1)/4n)^{2k}(C')^2$ and choose k satisfying $(n/(n-1))^k \ge n$. Continuing the above processes k times, we obtain

$$
\begin{aligned}
& \left(\int_{B_{r/2^k}(x\,,\,g_i)}|f|^{2(n/(n-1))^k}\,dV_{g_i}\right)^{((n-1)/n)^k} \\
(1.10)\qquad & \le \frac{C}{r^{n(1-((n-1)/n)^k)}}\left(\int_{B_{r/4}(x\,,\,g_i)}|f|^2\,dV_{g_i}\right)^{1/2}.
\end{aligned}
$$

Then (1.2) follows from Moser's iteration as in the proof of Theorem 8.17 in [16]. q.e.d.

We further observe that we may take the set $\{x_{i1}^{r/4}, \cdots, x_{iN}^{r/4}\}$ contained in the union of the balls $B_r(x_{i\beta}^r, g_i)$. Let $\{r_j\}_{j\ge 1}$ be a decreasing sequence of positive numbers such that $r_1 \le \frac{1}{4}$, $r_j \le r_{j-1}/4$. If we write $x_{i\beta}^j$ as $x_{i\beta}^{r_j}$

and define

$$\Omega_i^j = M_i \setminus \bigcup_{\beta=1}^{N} B_{2r_j}(x_{i\beta}^j, g_i), \tag{1.11}$$

then

$$\overline{\Omega}_i^j \subseteq \Omega_i^{j+1}\left(\frac{r_{j+1}}{8}\right) \quad \text{and} \quad \bigcup_{j\geq 1}^{j} \Omega_i^j = M_i \setminus \{x_{i1}, \cdots, x_{iN}\},$$

where $x_{i\beta} = \lim_{j\to\infty} x_{i\beta}^j$, and for any $1 \leq \beta \leq N$,

$$\Omega_i^{j+1}(\varepsilon) = \{x \in \Omega_i^{j+1} \mid \mathrm{dist}_{g_i}(x, \partial\Omega_i^{j+1}) > \varepsilon\}.$$

The following lemma is essentially a special case of the famous Gromov's compactness theorem (cf. [10], [12]).

Lemma 1.3. *Let $\{(X_i, h_i)\}$ be a sequence of n-dimensional Kähler-Einstein manifolds (maybe noncompact), and Ω_i a sequence of domains in X_i with boundary $\partial\Omega_i$. Suppose the following for all i:*

(i) *The norm $\|R(h_i)\|_{h_i}(x)$ of the bisectional curvatures $R(h_i)$ are uniformly bounded for x in Ω_i.*

(ii) $\mathrm{InjRad}(x) \geq c_i$ *for $x \in \Omega_i$ and for some constant depending only on i.*

(iii) $0 \leq C' \leq \mathrm{Vol}_{h_i}(\Omega_i) \leq C''$ *for some uniform constants C', C''.*

Then given any $\varepsilon > 0$, there is a subsequence $\{\Omega_{i_k}(\varepsilon), h_{i_k}\}_{k\geq 1}$ of Kähler-Einstein manifolds $\{\Omega_i(\varepsilon), h_i\}_{i\geq 1}$, where $\Omega_i(\varepsilon) = \{x \in \Omega_i \mid \mathrm{dist}_{h_i}(x, \partial\Omega_i) > \varepsilon\}$, and a Kähler-Einstein manifold $(\Omega_\infty(\varepsilon), h_\infty)$ such that for the compact subset $K \subset \Omega_\infty(\varepsilon)$, there is an $\varepsilon' > \varepsilon$ such that for k sufficiently large, there are diffeomorphisms ϕ_k of $\Omega_{i_k}(\varepsilon')$ into $\Omega_\infty(\varepsilon)$ satisfying:

(1) $K \subset \phi_k(\Omega_{i_k}(\varepsilon'))$ *for any $k \geq 1$,*

(2) $(\phi_k^{-1})^* h_i$ *converges uniformly to h_∞ on K,*

(3) $(\phi_k)_* \circ J_i \circ (\phi_k^{-1})_*$ *converges uniformly to J_∞ on K, where J_i, J_∞ are the almost complex structures of Ω_i, $\Omega_\infty(\varepsilon)$, respectively.*

Proof. By some standard computations and the assumption that the (X_i, h_i) are Kähler-Einstein manifolds, the bisectional curvature tensor $R(h_i)$ satisfies a quasi-linear elliptic system. The assumptions (i), (ii), and (iii) imply that the Sobolev inequalities hold on $\Omega_i(\varepsilon)$ with uniform Sobolev constants. It follows from some well-known elliptic estimates (cf. [27]) that

$$(1.12) \qquad \|D^l R(h_i)\|_{h_i}(x) \le C(l), \qquad l = 1, 2, \cdots, \infty,$$

where $D^l R(h_i)$ denotes the lth covariant derivative of $R(h_i)$ on Ω_i, and the $C(l)$ are uniform constants depending only on l. Then by Gromov's compactness theorem ([10], [12]), there is a subsequence $\{(\Omega_{i_k}(\varepsilon), h_{i_k})\}$ and a Riemannian manifold $(\Omega_\infty(\varepsilon), h_\infty)$ such that the above (1) and (2) hold. Let K be any compact subset in $\Omega_\infty(\varepsilon)$, and ϕ_k defined as in the statement of this proposition. For the almost complex structure J_i on Ω_i, it is clear that $(\phi_k)_* \circ J_{i_k} \circ (\phi_k^{-1})_*$ is almost complex on K. By taking the subsequence of $\{i_k\}$, we may assume that $(\phi_k)_* \circ J_{i_k} \circ (\phi_k^{-1})$ converges on K. Since K is arbitrary, we obtain an almost complex structure J_∞ on $\Omega_\infty(\varepsilon)$. It is easy to check that this J_∞ is integrable, and h_∞ is a Kähler-Einstein metric with respect to this J_∞. q.e.d.

Since $\operatorname{diam}(M_i, g_i) \le \mu$ and $\operatorname{Vol}(M_i, g_i) \ge \frac{1}{\mu}$ for all i, by an estimate on the injectivity radius in [4], one can prove that assumptions (i)–(iii) in Lemma 1.3 are fulfilled by (Ω_i^j, g_i), $i, j \ge 1$. Therefore, we have a sequence of open Kähler-Einstein manifolds $(\Omega_\infty^j, g_\infty^j)$. Furthermore, one can identify Ω_∞^j naturally with a subdomain in Ω_∞^{j+1} such that the restriction of g_∞^{j+1} to Ω_∞^j coincides with g_∞^j. Therefore the $\{(\Omega_\infty^j, g_\infty^j)\}$ can be glued together to be a Kähler-Einstein manifold (M'_∞, g_∞). By Fatou's lemma,

$$\int_{M'_\infty} \|Rm(g_\infty)\|_{g_\infty}^n \, dV_{g_\infty} \le \mu.$$

Also, it follows from the Volume Comparison Theorem [2] that M'_∞ has only finitely many connected components.

Let ρ_i be the distance function on $M_i \times M_i$ induced by g_i, and let ρ_∞ be the limit of ρ_i. Obviously, ρ_∞ is Lipschitz on $M_\infty = M'_\infty$. According to [10], one may attach finitely many points $x_{\infty 1}, \cdots, x_{\infty N}$ to M'_∞ such that $M_\infty = M'_\infty \cup \{x_{\infty 1}, \cdots, x_{\infty N}\}$ becomes a compact length space with length function ρ_∞ extending that ρ_∞ on $M'_\infty \times M'_\infty$. We need to give a Kähler orbifold structure on M_∞.

