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REGULARITY FOR THE HARVEY-LAWSON SOLUTIONS TO THE COMPLEX PLATEAU PROBLEM

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1. Introduction

It seems that one of the natural fundamental questions of complex geometry is the classical complex Plateau problem. Specifically, the problem asks which odd-dimensional, real submanifolds of $\mathbf{C}^N$ are boundaries of complex submanifolds in $\mathbf{C}^N$.

Recall that a C^1-submanifold M of a complex manifold X is said to be maximally complex if

$$\operatorname{codim}_{\mathbf{R}}(T_xM \cap J(T_xM)) = 1 \quad \text{for all } x \in M,$$

where J denotes the almost complex structure of X, and the codimension refers to M. It was a fundamental contribution to complex geometry by Harvey and Lawson [3] that if M is compact, oriented, and of dimension larger than 1, and if X is Stein, then maximal complexity implies that M forms the boundary of a holomorphic n-chain in X.

If M is a CR-manifold in the sense of Kohn [6], [2] (see Definition 2.1 below), then there is a natural filtration associated to the De Rham complex of M with complex coefficients [8], [9]. The $E_1^{p,q}$ term of the spectral sequence associated to this filtration is called the Kohn-Rossi cohomology group $H_{\mathrm{KR}}^{p,q}(M)$ of M [7], [8], [9]. In [9], we gave smooth solutions to the classical complex Plateau problem in the following cases.

Theorem 1. *Let M be a compact, orientable, connected CR-manifold of real dimension $2n-1$, $n \geq 3$, in a Stein manifold X of complex dimension $n+1$. Suppose that M is strongly pseudoconvex. Then M is a boundary of a complex submanifold $V \subseteq X - M$ if and only if Kohn-Rossi's cohomology groups $H_{\mathrm{KR}}^{p,q}(M)$ are zero for $1 \leq q \leq n-2$.*

However, for strongly pseudoconvex (see Definition 2.4 below) CR-manifolds of real dimension three in $\mathbf{C}^3$, the smoothness of Harvey-Lawson solutions to the classical complex plateau problem remains open.

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The purpose of this paper is to answer this question. In fact we shall introduce a numerical CR-invariant $\tau(M)$ for any compact connected $(2n-1)$-dimensional strongly pseudoconvex CR-manifold M in $\mathbf{C}^{n+1}$, $n \geq 2$. The vanishing of $\tau(M)$ will give a necessary and sufficient condition for the smoothness of Harvey-Lawson solutions to the classical complex Plateau problem for M. More precisely we have

Definition. Let M be a compact connected $(2n-1)$-dimensional strongly pseudoconvex CR-manifold in $\mathbf{C}^{n+1}$, $n \geq 2$. By a theorem of Harvey and Lawson (see [3]), M is the boundary of a complex variety V in the C^∞ sense. V is smooth except at finitely many isolated singular points $\{p_1, \cdots, p_k\}$. Let τ_i be the number of local moduli of V at p_i (see Definition 2.5 below). We define $\tau(M)$ to be $\tau_1 + \tau_2 + \cdots + \tau_k$.

Theorem 2. *Let M be a compact connected $(2n-1)$-dimensional strongly pseudoconvex CR-manifold in $\mathbf{C}^{n+1}$, $n \geq 2$. The $\tau(M)$ defined above is a CR-invariant in the sense that if $M' \subseteq \mathbf{C}^{n+1}$ is another $(2n-1)$-dimensional CR-manifold which is CR-diffeomorphic to M, then $\tau(M) = \tau(M')$. In fact for $n \geq 3$, $\tau(M) = \dim H^{p,q}_{\mathrm{KR}}(M)$ for $p+q = n-1, n$ and $1 \leq q \leq n-2$. Moreover, M is a boundary of the complex submanifold $V \subseteq \mathbf{C}^{n+1} - M$ if and only if $\tau(M) = 0$.*

Remark. It will be of extreme interest to give an intrinsic interpretation to $\tau(M)$ for $n = 2$. In fact, for $n = 2$, if M admits a transversal holomorphic S^1-action, Lawson and the present author [5] have proved that a necessary and sufficient condition for the regularity of the Harvey-Lawson solution to the complex Plateau problem is $\pi_1 = 0$.

2. Preliminaries and proof of Theorem 2

In this section, we shall recall some basic definitions which are needed in this paper.

Definition 2.1. Let M be a compact, connected, orientable real manifold of dimension $2n-1$. A CR-structure on M is an $(n-1)$-dimensional subbundle S of $\mathbf{C}TM$ such that the following are true:

(1) $S \cap \overline{S} = \{0\}$.

