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A CHARACTERIZATION OF THE 3RD STANDARD IMMERSIONS OF SPHERES INTO A SPHERE

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Dedicated to Professor Shigeru Ishihara on his 60th birthday

Introduction

Let M and $\overline{M}$ be complete connected Riemannian manifolds of dimension $n(\geqslant 2)$ and $n+p$ respectively. Hong [3] introduced a notion of planar geodesic immersions as follows: An isometric immersion f of M into $\overline{M}$ is called a *planar geodesic immersion* if each geodesic on M is locally mapped under the immersion into a 2-dimensional totally geodesic submanifold of $\overline{M}$. Planar geodesic immersions of M into an $(n+p)$-dimensional sphere $S^{n+p}(c)$ of constant curvature c have been completely classified by Little [4] and Sakamoto [9] independently, who stated that M is a compact symmetric space of rank one, and f is rigid to the 2nd or 1st standard immersion according as M is a sphere or not. In particular, concerning with isotropic immersions which are introduced by O'Neill [8], Sakamoto proves that the following properties are equivalent:

(1) f is nonzero constant isotropic and parallel,

(2) f is planar geodesic,

(3) for any geodesic γ on M, $f \circ \gamma$ is a circle on $\overline{M}$.

On the other hand, minimal immersions of compact symmetric spaces into a sphere have been investigated by Wallach [11]. Let $M = G/K$ be a compact symmetric space where the isotropy action of K is irreducible, and let Δ be the Laplacian operator for $(M, \langle\,,\rangle)$, where $\langle\,,\rangle$ is some G-invariant Riemannian structure up to scalar multiple. Let V_λ be an eigenspace with an eigenvalue λ of Δ, and for any real-valued functions g_1 and g_2 on M, let $(g_1, g_2) = \int_M g_1 g_2 dM$. Then V_λ is a vector space over $\mathbf{R}$ endowed with the inner product $(\,,)$. For each nonzero eigenvalue λ, let $\{g_1, \cdots, g_{q+1}\}$ be an orthonormal basis of V_λ, where

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$q+1=\dim V_\lambda$. We then define $f_\lambda\colon M \to \mathbf{R}^{q+1}$ by

$$f_\lambda(x) = \big(g_1(x),\cdots,g_{q+1}(x)\big)/(q+1)^{1/2}.$$

Then it is seen in [11] that f_λ is a minimal isometric immersion of $(M, \langle\,,\rangle)$ into S^{q+1}, which is called the *standard immersion* of M. With regard to the degree of the immersion in the sense of Wallach, if the degree of f_λ is not greater than 3, then it is rigid. In particular, in the case where M is a compact symmetric space of rank one, we denote by f_r the standard immersion corresponding to an rth eigenvalue λ_r. It is seen that a planar geodesic immersion is closely related the first standard immersion f_1 which is with degree 2 except for the sphere. When the rigidity of the standard immersion f_r is being carefully considered, it seems important to study the structure of the immersion with degree 3. As a matter of fact, the degree of the standard immersion f_r is calculated by do Carmo-Wallach [1] and Mashimo [5], [6], whose results imply that f_3 of S^n into $S^{N(3)}$ is the only one example with degree 3, where $N(3) = n(n+1)(n+5)/6 - 1$. In order to characterize geometrically the immersion, the author [7] introduces a notion of cubic geodesic immersions, which is more general than that of the planar geodesic immersions. An isometric immersion of M into $\overline{M}$ is called a *cubic geodesic immersion* if each geodesic in M is locally mapped under the immersion into a 3-dimensional totally geodesic submanifold of $\overline{M}$. The standard immersion f_3 of S^n into $S^{N(3)}$ is of course an easiest model of the cubic geodesic immersion. As a characterization of f_3, the following theorem for cubic geodesic immersions is proved in [7].

Theorem. *Let M be an $n(\geqslant 3)$-dimensional compact simply connected Riemannian manifold, and f be a full isometric immersion of M into $S^{n+p}(c)$, where $p \geqslant 2$. If f is minimal and isotropic, then the following statements hold*:

(1) *If f is properly cubic geodesic, then for each geodesic γ in M, $f\circ\gamma$ is a helix in $S^{n+p}(c)$.*

(2) *If $f\circ\gamma$ is a helix in $S^{n+p}(c)$ for any geodesic γ in M, then M is isometric to S^n, and f is rigid to the standard immersion f_3 of S^n into $S^{N(3)}$.*

In this paper, we shall be concerned with another analytical characterization of the standard immersion f_3 of S^n into $S^{N(3)}$, which generalizes the concept of parallelness of the second fundamental form. The purpose of this paper is to prove the following.