Lemma 1.4. *There is a decreasing positive function* $\varepsilon(r)$, *satisfying* $\lim_{r\to\infty} \varepsilon(r) = 0$ *such that for any point* x *in* M'_∞, *we have*

$$\|Rm(g_\infty)\|(x) \le \frac{\varepsilon(r(x))}{r^2(x)},$$

where $r(x) = \min\{\rho_\infty(x_{\infty j}, x) | 1 < j \le N\}$.

This is simply a corollary of Lemma 1.2. Using the trick of blowing up and the curvature estimate in Lemma 1.4, one can endow M_∞ with a topological orbifold structure at $x_{\infty\beta}$ $(1 \le \beta \le N)$. Precisely, for each β, there is an open neighborhood U_β of $x_{\infty\beta}$ such that each connected component $U_{\beta j}$ $(1 \le j \le l_\beta)$ of $U_\beta \cap M'_\infty$ is covered by a smooth manifold $\widetilde{U}_{\beta j}$ diffeomorphic to the punctured ball D^*_r in C^n. The covering group $\Gamma_{\beta j}$ is isomorphic to a finite group in $U(n)$. Moreover, let $\phi_{\beta j}$ be the diffeomorphism from D^*_r onto $\widetilde{U}_{\beta j}$ and let $\pi_{\beta j}\colon \widetilde{U}_{\beta j} \to U_{\beta j}$ be the covering map. Then $\phi^*_{\beta j} \circ \pi^*_{\beta j} g_\infty$ extends to be a C^0-metric on D^n_r, where $D^n_r = \{x | \exists C^n, |x| < r\}$, $D^*_r = D^n_r \backslash \{0\}$. We refer readers to §3 in [20] for the details of its proof.

In order to obtain a Kähler orbifold structure on M_∞, we have to prove that the curvature tensor $Rm(g_\infty)$ is in fact bounded. From Lemma 1.4 follow the topological orbifold structure of M_∞ and the analogy of Uhlenbeck's removable singularity theorem [27]. In §4 of [20], this boundedness of $Rm(g_\infty)$ is proved for surfaces, i.e., for $n = 2$. However, the whole argument can be generalized to higher dimensions without substantial change. Next, as the author did in Lemma 4.4 and 4.5 of [20], one can construct a diffeomorphism ψ from D^*_r into itself such that $\psi^* \circ \phi^*_{\beta j} \circ \pi^*_{\beta j} g_\infty$ extends smoothly across the origin, where $\phi_{\infty j}$ and $\pi_{\beta j}$ are the same as in last paragraph. Therefore, (M_∞, g_∞) is a Kähler-Einstein orbifold with $\mathrm{Ric}(g_\infty) = \omega_{g_\infty}$.

Note that M_∞ is in fact connected (cf. [20]). However, we do not need this fact in the following arguments, and the sketched proof of Theorem 1.1 is finished.

2. Convergence of pluricanonical or plurianticanonical divisors

Let $\{(M_i, g_i)\}_{i \ge 1}$ be a sequence of Kähler-Einstein manifolds in either $K_+(\mu, n)$ or $K_-(\mu, n)$. By Theorem 1.1, we may assume that (M_i, g_i) converges to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense of Cheeger-Gromov. In this section we will apply the L^2-estimate for $\overline{\partial}$ operators to show the convergence of $H^0(M_i, K_{M_i}^{-m})$ to $H^0(M_\infty, K_{M_\infty}^{-m})$ for any integer m as (M_i, g_i) approaches (M_∞, g_∞). Recall that M_∞ is a Kähler orbifold with only isolated quotient singularities.

A line bundle L on M_∞ is a line bundle on the regular part M'_∞ such that for each local uniformization $\pi_x\colon \widetilde{U}_x \to M_\infty$ of a singular point x, the pullback $\pi_x^* L$ on $\widetilde{U}_x \backslash \pi^{-1}(x)$ can be extended to the whole $\widetilde{U}_x$. The natural line bundles on M_∞ are pluricanonical and plurianticanonical ones $K^m_{M_\infty}$ $(m \in Z)$. A global section of $K^m_{M_\infty}$ is an element in $H^0(M'_\infty, K^m_{M_\infty})$, which can be extended across the singular set in the above sense. Then $H^0(M_\infty, K^m_{M_\infty})$ is just the linear space of all the global sections of $K^m_{M_\infty}$. Note that the metric g_∞ induces natural hermitian orbifold metrics on $K^m_{M_\infty}$.

Lemma 2.1. *Let $\{(M_i, g_i)\}$ be the sequence of Kähler-Einstein manifolds given at the beginning of this section and let S^i be a global holomorphic section in $H^0(M_i, K^{-m}_{M_i})$ with $\int_{M_i} \|S^i\|^2_{g_i} dV_{g_i} = 1$, where m is a fixed positive integer. Then there is a subsequence $\{i_k\}$ of $\{i\}$ such that the sections S^{i_k} converge to a global holomorphic section S^∞ in $H^0(M_\infty, K^{-m}_{M_\infty})$. In particular, if $\{S^i_\beta\}_{0\le\beta\le N_m}$ is an orthogonal basis of $H^0(M_i, K^{-m}_{M_i})$ with respect to the induced inner product by g_i, then by taking a subsequence, we may assume that $\{S^i_\beta\}_{0\le\beta\le N_m}$ converges to an orthonormal basis of a subspace in $H^0(M_\infty, K^{-m}_{M_\infty})$, where $N_m + 1 = \dim_C H^0(M_i, K^{-m}_{M_i})$.*

Remark. Before we prove this lemma, we should justify the meaning of the convergence of $\{S^i\}$ in the above lemma since these sections are no longer on the same Kähler manifold. Recall that for any compact subset $K \subset M_\infty \backslash \operatorname{Sing}(M_\infty)$, there are diffeomorphisms ϕ_i from compact subsets $K_i \subset M_i$ onto K such that $(\phi_i^{-1})^* g_i$ and $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converge to g_∞ and J_∞ on K, respectively. Now with ϕ_i as above, we can push the sections S^i down to the sections $\phi_{i*}(S^i)$ of $\bigotimes^m(\Lambda^n(TM_\infty \oplus \overline{TM_\infty}))$ on K. The convergence in Lemma 2.1 means that for any compact subset K of $M_\infty \backslash \operatorname{Sing}(M_\infty)$ and ϕ_i as above, the sections $\phi_{i_k *}(S^{i_k})$ converge to a section S^∞ of $K^{-m}_{M_\infty}$ on K in the C^∞-topology. Note that the limit S^∞ is automatically holomorphic.

Proof of Lemma 2.1. Let Δ_i be the laplacian of the metric g_i. Then by a direct computation, we have

$$\Delta_i(\|S^i\|^2_{g_i})(x) = \|D_i S^i\|^2_{g_i}(x) - nm\|S^i\|^2_{g_i}(x), \tag{2.1}$$

where D_i is the covariant derivative with respect to g_i. Since $\int_{M_i} \|S^i\|^2_{g_i}(x)\, dV_{g_i} = 1$, by Lemma 1.1 and applying Moser's iteration to (2.1), there is a constant $C(n, m)$ depending only on m such that

$$\sup_{M_i}(\|S^i\|^2_{g_i}(x)) \le C(n, m). \tag{2.2}$$

Let K be a compact subset in $M_\infty \setminus \mathrm{Sing}(M_\infty)$, and ϕ_i the diffeomorphism from K_i onto K as in the above remark. To prove the lemma, it suffices to show

$(*)$: for any integer $l > 0$, the lth covariant derivatives of $\phi_{i*}(S^i)$ with respect to g_∞ are bounded in K by a constant C_l' depending only on l and K.