(2) If L, L' are local sections of S, then so is $[L, L']$.

Definition 2.2. Let $L_1, \cdots, L_{n-1}$ be the local basis for sections of S over an open subset $U \subset M$ so that $\overline{L}_1, \cdots, \overline{L}_{n-1}$ form a local basis for sections of $\overline{S}$. Since $S \oplus \overline{S}$ has complex codimension one in $\mathbf{C}TM$, we may choose a local section N of $\mathbf{C}TM$ such that $L_1, \cdots, L_{n-1}, \overline{L}_1, \cdots, \overline{L}_{n-1}, N$ span $\mathbf{C}TM$. We may assume that N is purely imaginary. Then the matrix (c_{ij}), defined by

$$[L_i, L_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \overline{L}_k + c_{ij} N,$$

is Hermitian and is called the Levi form.

The Levi form is noninvariant; however, its essential features are invariant.

Proposition 2.3. *The number of nonzero eigenvalues and the absolute value of the signature of* (c_{ij}) *at each point are independent of the choice of* $L_1, \cdots, L_{n-1}, N$.

Definition 2.4. Let M be a CR-manifold. Then M is strongly pseudoconvex if the Hermitian matrix (c_{ij}) obtained in Definition 2.2 is always nonsingular and its eigenvalues are of the same sign.

Definition 2.5. Let f be a holomorphic function in $\mathbf{C}^{n+1}$. Suppose that $V = \{z \in \mathbf{C}^{n+1} : f(z) = 0\}$ has an isolated singularity at the origin. Then the number of local moduli τ of V at 0 is given by

$$\tau = \dim \mathbf{C}[[z_0, z_1, \cdots, z_n]] \Big/ \left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right).$$

Proof of Theorem 2. By a theorem of Harvey and Lawson, there exist complex varieties V and V' in $\mathbf{C}^{n+1}$ such that $\partial V = M$ and $\partial V' = M'$ in the C^∞ sense. V is smooth except at finitely many isolated singular points $\{p_1, \cdots, p_k\}$ while V' is smooth except at finitely many isolated singular points $q_1, \cdots, q_{k'}$. Since M is strongly pseudoconvex, we can take a 1-convex exhaustion function φ on V such that $\varphi \geq 0$ on V and $\varphi(y) = 0$ if and only if $y \in \{p_1, \cdots, p_k\}$. Put $V_r = \{y \in V : \varphi(y) \leq r\}$. Let $\sigma = (\sigma_0, \sigma_1, \cdots, \sigma_n)$ be the CR-diffeomorphism from M to M'. By the strong pseudoconvexity of $M = \partial V$, we see that the σ_i, $0 \leq i \leq n$, extend to a holomorphic function defined in a neighborhood U of M in V. Since $V - V_r \subseteq U$ for r large enough, the extension of σ_i is holomorphic on $V - V_r$ for $0 \leq i \leq n$. On the other hand, by Andreotti and Grauert (Théoréme 15 of [1]), $H^0(V - \{p_1, \cdots, p_k\}, \mathscr{O}) \to H^0(V - V_r, \mathscr{O})$ is an isomorphism. So the extension of σ_i is holomorphic on $V - \{p_1, \cdots, p_k\}$ for $0 \leq i \leq n$. Observe that V is a complex hypersurface. Therefore, $p_1, \cdots, p_k$ are hypersurface singularities, and in particular they are normal singularities. Hence $\bar{\sigma}_i$, the extension of σ_i, is actually holomorphic on V for all $0 \leq i \leq n$, and we have a holomorphic map $\bar{\sigma}: V \to \mathbf{C}^{n+1}$ such that the restriction of $\bar{\sigma}$ to M is σ. Clearly $\bar{\sigma}(V)$ and V' are both complex varieties in $\mathbf{C}^{n+1}$, which have the same boundary M'. By the uniqueness of the solution to the complex