Theorem. *Let M be an $n(\geqslant 2)$-dimensional complete simply connected Riemannian manifold, and let f be a full isometric immersion of M into $S^{n+p}(c)$. If f is minimal and isotropic, and the second fundamental form σ of f satisfies*

$$(0.1) \qquad (\nabla'^2\sigma)(u,u,u,u) + l^2\sigma(u,u) = 0$$

for any unit vector u at any point x in M, where l is a positive constant, then M is isometric to S^n, and f is rigid to f_3.

1. Preliminaries

Let M be an $n(\geqslant 2)$-dimensional connected orientable Riemannian manifold equipped with the Riemannian metric g, and let f be an isometric immersion of (M, g) into $(n + p)$-dimensional Riemannian manifold $(\overline{M}, \bar{g})$. The metrics on the tangent bundles TM and $T\overline{M}$ are denoted by $\langle\,,\rangle$. Let ∇ and $\overline{\nabla}$ be the Riemannian connection on (M, g) and $(\overline{M}, \bar{g})$ respectively. The metric and the connection on the pull back $f^{-}T\overline{M}$ induced from $\langle\,,\rangle$ and $\overline{\nabla}$ are also denoted by $\langle\,,\rangle$ and $\overline{\nabla}$. Moreover, we have an orthogonal sum $f^{-}T\overline{M} = TM \oplus NM$, where NM denotes the normal bundle for f. Let D denote the normal connection on NM induced from $\overline{\nabla}$. Now denote by X and Y (resp. ξ and η) vector fields tangent (resp. normal) to M. We then recall the following equations, which are called *Gauss and Weingarten formulas*:

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{1.1}$$

$$\overline{\nabla}_X \xi = -A_\xi(X) + D_X \xi \tag{1.2}$$

for any vector fields X, Y and ξ on M. The tensors σ and A_ξ are called a *second fundamental form of f* and a shape operator in the direction of ξ respectively. $A_\xi(X)$ is bilinear in ξ and X, and for each normal vector field ξ on M we have

$$\langle A_\xi(X), Y\rangle = \langle \sigma(X, Y), \xi\rangle \tag{1.3}$$

for any $X, Y \in TM$. Hence A_ξ is symmetric and self-adjoint with respect to $\langle\,,\rangle$. We define a linear mapping A of NM into a set $\mathrm{End}(TM, TM)$ of all symmetric linear transformations of TM by $A(\xi) = A_\xi$.

We next define a connection ∇' induced on the vector bundle $f^{-}T\overline{M}$ as follows: For any NM-valued tensor field T of type $(0, k)$ we define $\nabla'_X T$ by

$$\begin{aligned}(\nabla'_X T)&(Y_1,\cdots,Y_k)\\ &= D_X(T(Y_1,\cdots,Y_k)) - \sum_{r=1}^{k} T(Y_1,\cdots,\nabla_X Y_r,\cdots,Y_k)\end{aligned} \tag{1.4}$$

for any vector fields $X, Y_1,\cdots,Y_k$ on M, and $\nabla' T$ is defined by $(\nabla' T)(Y_1,\cdots,Y_k, X) = (\nabla'_X T)(Y_1,\cdots,Y_k)$. It is an NM-valued tensor field of type $(0, k + 1)$. Furthermore we denote by $\nabla'^2 T$ the covariant derivative of $\nabla' T$ with respect to the induced connection ∇', and then we can inductively define $\nabla'^m T$. Denote by R and $\overline{R}$ the Riemannian curvature tensors for ∇ and $\overline{\nabla}$ respectively. We recall following fundamental equations which are called the *equation of Gauss and Codazzi* respectively:

$$\begin{aligned}\langle R(X, Y)Z, W\rangle = \langle \overline{R}(X, Y)Z, W\rangle &+ \langle \sigma(X, W), \sigma(Y, Z)\rangle\\ &-\langle \sigma(X, Z), \sigma(Y, W)\rangle,\end{aligned} \tag{1.5}$$

$$(1.6)\qquad \{\bar{R}(X,Y)Z\}^N = (\nabla'\sigma)(Y,Z,X) - (\nabla'\sigma)(X,Z,Y)$$

for any vector fields on M, where the superscript N denotes the normal component. In particular, for the second fundamental form σ it follows from (1.2) and (1.4) that

$$(1.7)\qquad \begin{aligned}\bar{\nabla}_X(\sigma(Y,Z)) = (\nabla'\sigma)(Y,Z,X) + \sigma(\nabla_X Y, Z)\\ +\sigma(Y,\nabla_X Z) - A_{\sigma(Y,Z)}(X).\end{aligned}$$

The immersion f is said to be *parallel* if $\nabla'\sigma = 0$.