There is an $r > 0$, depending only on K, such that for any point x in K_i, the geodesic ball $B_r(x, g_i)$ is uniformly biholomorphic to an open subset in C^n. On each $B_r(x, g_i)$, the section S_i is represented by a holomorphic function $f_{i,x}$. By (2.1), the function $f_{i,x}$ is uniformly bounded. Therefore, by the well-known Cauchy integral formula, one can easily prove that at x the lth covariant derivative of S^i is uniformly bounded by a constant depending only on l, K. $(*)$ follows since $(\phi_i^{-1})^* g_i$ uniformly converges to g_∞ in K. Hence the lemma is proved. q.e.d.

The following proposition can be easily proved by modifying the proof of [14, p. 92, Theorem 4.4.1] with the use of the Bochner-Kodaira Laplacian formula (see, e.g., [16]).

Proposition 2.1. *Suppose that (X, g) is a complete Kähler orbifold of complex dimension n, L a line bundle on X with the hermitian orbifold metric h, and ψ a function on X which can be approximated by a decreasing sequence of smooth functions $\{\psi_l\}_{1 \le l < +\infty}$. If, for any tangent vector v of type $(1, 0)$ at any point of X and for each l,*

$$\left\langle \partial\bar\partial\psi_l + \frac{2\pi}{\sqrt{-1}}(\mathrm{Ric}(h) + \mathrm{Ric}(g)), v \wedge \bar v \right\rangle_g \ge C\|v\|_g^2, \tag{2.3}$$

where C is a constant independent of l, and $\langle\ ,\ \rangle_g$ is the inner product induced by g, then for any C^∞ L-valued $(0, 1)$-form w on X with $\bar\partial w = 0$ and $\int_X \|w\|^2 e^{-\psi}\, dV_g$ finite, there exists a C^∞ L-valued function u on X such that $\bar\partial u = w$ and

$$\int_X \|u\|^2 e^{-\psi}\, dV_g \le \frac{1}{C}\int_X \|w\|^2 e^{-\psi}\, dV_g, \tag{2.4}$$

where $\|\cdot\|$ is the norm induced by h and g.

Lemma 2.2. *Any section S in $H^0(M_\infty, K_{M_\infty}^{-m})$ is the limit of some sequence $\{S^i\}$ with S^i in $H^0(M_i, K_{M_i}^{-m})$. In particular, this implies that the dimension of $H^0(M_\infty, K_{M_\infty}^{-m})$ is the same as that of $H^0(M_i, K_{M_i}^{-m})$, that is, plurianticanonical dimensions are invariant under the degeneration of Kähler-Einstein manifolds in either $K_+(\mu, n)$ or $K_-(\mu, n)$.*

Proof. We may assume that $\int_{M_\infty} \|S\|^2_{g_\infty}(x)\, dV_{g_\infty} = 1$. Let $\{r_i\}$ be a sequence of positive numbers with $\lim_{i\to\infty} r_i = 0$ such that for each i, there is a diffeomorphism ϕ_i from $M_i \setminus \bigcup_{\beta=1}^N B_{r_i}(x_{i\beta}, g_i)$ into $M_\varepsilon \setminus \mathrm{Sing}(M_\infty)$ as given in Theorem 1.1, where N is defined in Lemma 1.2, and $x_{i\beta}$ are defined in (1.3). Then ϕ_i satisfies the following facts:

(1) $\lim_{i\to\infty}(\mathrm{Im}(\phi_i))$ is just $M_\infty \setminus \mathrm{Sing}(M_\infty)$,

(2) $(\phi_i^{-1})^* g_i$ uniformly converges to g_∞ on any compact subset of $M_\infty \setminus \mathrm{Sing}(M_\infty)$ in the C^∞-topology,

(3) $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converges to J_∞, where J_i, J_∞ are the almost complex structures on M_i, M_∞, respectively.

Define a cut-off function $\eta: R^1 \to R^1_+$ satisfying $\eta(t) = 0$ for $t \le 1$, and $\eta(t) = 1$ for $t \ge 2$ and $|\eta'| \le 1$. Also let π_i be the natural projection from the bundle $\bigotimes^m(\Lambda^n(TM_i \oplus \overline{TM_i}))$ onto $K_{M_i}^{-m} = \bigotimes^m(\Lambda^n TM_i)$. For each i, we have a smooth section $v_i = \eta(\rho_i(x)/2r_i) \cdot \pi_i((\phi_i^{-1})_* S)$ of $K_{M_i}^{-m}$ on M_i, where $\rho_i(x)$ is a Lipschitz function defined by $\rho_i(x) = \min_{1\le\beta\le N}\{\mathrm{dist}_{g_i}(x, x_{i\beta})\}$. Then by facts (2) and (3) above, there is a decreasing function $\varepsilon_3(r)$ on r with $\lim_{r\to 0}\varepsilon_3(r) = 0$ such that

$$\sup\left\{ \|\overline{\partial}_i \pi_i((\phi_i^{-1})_* S)\|_{g_i}(x) | x \in M_i \setminus \bigcup_{\beta=1}^N B_{2r_i}(x_{i\beta}, g_i) \right\} \le \varepsilon_3(r_i), \tag{2.5}$$

$$\left| \int_{M_i} \|v_i\|^2_{g_i}(x)\, dV_{g_i} - 1 \right| \le \varepsilon_3(r_i), \tag{2.6}$$

where $\overline{\partial}_i$ is the corresponding $\overline{\partial}$- operator on M_i.

By (2.5), we have

$$\begin{aligned} \int_{M_i} \|\overline{\partial}_i v_i\|^2_{g_i}(x)\, dV_{g_i} &\le \varepsilon_3(r_i) \mathrm{Vol}_{g_i}(M_i) \\ &\quad + \sum_{\beta=1}^N \int_{B_{4r_i}(x_{i\beta}, g_i)} \left\| \overline{\partial}_i \left(\eta\left(\frac{\rho_i}{2r_i}\right)\right) \cdot \pi_i((\phi_i^{-1})_* S) \right\|^2_{g_i} (x)\, dV_{g_i} \\ &\le \varepsilon_3(r_i) \mathrm{Vol}_{g_i}(M_i) \sum_{\beta=1}^N \frac{1}{4r_i^2} \mathrm{Vol}(B_{4r_i}(x_{i\beta}, g_i)) \\ &\quad \times \sup\left\{ \|(\phi_i^{-1})_* S\|^2_{g_i}(x) | x \in M_i \Big\backslash \bigcup_{\beta=1}^N B_{2r_i}(x_{i\beta}, g_i) \right\}. \end{aligned} \tag{2.7}$$

As in the proof of Lemma 2.1, one may bound $\sup_{M_\infty}(\|S\|^2_{g_\infty}(x))$ by the constant $C(n, m)$ in (2.2). Thus by (2.7), the Volume Comparison Theorem, and the convergence of $(\phi_i^{-1})^* g_i$ in fact (2) above, there is a constant C independent of i such that

$$(2.8) \qquad \int_{M_i} \|\overline{\partial}_i v_i\|^2_{g_i}(x)\, dV_{g_i} \leq C(r_i^{2n-2} + \varepsilon_3(r_i)).$$

Now applying Proposition 2.1, i.e., the L^2-estimate of $\overline{\partial}$-operators, we have a C^∞-smooth $K_{M_i}^{-m}$-valued function u_i such that

$$(2.9) \qquad \overline{\partial} u_i = \overline{\partial} v_i,$$

$$(2.10) \qquad \begin{aligned} \int_{M_i} \|u_i\|^2_{g_i}(x)\, dV_{g_i} &\leq \frac{1}{m+1} \int_{M_i} \|\overline{\partial}_i v_i\|^2_{g_i}(x)\, dV_{g_i} \\ &\leq \frac{C}{m+1}(r_i^{2n-2} + \varepsilon(r_i)). \end{aligned}$$