Plateau problem (see [3]), we see that $\bar{\sigma}(V) = V'$. Similarly let σ' be the inverse mapping of σ, which is a CR-diffeomorphism from M' to M. Since M' is also strongly pseudoconvex, the same argument as before shows that σ' extends to a holomorphic map $\bar{\sigma}' : V' \to \mathbf{C}^{n+1}$ such that $\bar{\sigma}'(V') = V$. $\bar{\sigma}' \circ \bar{\sigma} : V \to V$ is a holomorphic mapping which extends the identity map $\mathrm{Id} : \partial V \to \partial V$. By the uniqueness of the extension, we conclude that $\bar{\sigma}' \circ \bar{\sigma} : V \to V$ is an identity map. Similarly, $\bar{\sigma} \circ \bar{\sigma}' : V' \to V'$ is an identity map. Hence, $\bar{\sigma} : V \to V'$ is biholomorphic and $k = k'$. Without loss of generality, we may assume that $\bar{\sigma}(p_i) = q_i$ and hence $\tau_i = \tau_i'$ for all $1 \leq i \leq k$, where τ_i and τ_i' are local moduli of V and V' at p_i and q_i, respectively. It follows that

$$\tau(M) = \sum_{i=1}^{k} \tau_i = \sum_{i=1}^{k} \tau_i' = \tau(M').$$

For $n \geq 3$, $\tau(M) = \dim H_{\mathrm{KR}}^{p,q}(M)$ for $p+q = n-1$, n and $1 \leq q \leq n-2$. This was proved in our previous paper [9].

Finally, it is easy to see that τ_i vanishes if and only if V is smooth at p_i. Hence $\tau(M) = 0$ if and only if $\tau_i = 0$ for all $1 \leq i \leq k$ if and only if V is smooth. q.e.d.

The following well-known remark, which is included here for the convenience of the readers, implies that $\tau_i = \tau_i'$ for all $1 \leq i \leq k$ in the above proof of Theorem 2.

Let $\mathscr{O}_{n+1}$ denote the ring of germs at the origin of holomorphic functions $(\mathbf{C}^{n+1}, 0) \to \mathbf{C}$. If $(V, 0)$ is a germ at the origin of a hypersurface in $\mathbf{C}^{n+1}$, let $I(V)$ be the ideal of functions in $\mathscr{O}_{n+1}$ vanishing on V, and let f be a generator of $I(V)$. It is well known that $V - \{0\}$ is nonsingular if and only if the $\mathbf{C}$-vector space

$$A(V) = \mathscr{O}_{n+1} \Big/ \left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right) \mathscr{O}_{n+1}$$

is finite dimensional. In this case,

$$A(V) \cong \mathbf{C}[[z_0, z_1, \cdots, z_n]] \Big/ \left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right).$$

The germ at 0 of $A(V)$ is the base space for the miniversal deformation of $(V, 0)$. So $A(V)$, provided with the obvious $\mathbf{C}$-algebra structure, is called the *moduli algebra* of V.

Remark 2.6. Suppose $(V, 0)$ and $(W, 0)$ are germs of hypersurfaces in $\mathbf{C}^{n+1}$, and $V - \{0\}$ is nonsingular. If $(V, 0)$ is biholomorphically equivalent to $(W, 0)$, then $A(V)$ is isomorphic to $A(W)$ as a $\mathbf{C}$-algebra.

Proof. Let f and g be generators of $I(V)$ and $I(W)$, respectively. Let $h\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ be the germ at the origin of a biholomorphically mapping such that $h(V) = W$. Then there exists $u \in \mathscr{O}_{n+1}$ such that $f = u(g \circ h)$ and $u(0) = 0$. Write $h = (h_0, h_1, \cdots, h_n)$, where $h_i\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$. Then

$$\frac{\partial f}{\partial z_i} = \frac{\partial u}{\partial z_i}(g \circ h) + u \sum_{j=0}^{n} \left(\frac{\partial g}{\partial z_j} \circ h \right) \frac{\partial h}{\partial z_i}.$$

Hence, $\frac{\partial f}{\partial z_i}$ is in the ideal generated by $g \circ h$, $\frac{\partial g}{\partial z_0} \circ h, \cdots, \frac{\partial g}{\partial z_n} \circ h$. A similar argument shows that $\frac{\partial g}{\partial z_i}$ is in the ideal generated by $f \circ h^{-1}$, $\frac{\partial g}{\partial z_0} \circ h^{-1}, \cdots, \frac{\partial g}{\partial z_n} \circ h^{-1}$. From this, it follows immediately that

$$h^* \left(g, \frac{\partial g}{\partial z_0}, \cdots, \frac{\partial g}{\partial z_n} \right) \mathscr{O}_{n+1} = \left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n} \right) \mathscr{O}_{n+1},$$

where $h^*\colon \mathscr{O}_{n+1} \to \mathscr{O}_{n+1}$ is the $\mathbf{C}$-algebra isomorphism defined by $h^* u = u \circ h$. Hence h^* induces the $\mathbf{C}$-algebra isomorphism $A(V) \cong A(W)$.

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