We denote by $M^m(c)$ an m-dimensional complete simply connected Riemannian manifold of constant curvature c, which is called a *real space form*; it consists of a sphere $S^m(c)$, a Euclidean space $\mathbf{R}^m$ and a hyperbolic space $H^m(c)$. Fron now on we assume that *the ambient space is a real space form of constant curvature c*. Then the equations of Gauss and Codazzi are reduced to

$$(1.5)'\qquad \begin{aligned}\langle R(X,Y)Z,W\rangle = c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)\\ +\langle\sigma(X,W),\sigma(Y,Z)\rangle - \langle\sigma(X,Z),\sigma(Y,W)\rangle,\end{aligned}$$

$$(1.6)'\qquad (\nabla'\sigma)(Y,Z,X) = (\nabla'\sigma)(X,Z,Y)$$

for any vector fields X, Y, Z on W on M. The normal vector field defined by $\mathfrak{h} = \operatorname{Tr}\sigma/n$ is called a *mean curvature vector* of the immersion. In the case where the mean curvature vector $\mathfrak{h}$ vanishes identically, f is said to be *minimal*.

Now for any fixed point x in M and any unit vector u at x, the vector $\sigma(u,u)$ is called a *normal curvature vector* in the direction of u. If every normal curvature vector has the same length for any unit vector u at x, then the immersion is said to be *isotropic* at x. If f is isotropic at any point on M, namely if the length of a normal curvature vector depends only on the initial point, then the immersion is said to be λ-isotropic, where λ is the length. The isotropy λ is continuous, the square of which is smooth on M. The immersion f is λ-isotropic at x if and only if the second fundamental form satisfies

$$(1.8)\qquad \mathfrak{S}_3\langle\sigma(u_1,u_2),\sigma(u_3,v)\rangle = \lambda^2\mathfrak{S}_3\langle u_1,u_2\rangle\langle u_3,v\rangle$$

for any unit vectors u_i $(i = 1,2,3)$ and v, where $\mathfrak{S}_m$ denotes the cyclic sum with respect to vectors $u_1,\cdots,u_m$. This is equivalent to

$$(1.9)\qquad \langle\sigma(u,u),\sigma(u,v)\rangle = 0$$

for any orthogonal vectors u and v at x [7]. If λ is constant on M, then f is said to be *constant isotropic*. In the sequel we assume that *the immersion f is nonzero constant isotropic on M with the constant isotropy k, and moreover it is minimal*. The minimalness implies that for any orthonormal basis $\{e_1,\cdots,e_n\}$ of the tangent space M_x at any point x, $\Sigma_i(\nabla'\sigma)(e_i,e_i,u) = 0$ and $\Sigma_i(\nabla'^2\sigma)(e_i,e_i,u,v) = 0$ for any vectors u and v at x.

Remark. In the case where $n \geqslant 3$, if f is isotropic and minimal, then it is constant isotropic. In fact, M is an Einstein manifold, and the Ricci curvature S is given by $S(X, Y) = \{(n-1)c - \frac{1}{2}(n+2)\lambda^2\}\langle X, Y\rangle$ for any vector fields X and Y.

From the definition of the connection ∇' and the fact that f is k-isotropic it follows that

$$\begin{aligned} s\mathfrak{S}_3\{\langle(\nabla'\sigma)(X_1, X_2, Y), \sigma(X_3, X)\rangle \\ +\langle\sigma(X_1, X_2), (\nabla'\sigma)(X_3, X, Y)\rangle\} = 0, \end{aligned} \tag{1.10}$$

and it implies

$$\langle(\nabla'\sigma)(X, X, Y), \sigma(X, X)\rangle = 0$$

and $\langle(\nabla'\sigma)(X, X, X), \sigma(X, Y)\rangle = 0$ for any vector fields. Accordingly, they yield $\mathfrak{S}_4\langle(\nabla'\sigma)(X_1, X_2, X_3), \sigma(X_4, Y)\rangle = 0$, which together with (1.10) implies that

$$\langle(\nabla'\sigma)(X_1, X_2, X_3), \sigma(X, Y)\rangle = \mathfrak{S}_3\langle\sigma(X_1, X_2), (\nabla'\sigma)(X_3, X, Y)\rangle \tag{1.11}$$

for any vector fields.

2. Locally symmetric spaces

Let f be an isotropic and minimal immersion of M into $\overline{M} = M^{n+p}(c)$, and A the shape operator of NM into End(TM, TM) which is a symmetric and self-adjoint mapping. This section is devoted to finding a sufficient condition for M to be locally symmetric. We suppose that A satisfies

$$\operatorname{Tr} A_\xi A_{\sigma(X,Y)} = L\langle A_\xi(X), Y\rangle \tag{2.1}$$

for any vector fields X, Y and ξ, where L is a constant. For any fixed point x on M, we consider a sufficiently small neighborhood of x. For a vector field Z, let $c = c(t)$ be a smooth curve passing through $c(0) = x$ and satisfying $c'(0) = Z(x)$. For any vector field X the value of $\nabla_Z X$ at x depends only on the vector $Z(x)$ and the value of X along the curve c. Accordingly we may suppose that $\nabla_Z e_i = 0$ for any orthonormal basis $\{e_1, \cdots, e_n\}$ of M_x, because each of them is extended to a parallel vector field along c for small values of t. Differentiating (2.1) in the direction of Z and taking account of the derivative of the function $\langle A_\xi(X), Y\rangle$ with respect to Z, we have