By (2.9), for each i, the norm function $\|u_i\|^2_{g_i}$ satisfies the elliptic equation

$$(2.11) \qquad \begin{aligned} &\Delta_i(\|u_i\|^2_{g_i}(x)) \\ &\quad = \|D_i u_i\|^2_{g_i}(x) - nm\|u_i\|^2_{g_i}(x) + 2\operatorname{Re}(h_i^m(u_i, \overline{\partial}_i^* \overline{\partial}_i v_i))(x), \end{aligned}$$

where $\overline{\partial}_i^*$ is the adjoint operator of $\overline{\partial}_i$ on a $K_{M_i}^{-m}$-valued function with respect to g_i. As in (2.5), we also have

$$(2.12) \qquad \sup\{\|\overline{\partial}_i^* \overline{\partial}_i v_i\|^2_{g_i}(x) | x \in M_i \backslash B_{4r_i}(x_{i\beta}, g_i)\} \to 0 \quad \text{as } i \to \infty.$$

Using (2.9), (2.10), (2.11), and (2.12), we see that u_i converges uniformly to zero in the sense of the remark after Lemma 2.1 as i goes to infinity. Put

$$(2.13) \qquad S^i(x) = \frac{(v_i(x) - u_i(x))}{(\int_{M_i} \|v_i - u_i\|^2_{g_i}(x)\, dV_{g_i})^{1/2}}.$$

Then $\{S^i\}$ is the required sequence.

Lemma 2.3. *Let $\{(M_i, g_i)\}$ and (M_∞, g_∞) be given as in Theorem 1.1. For each integer $m > 0$, we have orthonormal bases $\{S^i_{m\beta}\}_{0\leq\beta\leq N_m}$ (resp. $\{S^\infty_{m\beta}\}$) of $H^0(M_i, K_{M_i}^{-m})$ (resp. $H^0(M_\infty, K_{M_\infty}^{-m})$). Then*

$$(2.14) \qquad \varliminf_{i\to\infty} \left(\inf_{M_i} \left\{ \sum_{\beta=0}^{N_m} \|S^i_{m\beta}\|^2_{g_i}(x) \right\} \right) \geq \inf_{M_\infty} \left\{ \sum_{\beta=0}^{N_m} \|S^\infty_{m\beta}\|^2_{g_\infty}(x) \right\}.$$

Proof. By direct computations, we have

$$(2.15)\quad \Delta_i(\|D_iS^i_{m\beta}\|^2_{g_i})(x) = \|D_iD_iS^i_{m\beta}\|^2_{g_i}(x) - ((n+1)m-2)\|D_iS^i_{m\beta}\|^2_{g_i}(x),$$

where Δ_i (resp. D_i) is the laplacian (resp. covariant derivative) with respect to g_i. Then by (2.1), Lemma 1.1, and a standard Moser's iteration, there is a constant $C'(n, m)$ depending only on n, m such that

$$(2.16)\qquad \sup\{\|D_iS^i_{m\beta}\|^2_{g_i}(x) | 0 \le \beta \le N_m,\ x \in M_i\} \le C'(n, m).$$

Combining this with (2.2), we conclude that the first derivatives of $\sum_{\beta=0}^{N_m} \|S^i_{m\beta}\|^2_{g_i}(x)$ are uniformly bounded independent of i. Then (2.14) follows from this and Lemmas 2.1 and 2.2.

Theorem 2.1. *There exist a universal integer $m_0 > 0$ and a universal constant $C > 0$ such that for any Kähler-Einstein surface (M', g') in either $K_+(\mu, n)$ or $K_-(\mu, n)$, we have*

$$(2.17)\qquad \inf_{M'}\left\{\sum_{\beta=0}^{N_m} \|S'_\beta\|^2_{g'}\right\} \ge C > 0,$$

where N_m+1 is the complex dimension of $H^0(M', K_{M'}^{-m_0})$, and $\{S'_\beta\}_{0\le\beta\le N}$ is an orthonormal basis of $H^0(M', K_{M'}^{-m_0})$ with respect to the inner product induced by g'.

Proof. It suffices to prove that for any sequence of a Kähler-Einstein surface $\{(M_i, g_i)\}$ converging to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense of Theorem 1.1, there exist $m_0 > 0$ and $C > 0$ such that (2.17) holds for these (M_i, g_i). By Lemma 2.3, it is sufficient to find a large m such that

$$(2.18)\qquad \inf\left\{\sum_{\gamma=0}^{N_m} \|S^\infty_{m\gamma}\|^2(x) | x \in M_\infty\right\} > 0,$$

where $\{S^\infty_{m\gamma}\}$ and N_m are given as in Lemma 2.3. This is equivalent to the fact that for any point x in M_∞, there is a holomorphic global section S in $H^0(M_\infty, K_{M_\infty}^{-m})$ such that $S(x) \neq 0$. The latter can be achieved by the application of an L^2-estimate (Proposition 2.1) as follows. Let $x_{\infty 1}, \cdots, x_{\infty N}$ be the singular points of M_∞. There is a small positive number r independent of β such that for any $x_{\infty\beta}$ in M_∞, the closure of each connected component in $B_r(x_{\infty\beta}, g_\infty)\backslash\{x_{\infty\beta}\}$ is locally uniformized by a neighborhood $\widetilde{U}_{\beta j}$ $(1 \le j \le l_\beta)$ of the origin o in C^n with finite uniformization group Γ_β. Let $\pi_{\beta_j}: \widetilde{U}_{\beta_j} \to B_r(x_{\infty\beta}, g_\infty)$ be the natural projection with $\pi_{\beta j}(o) = x_{\infty\beta}$ and $q = \prod_{1\le\beta\le N}(\prod_{1\le j\le R_\beta} q_{\beta j})$, where $q_{\beta j}$ is the order of the finite group $\Gamma_{\beta j}$. Let $m = pq$. We will choose p later. We may take r to be sufficiently small such that the function $\rho_\beta = \mathrm{dist}(\cdot, x_{\infty i})$ is smooth on $B_r(x_{\infty\beta}, g_\infty)\backslash\{x_{\infty\beta}\}$ for any β. Now fix an $x_{\infty\beta}$ and $\widetilde{U}_{\beta j}$.

Let $(z_1, \dots, z_n)$ be a coordinate system on $\widetilde{U}_{\beta j}$, and define a q-anticanonical section v by

$$v(y) = \sum_{\sigma\in\Gamma_{\beta j}} \sigma^*\left(\left(\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_n}\right)^q\right)(y), \qquad y\in\widetilde{U}_{\beta j}.$$

By the definition of q, we have $v(o)\neq 0$. Let $\eta: R^1 \to R^1_+$ be a cut-off function such that $\eta(t) = 1$ for $t\le 1$, and $\eta(t) = 0$ for $t\ge 2$ and $|\eta'(t)|\le 1$. Then $w = \eta(4\rho_\beta/r^2)(\pi_{\beta j})_*(v^p)$ is a C^∞-global section of the line bundle $K_{M_\infty}^{-m}$. Choose a large p depending only on r such that for tangent vector ν of type $(1, 0)$,

$$\left\langle \partial\overline{\partial}\left(4n\eta\left(\frac{4\rho_\beta}{r^2}\right)\log\left(\frac{\rho_\beta}{r^2}\right)\right) + \frac{2\pi i}{\sqrt{-1}}\omega_{g_\infty}, \nu\wedge\overline{\nu}\right\rangle_{g_\infty} \ge \|\nu\|^2_{g_\infty}. \tag{2.19}$$

Applying Proposition 2.1, we obtain a C^∞ smooth $K_{M_\infty}^{-m}$-valued function u satisfying $\overline{\partial}u = \overline{\partial}w$ and

$$\int_{M_\infty}\|u\|^2_{g_\infty}e^{-4n\eta\log(\rho_\beta/r^2)}\,dV_{g_\infty} \le \int\|\overline{\partial}w\|^2_{g_\infty}e^{-4n\eta\log(\rho_\beta/r^2)}\,dV_{g_\infty} < +\infty.$$

It follows that the pullback $\pi^*_{\beta j}u$ of u vanishes up to order 2 at the origin in $\widetilde{U}_{\beta j}\subset C^n$. Put

$$S_{\beta j} = \frac{w-u}{(\int_{M_\infty}\|w-u\|^2_{g_\infty}\,dV_{g_\infty})^{1/2}}; \tag{2.20}$$

then $S_{\beta j}\in H^0(M_\infty, K^m_{M_\infty})$ and $\inf_{\widetilde{U}_{\beta j}}\{\pi^*_{\beta j}\|S_{\beta j}\|_{g_\infty}(x)\} > 0$. By the same arguments as in the proof of Lemma 2.3, one can bound the first derivatives of these $S_{\beta j}$ by a uniform constant. So if r is taken sufficiently small, we have

$$\inf\left\{\sum_{\gamma=0}^{N_m}\|S_{m\gamma}^{\infty}\|_{g_\infty}^2(x)\;\middle|\; x\in B_r(x_{\infty\beta}, g_\infty),\ 1\le\beta\le N_m\right\}$$
$$\ge \inf\left\{\|S_{\beta j}\|_{g_\infty}^2(x)\;\middle|\; x\in\pi_{\beta j}(\widetilde{U}_{\beta j}),\ 1\le\beta\le N_m,\ 1\le j\le l_\beta\right\}>0.$$

For any point x in $M_\infty\setminus\bigcup_{\beta=1}^N B_r(x_{\infty\beta}, g_\infty)$, define $\rho_x=\mathrm{dist}(\cdot, x)$. As above, by applying Proposition 2.1 to $K_{M_\infty}^{-m}$-valued $\overline{\partial}$-equation with the weight function $4n\eta(4\rho_x^2/r^2)\log(\rho_x^2/r^2)$, one can easily construct a holomorphic section S_x in $H^0(M_\infty, K_{M_\infty}^{-m})$ such that $S_x(x)\ne 0$. Thus the inequality (2.18) is proved, and so is Theorem 2.1.

Corollary 2.1. *The Kähler-Einstein orbifold (M_∞, g_∞) is irreducible.*

Since we do not need this result, we omit its proof here and refer readers to Proposition 5.2 in [20].

3. Application of Kohn's estimates of CR-manifolds

Let $\{(M_i, g_i)\}$ be the sequence of Kähler-Einstein manifolds in either $K_+(\mu, n)$ or $K_-(\mu, n)$ as in §1. By Theorem 1.1 and Corollary 2.1, these (M_i, g_i) converge to a Kähler-Einstein orbifold (M_∞, g_∞). Precisely, there are points $x_{i1}, \cdots, x_{iN}$ in M_i and $x_{\infty 1}, \cdots, x_{\infty N}$ in M_∞ satisfying: for $r>0$, there are diffeomorphisms $\phi_i^* g_i$ and $\phi_i^*\circ J_i\circ(\phi_i^{-1})^*$ converging to g_∞ and J_∞, respectively, in C^5-norms. The purpose of this section is to study the holomorphic structure of $B_r(x_{i\beta}, g_i)$ for sufficiently small r and large i. The main analytic tool is Kohn's estimate for $\Box_b$-operators.

Let $\rho_\infty(\cdot, \cdot)$ be the distance function on $M_\infty\times M_\infty$. For simplicity, we may assume that $N=1$ and write x_i for x_{i1}, and x_∞ for $x_{\infty 1}$. For each sufficiently small $r>0$, the level surface $\partial B_r(x_\infty, g_\infty)$ of $\rho_\infty(\cdot, x_\infty)$ is smooth. The Levi form on $\partial B_r(x_\infty, g_\infty)$ is the natural hermitian form on the $(n-1)$-dimensional space $T^{(1,0)}M_\infty\cap(T_R H_{\infty r}\otimes C)$ given by

$$(L_1, L_2)=2(\partial\overline{\partial}\rho_\infty(\cdot, x_\infty), L_1\wedge\overline{L}_2),$$

where $H_{\infty r}$ denotes the level surface $\partial B_r(x_\infty, g_\infty)$.

It is easy to see that this form is positive definite for r small. In fact, $\rho_\infty(x_\infty, \cdot)$ is convex near x_∞. Therefore, each $H_{\infty r}$ is a strongly pseudoconvex CR-manifold. Similarly, if we define H_{ir} to be the level surface

$$\{x \in M_i | \rho_\infty(x_\infty, \phi_i^{-1}(x)) = r\},$$

then the H_{ir} are also smooth strongly pseudoconvex CR-manifolds.

Define the following for $r > 0$:

$$\tilde{g}_{\infty r} = \frac{1}{r^2} g_\infty, \qquad \tilde{g}_{ir} = \frac{1}{r^2} g_i,$$

$$(L_1, L_2)_{\infty r} = \frac{2}{r^2}(\partial\overline{\partial}\rho_\infty, L \cap \overline{L_2}) \quad \forall L_1, L_2 \in T^{(1,0)}M_\infty \cap (T_R H_{\infty r} \otimes C),$$

$$(L_1, L_2)_{ir} = \frac{2}{r^2}(\partial\overline{\partial}(\rho_\infty \cdot \phi_i^{-1}), L_1 \wedge \overline{L}_2) \quad \forall L_1, L_2 \in T^{(1,0)}M_i \cap (T_R H_{ir} \otimes C).$$

Lemma 3.1. *As r goes to zero, $(H_{\infty r}, \tilde{g}_{\infty r}, (\cdot, \cdot)_{\infty r})$ converges to $(S^{2n-1}/\Gamma, ds^2, (\cdot, \cdot)_s)$, where $\Gamma \subset U(n)$ is a finite group, ds^2 is the metric with constant curvature $+1$, and $(\cdot, \cdot)_s$ is induced by the standard Levi-form on the unit sphere.*

Proof. It follows trivially from the boundedness of the curvature tensor $Rm(g_\infty)$.

Lemma 3.2. *There is a subsequence $\{i_j\}$ such that there are diffeomorphisms ψ_j from S^{2n-1} onto H_{i_j, r_j}, where $r_j = 1/j$, satisfying:*

(1) $\|\psi_j^* \tilde{g}_{i_j r_j} - ds^2\|_{C^t(S^{2n-1})} \leq \varepsilon(j)$, *and*

(2) $\|\psi_j^*(\cdot, \cdot)_{i_j r_j} - (\cdot, \cdot)_s\|_{C^5(S^{2n-1})} \leq \varepsilon(j)$,

where $\varepsilon(j) \to 0$ as $j \to \infty$.

In other words, $(H_{i_j r_j}, \tilde{g}_{i_j r_j}, (\cdot, \cdot)_{i_j r_j})$ converges to $(S^{2n-1}, ds^2, (\cdot, \cdot_s))$ as j tends to infinity.

Proof. Because of the convergence of (M_i, g_i) to (M_∞, g_∞), for each j there is a diffeomorphism ϕ_j from $M_\infty \backslash B_{r_j/10}(x_\infty, g_\infty)$ into M_{i_j} for some i_j satisfying:

(1) $M_{i_j} \backslash B_{1/2r_j}(x_{i_j}, g_{i_j}) \subset \mathrm{Im}(\phi_i)$,

(2) $\|\phi_j^* g_{i_j} - g_\infty\|_{C^5(M_\infty) \leq 1/j}$, and

(3) $\|\phi_j^* J_{i_j} - J_\infty\|_{C^5(M_\infty)} \leq \frac{1}{j}$, where J_{i_j} and J_∞ are almost complex structures on M_{i_j} and M_∞, respectively.

By Lemma 3.1, there are diffeomorphisms θ_j from S^{2n-1} onto $H_{\infty r_j}$ such that

(i) $\|\theta_j^* \tilde{g}_{\infty r_j} - ds^2\|_{C^5(S^{2n-1})} \leq \varepsilon'(j)$, and

(ii) $\|\theta_j^*(\cdot, \cdot)_{\infty r_j} - (\cdot, \cdot)_S\|_{C^5(S^{2n-1})} \leq \varepsilon'(j)$, where $\varepsilon'(j) \to o$ as $j \to \infty$. Now our ψ_j are just the compositions of ϕ_j with θ_j. q.e.d.

Given a complex manifold X with strongly pseudoconvex boundary Y, we define $\mathscr{B}^{p,q}(Y)$ to be the space of smooth sections of the vector bundle $\Omega^{p,q}(X) \cap \Lambda^{p,q}(T_R^* Y \otimes C)$ on Y. The $\overline{\partial}$-operator of X induces the $\overline{\partial}_b$-operator from $\mathscr{B}^{p,q}(Y)$ into $\mathscr{B}^{p,q+1}(Y)$, explicitly, $\overline{\partial}_b \phi$ is the projection of $\overline{\partial}\phi$ onto $\mathscr{B}^{p,q+1}(Y)$. Let $\overline{\partial}_b^*$ be the adjoint operator of $\overline{\partial}_b$ on Y with respect to the induced metric on Y from X and the Levi form.

Since $\overline{\partial}^2 = 0$, it follows that $\overline{\partial}_b^2 = 0$, so we have the boundary complex

$$0 \to \mathscr{B}^{p,0} \xrightarrow{\overline{\partial}_b} \mathscr{B}^{p,1} \longrightarrow \dots \xrightarrow{\overline{\partial}_b} \mathscr{B}^{p,n-1} \longrightarrow 0.$$

Then the cohomology of the above boundary complex is called the Kohn-Rossi cohomology and is denoted by $H^{p,q}(\mathscr{B})$. We recall the following proposition.

Proposition 3.1. *Let X, Y be as above. Then for $1 \le q \le n-2$, the cohomology $H^{p,q}(\mathscr{B})$ is finite dimensional, and the range of $\overline{\partial}_b \colon \mathscr{B}^{p,q-1} \to \mathscr{B}^{p,q}$ is closed in the C^∞-topology.*

Let $\widetilde{H}_j$ be the universal covering of $H_{i_j r_j}$; then they are diffeomorphic to S^{2n-1}. In fact, ψ_j induces these diffeomorphisms from S^{2n-1} onto $\widetilde{H}_j$, still denoted by ψ_j.

Lemma 3.3. *Let $n \ge 3$. There is a uniform constant $C > 0$ such that for j sufficiently large,*

$$C\|u\|_2^2 \le \|\overline{\partial}_b u\|_2^2 + \|\overline{\partial}_b^* u\|_2^2 \tag{3.1}$$

for any u in $\mathscr{B}^{0,1}(\widetilde{H}_j)$, where $\|\cdot\|_2$ denotes the L^2-norm induced by the metric $g_{i_j r_j}$ and Levi form $(\cdot\,,\,\cdot)_{i_j r_j}$.

Proof. Let λ_j be the smallest eigenvalue of the operator of $\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$ on $\mathscr{B}^{0,1}(\widetilde{H}_j)$. Then (3.1) is equivalent to $\lambda_j \ge c > 0$.

Suppose that the lemma is false. Then we may assume that $\lambda_j \to 0$ as $j \to \infty$. By Proposition 3.1, the eigenspace of λ_j is of finite dimension. Pick up an eigenfunction u_j for λ_j with $\|u_j\|_2 = 1$. Then

$$\lambda_j \|u_j\|_2^2 = \|\overline{\partial}_b u_j\|_2^2 + \|\overline{\partial}_b^* u_j\|_2^2.$$

Since $(\widetilde{H}_j, \tilde{g}_{i_j r_j}, (\cdot\,,\,\cdot)_{i_j r_j})$ converges to $(S^{2n-1}, ds^2, (\cdot\,,\,\cdot)_s)$ in the C^5-topology, by Kohn's estimate for $\Box_b$, these u_j converge to u_∞ in $\mathscr{B}^{0,1}(S^{2n-1})$ satisfying

$$\|u_\infty\|_2 = 1 \quad \text{and} \quad \Box_b u_\infty = 0.$$

In particular, u_∞ gives a nontrivial cohomological class in $H^{0,1}(\mathscr{B}(S^{2n-1}))$. However, it follows from Theorem A in [26] that $H^{0,1}(\mathscr{B}(S^{2n-1}))=0$ for $n\geq 3$, a contradiction. Therefore, (3.1) holds.

Lemma 3.4. *There exist embeddings $\iota_j\colon \widetilde{H}_j\to C^n$ such that the $\iota_j(\widetilde{H}_j)$ converge to S^{2n-1} as submanifolds in C^n in the C^4-topology.*

Proof. Let $z_1, \cdots, z_n$ be the standard coordinates in $\mathscr{C}^n$. The restrictions of these to S^{2n-1} are CR-functions denoted by the same letters for simplicity. Define

$$z_{ji}=z_i\circ\psi_j^{-1},\qquad 1\leq i\leq n,\ j\gg 0.$$

Then $\sup_{1\leq i\leq n}\{\|\overline{\partial}_b(z_i\circ\psi_j)\|_{C^4(\widetilde{H}_j)}\}\leq C\varepsilon(j)$, where C is a uniform constant, and $\varepsilon(j)\to 0$ as $j\to\infty$.

By Lemma 3.3, there are v_{ij} solving

$$\Box_b v_{ij}=\overline{\partial}_b(z_i\circ\psi_j^{-1})\quad\text{on } \widetilde{H}_j$$

with

$$\|\overline{\partial}_b v_{ij}\|_2^2+\|\overline{\partial}_b^* v_{ij}\|_2^2+\|v_{ij}\|_2^2\leq C_1\|\overline{\partial}_b(z_i\cdot\psi^{-1})\|_2^2\leq C_1C\varepsilon(j).$$

Define $z_{ij}=z_i\circ\psi_j^{-1}-\overline{\partial}_b^* v_{ij}$; then $\overline{\partial}_b z_{ij}=0$.

Using Kohn's estimate for the $\overline{\partial}_b$-operator, we have

$$\sup_{i\leq i\leq j}\{\|\overline{\partial}_b^* v_{ij}\|_{C^4(\widetilde{H}_j)}\}\leq C_3\sup_{1\leq i\leq j}\|\overline{\partial}_b(z_i\circ\psi_j^{-1})\|_{C^4(\widetilde{H}_j)}\leq C_4\varepsilon(j).$$

The required maps ι_j assign x in $\widetilde{H}_j$ to $(z_{ij}(x), \cdots, z_{nj}(x))$ in C^n. Since $\overline{\partial}_b z_{ij}=0$ and $(\widetilde{H}_j, g_{i_j r_j}, (\cdot,\cdot)_{i_j r_j})$ converge to $(S^{2n-1}, ds^2, (\cdot,\cdot)_s)$ through ψ_j, these ι_j are CR-embeddings of $\widetilde{H}_j$ such that the images approach S^{2n-1}. Hence the lemma is proved. q.e.d.

Choose a large m such that the basis $\{s_0^\infty, \ldots, S_{N_m}^\infty\}$ of $H^0(M_\infty, K_{M_\infty}^{-m})$ gives a Kodaira's embedding of M_∞ into CP^{N_m}, where $N_m=\dim_C H^0(M_\infty, K_{M_\infty}^{-m})-1$. Moreover, we may arrange these S_β^∞ such that $S_0^\infty(x_\infty)\neq 0$, and $S_\beta^\infty(x_\infty)=0$ for $\beta\geq 1$. By Theorem 2.3 in the previous section, there are bases $\{S_\beta^j\}$ of $H^0(M_{i_j}, K_{M_{ij}}^{-m})$ converging to $\{S_\beta^\infty\}$. In particular, for j sufficiently large, these bases $\{S_\beta^j\}$ give embeddings of M_{i_j} into CP^{N_m}. Fix a small $r>0$; then for j large we have local embeddings

$$\tau_j\colon B_r(x_{i_j})\to C^{N_m},\qquad \tau_\infty\colon B_r(x_\infty)\to C^{N_m}.$$

Denote by $w_1, \cdots, w_{N_m}$ the coordinate functions. Let $\pi_j : \widetilde{H}_j \to H_{i_j}$ be the covering maps. Then the compositions $w_\beta \circ \pi_j$ $(1 \le \beta \le N_m)$ are CR-functions on $\widetilde{H}_j$. Now by the previous lemma, $\widetilde{H}_j$ bound strongly pseudo-convex domains B_j in C^n. Moreover, these B_j converge to the unit ball in C^n as j approaches infinity.

Lemma 3.5. *Each $w_\beta \circ \pi_j$ can be extended to be a holomorphic function $h_{\beta j}$.*

Proof. Since B_j is a domain in C^n, there is a nonconstant holomorphic function on B_j. This lemma then follows from Theorem 5.3.2 in [6]. q.e.d.

Define

$$\tilde{\tau}_j = (h_{ij}, \cdots, h_{N_m j}) :\to C^{N_m}.$$

Then $\tilde{\tau}_j$ coincides with $\tau_j \circ \pi_j$ on $\widetilde{H}_j$, so by the analytic unique continuation, the image $\tilde{\tau}_j(B_j)$ coincides with part of $\tau_j(B_r(x_{ij}))$. It follows that there are holomorphic maps $\tau_j^{-1} \circ \tilde{\tau}_j$ from B_j onto the domain in $B_r(x_{ij})$ enclosed by H_{i_j}, in particular, $\tau_j^{-1} \circ \tilde{\tau}_j$ immersions near $\widetilde{H}_j$ and finite maps on B_j. For simplicity, denote $\tau_j^{-1} \circ \tilde{\tau}_j$ by π_j.

Lemma 3.6. *Let Γ be the fundamental group of H_{ij}. Then Γ acts on $\widetilde{H}_j$ as a CR-isomorphism group, and can be extended to be automorphisms of B_j. In particular, $\Gamma \subset U(n)$.*

Proof. It is clear that each $\sigma \in \Gamma$ preserves the CR-structure of $\widetilde{H}_j$ as a deck transformation. Therefore, the CR-functions $z_1 \circ \sigma, \cdots, z_n \circ \sigma$ can be extended to be holomorphic ones in B_j (cf. proof of Lemma 3.5), that is, σ extends to be a holomorphic map from B_j into itself. The extension must be an automorphism since σ has degree one near $\widetilde{H}_j$. q.e.d.

As a finite group in $U(n)$, Γ has at least a fixed point in B_j if it is nontrivial. This implies that $\mathscr{B}_j/\Gamma$ is singular, contradicting to the fact that $B_r(x_{i_j})$ is smooth for each j. Therefore, $\Gamma = \{\mathrm{id}\}$, and M_∞ is in fact smooth.

Summarizing the above, we have

Theorem 3.1. *Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein manifolds in $K_+(\mu, n)$ (resp. $K_-(\mu, n)$). Then either (M_i, g_i) converges to a Kähler-Einstein manifold in the C^5-topology, or there is a smooth Kähler-Einstein manifold (M_∞, g_∞) in $K_+(\mu, n)$ (resp. $K_-(\mu, n)$) such that a subsequence of $\{(M_i, g_i)\}$, say $\{(M_i, g_i)\}$ itself, converges to (M_∞, g_∞) outside finitely many points in the C^5-topology.*

4. Proof of Theorem 2

In this section, we classify all complete Ricci-flat Kähler manifolds (X, g) with euclidean volume growth and $\int_X |Rm(g)|^n \, dV_g < \infty$, where $n = \dim_C X$. Let us fix one of them, say (X, g).

Lemma 4.1. *There is a decreasing positive function* $\varepsilon(r)$ *with* $\lim_{r\to\infty} \varepsilon(r) = 0$ *such that*

$$\|Rm(g)\|_g(x) \le \frac{\varepsilon(r(x))}{r(x)^2}, \tag{4.1}$$

where $r(x)$ *is the distance function from some fixed points.*

Proof. Choose $\varepsilon(r)$ to be a decreasing positive function such that $\lim_{r\to\infty} \tilde{\varepsilon}(r) = 0$ and

$$\int_{B_r(x, g)} \|Rm(g)\|_g^n \, dV_g \le \tilde{\varepsilon}(r) \quad \text{for } x \subset \partial B_{2r}(x_0).$$

Now for each fixed x in $\partial B_{2r}(x_0)$, define a new metric $g_x = g/r^2$; then g_x has vanishing Ricci curvature, and

$$\int_{B_1(x, g_x)} \|Rm(g_x)\|_{g_x}^n \, dV_{g_x} \le \tilde{\varepsilon}(r).$$

On the other hand, since (X, g) has the euclidean volume growth, there is a constant C', independent of r, such that

$$\mathrm{Vol}_g(B_{2r}(X_0, g_x)) \ge C' r^{2n},$$

so by the Volume Comparison Theorem [2],

$$\begin{aligned} \mathrm{Vol}_g(B_1(X, g)) &\ge \frac{1}{4^{2n}} \mathrm{Vol}_g(B_{4r}(X, g)) \\ &\ge \frac{1}{4^{2n}} \mathrm{Vol}_g(B_{2r}(X_0, g)) \ge \frac{C'}{4^{2n}} r^{2n}. \end{aligned}$$

It follows that $\mathrm{Vol}_{g_x}(B_1(x, g_x)) = \mathrm{Vol}_g(B_r(x, g_x))/r^2$ is not less than a uniform positive constant $C'/4^{2n}$. So we can apply Lemma 1.2 to $(B_1(x, g_x), g_x)$ and obtain

$$\|Rm(g_x)\|_{g_x} \le C\tilde{\varepsilon}(r), \tag{4.2}$$

where C is a constant independent of x. Take $\varepsilon(r) = C\tilde{\varepsilon}(r)$. Then (4.1) is nothing else but (4.2), and the lemma is proved. q.e.d.

Consider a sequence of complete Ricci-flat Kähler manifolds $(X_i, g_i) = (X, g/i^2)$. By Lemma 4.1, $\|Rm(g_i)\|_{g_i}$ are bounded by $\varepsilon(i)/\delta$ outside $B_\delta(x_0, g_i)$ for any $\delta > 0$. Therefore, we can proceed as in §1 to show that (X_i, g_i), by taking subsequences, converges to a complete Kähler orbifold (X_∞, g_∞). In fact, the proof in this case is much similar, and (X_∞, g_∞) is flat because of Lemma 4.1. Therefore, $X_\infty = C^n/\Gamma$ with unique singular point o in $U(n)$. In particular, there are smooth diffeomorphisms ψ_i from $X_i \backslash B_{1/2}(x_0, g)$ into $x_\infty \backslash B_{1/4}(0, g_\infty)$ such that $\|(\psi_i^{-1})^* g_i - g_\infty\|_{C^5(X_\infty, g_\infty)} = o(1)$ as i goes to infinity. Put $\Sigma_i = \psi_i^{-1}(\partial B_1(0, g_\infty))$, and let $\widetilde{\Sigma}_i$ be its universal covering. Then the $\widetilde{\Sigma}_i$ are strongly pseudoconvex CR-manifolds and converge to S^{2n-1} in C^n. Thus by Lemma 3.4, for i sufficiently large, these $\widetilde{\Sigma}_i$ can be holomorphically embedded into C^n and bound domains B_i^n there. Moreover, Γ acts on B_i^n by holomorphic transformations.

On the other hand, if we denote by ρ^2 the square of the euclidean distance function from o in C^n/Γ, then the $\psi_i^* \rho^2$ are convex functions near Γ_i. So by Grauert's theorem [8], for each large i, there is a holomorphic map $v_i \colon E_i \to C^{N_i}$ which is actually an embedding near $\Gamma_i = \partial E_i$, where E_i is the bounded domain enclosed by Σ_i.

Lemma 4.2. *For each fixed i, if $w_1, \cdots, w_{N_i}$ are coordinate functions of C^{N_i}, then the CR-functions $w_j \circ \pi_i \colon \widetilde{\Sigma}_i \to C^{N_i}$ can be extended to be holomorphic ones in B_i^n, where $\pi_i \colon \widetilde{\Sigma}_i \to \Sigma_i$ are natural projections.*

We omit its proof (cf. Lemma 3.5).

It follows that there are holomorphic maps $\phi_i \colon B_i^n/\Gamma \to v_i(E_i)$, which are embeddings in the neighborhoods of Σ_i.

Lemma 4.3. *For each i, there is a holomorphic map $p_i \colon E_i \to B_i^n/\Gamma$ such that $v_i = \phi \circ p_i$.*

Proof. It is easy to see that $\phi_i^{-1}(x)$ contains exactly one point in B_i^n/Γ for x in $v_i(E_i)$. Let $D_1, \cdots, D_{il_i} \in E_i$ be analytic subvarieties such that $v_i^{-1} \circ v_i(D_{ij})$ contains more than one point. Then the $v_i(D_{ij})$ are isolated points. Define $p_i = \phi_i^{-1} \circ v_{i_l}$ outside these $D_{i1}, \cdots, D_{il_i}$; then p_i is a holomorphic map from $E_i \backslash \bigcup_{\beta=1}^{l_i} D_{i\beta}$ into B_i^n. Since B_i^n is bounded, the map p_i can be extended across $D_{i\beta}$. In particular, this implies that $l_1 = 1$, i.e., there is only one connected component, and $v_i(E_i)$ has only one singular point, so $v_i(E_i) \cong B_i^n/\Gamma$. q.e.d.

It follows that X is the resolution of C^n/Γ. Hence Theorem 2 is proved.

5. Proof of Theorem 1

In this section, we will finish the proof of Theorem 1.

Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein manifolds either in $K_+(\mu, n)$ or $K_-(\mu, n)$. By Theorem 3.1, (M_i, g_i) converges to a smooth Kähler-Einstein manifold (M_∞, g_∞) outside finitely many points. Precisely, there are $x_{i1}, \cdots, x_{iN}$ satisfying: for each $r > 0$, there are diffeomorphisms ϕ_{ir} from $M_\infty \backslash \bigcup_{\beta=1}^N B_r(x_{\infty\beta}, g_\infty)$ into M_i containing $M_i \backslash \bigcup_{\beta=1}^N B_{2r}(x_{i\beta}, g_i)$ such that $\phi_{ir}^* g_i$ converges to g_∞ in the C^5-topology. Each $B_r(x_{\infty\beta}, g_\infty)$ with small r is a smooth ball in C^n. So $B_r(x_{i\beta}, g_i)$ are smooth balls in C^n, too.

We need to show that the $Rm(g_i)$ are uniformly bounded in $\bigcup_{\beta=1}^N B_{2r}(x_{i\beta} g_i)$. Suppose it is not true. Then by taking the subsequence, we may assume that $\mu_i^2 = \|Rm(g_i)\|_{g_i}(y_i) \to +\infty$ for some y_i in $B_{r_i}(x_{i1}, g_i)$, where $\lim_{i\to\infty} r_i = 0$. Define new metrics on M_i by

$$h_i = \mu_i^2 g_i .$$

Then the pointed manifolds $(B_r(x_{i1}, g_i), h_i, y_i)$ converge to a complete Ricci-flat Kähler manifold (X, h) with $\int_X \|Rm(h)\|_h^n \, dV_h < \infty$, where r is a fixed small positive number.

Lemma 5.1. *X is a Stein manifold.*

Proof. Let (M_i, g_i) be in $K_+(\mu, n)$ for all i. The proof of the other case is identical.

Fix an $m > 0$ such that the basis of $H^0(M_\infty, K_{M_\infty}^{-m})$ gives an embedding of M_∞ into some projective space. In particular, there is a positive constant C satisfying

$$\min_{M_\infty} \left\{ \sum_{\beta=0}^N \|S_\beta^\infty\|_{g_\infty}^2 (x) \right\} \geq 2C > 0,$$

where $N = \dim_C H^0(M_\infty, K_{M_\infty}^{-m})$, and $\{S_\beta^\infty\}$ is an orthonormal basis of $H^0(M_\infty, K_{M_\infty}^{-m})$ with respect to g_∞.

By Theorem 2.1, for i sufficiently large,

$$\min_{M_i} \left\{ \sum_{\beta=0}^N \|S_\beta^i\|_{g_i}^2 (x) \right\} \geq C > 0, \tag{5.1}$$

where the $\{S_\beta^i\}$ are orthonormal bases of $H^0(M_i, K_{M_i}^{-m})$ with respect to

g_i. Let $\widetilde{S}_i$ be the section of $K_{M_i}^{-m}$ satisfying:

(1) $\int_{M_i} \|\widetilde{S}_i\|_{g_i}^2 \, dV_{g_i} = 1$,

(2) $\|\widetilde{S}_i\|_{g_i}^2(y_i) = \sup\{\|S\|_{g_i}(y_i) | \int_{M_i} \|S\|_{g_i}^2 \, dV_{g_i} = 1\}$.

Then for i sufficiently large and r sufficiently small,

$$\min_{B_r(x_{i1}, g_i)} (\|\widetilde{S}_i\|) \geq C > 0. \tag{5.2}$$

Define $u_i(x) = -\log(\|\widetilde{S}_i\|_{g_i}(x)/\|\widetilde{S}_i\|_{g_i}(y_i))$. Then the u_i are uniformly bounded smooth functions in $B_r(x_{i1}, g_i)$ satisfying:

$$\omega_{g_i} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_i \quad \text{in } B_r(x_{i1}g_i),$$
$$u_i(y_i) = \min_{B_r(x_{i1}, g_i)} u_i = 0.$$

Therefore, $\omega_{h_i} = \sqrt{-1}\partial\overline{\partial}(\mu_i^2 u_i)/(2\pi)$, and $\mu_i^2 u_i$ converge to a smooth function u in X such that $\omega_h = \sqrt{-1}\partial\overline{\partial}u/2\pi$. This implies that X is Stein, and hence the lemma is proved. q.e.d.

By Theorem 2, X is a smooth resolution of some C^n/Γ. Therefore, X has to be C^n/Γ, and Γ is trivial since X is Stein.

Thus (X, h) must be flat, contradicting that $\max_x \|Rm(h)\|_h = 1$. This finishes the proof of Theorem 1.

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