$$\begin{aligned} Z(\operatorname{Tr} A_\xi A_{\sigma(X,Y)}) &= \operatorname{Tr} A_{(\nabla'\sigma)(X,Y,Z)} A_\xi + \operatorname{Tr} A_{\sigma(\nabla_Z X, Y)} A_\xi \\ &\quad + \operatorname{Tr} A_{\sigma(X, \nabla_Z Y)} A_\xi + \sum_i \langle\sigma(X, Y), (\nabla'\sigma)(A_\xi(e_i), e_i, Z) \\ &\quad + \sigma(\nabla_Z(A_\xi(e_i)), e_i)\rangle. \end{aligned} \tag{2.2}$$

Under the assumption (2.1), from (2.2) it follows that

$$\begin{aligned} &L\langle(\nabla'\sigma)(X,Y,Z),\xi\rangle + L\langle\sigma(X,Y),D_Z\xi\rangle \\ &\quad = \operatorname{Tr} A_{(\nabla'\sigma)(X,Y,Z)}A_\xi + \sum_i \langle\sigma(X,Y),(\nabla'\sigma)\big(A_\xi(e_i),e_i,Z\big) \\ &\qquad + \sigma\big(\nabla_Z\big(A_\xi(e_i)\big),e_i\big)\rangle. \end{aligned}$$

By virtue of the above equation at x and the property $\langle A_\xi(X),Y\rangle = \langle\xi,\sigma(X,Y)\rangle$, we have

$$\begin{aligned} L\langle(\nabla'\sigma)(X,Y,Z),\xi\rangle = {}& \operatorname{Tr} A_{(\nabla'\sigma)(X,Y,Z)}A_\xi \\ &+ \sum_i \left\{\langle\sigma(X,Y),(\nabla'\sigma)\big(Z,A_\xi(e_i),e_i\big)\rangle\right. \\ &\left. + \langle\xi,(\nabla'\sigma)\big(Z,A_{\sigma(X,Y)}(e_i),e_i\big)\rangle\right\}. \end{aligned}$$

Thus using (1.11) and taking the cyclic sum with respect to X, Y and Z give the following relation at x:

$$\begin{aligned} 3L\langle(\nabla'\sigma)(X,Y,Z),\xi\rangle = {}& 4\operatorname{Tr} A_{(\nabla'\sigma)(X,Y,Z)}A_\xi \\ &+ \mathfrak{S}_3\Sigma_i\langle\xi,(\nabla'\sigma)\big(Z,A_{\sigma(X,Y)}(e_i),e_i\big)\rangle \end{aligned} \tag{2.3}$$

for any vector fields.

Now we consider a normal vector $\xi = \Sigma_j\sigma(e_j, A_{(\nabla'\sigma)(X,Y,Z)}(e_j))$ at x. Then the left-hand side of (2.3) is equal to $3L\operatorname{Tr} A_{(\nabla'\sigma)(X,Y,Z)}{}^2$, and the second term of the right-hand side of (2.3) is equal to $3\Sigma_{i,j}\langle\sigma(e_j, A_{(\nabla'\sigma)(X,Y,Z)}(e_j))$, $(\nabla'\sigma)(Z, A_{\sigma(X,Y)}(e_i), e_i)\rangle$, because $\nabla'\sigma$ is symmetric. Consequently, combining this relation together with (2.1) we have

$$\begin{aligned} 0 = L\operatorname{Tr} A_{(\nabla'\sigma)(X,Y,Z)}{}^2 &+ 3\sum_{i,j,k,l}\langle\sigma(e_i,e_j),(\nabla'\sigma)(X,Y,Z)\rangle \\ &\cdot\langle\sigma(e_i,e_j),(\nabla'\sigma)(e_k,k_1,Z)\rangle\langle\sigma(e_k,e_1),\sigma(X,Y)\rangle, \end{aligned}$$

from which it follows that

$$\begin{aligned} 0 = L&\sum_{i,j,k,l}\|A_{(\nabla'\sigma)(e_i,e_j,e_k)}(e_1)\|^2 \\ &+3\sum_{i,j,m}\|\Sigma_{k,l}\langle\sigma(e_i,e_j),(\nabla'\sigma)(e_k,e_1,e_m)\rangle\sigma(e_k,e_1)\|^2. \end{aligned} \tag{2.4}$$

Given an orthonormal normal frame $\{\xi_{n+1},\cdots,\xi_{n+p}\}$, we denote simply by A_α the shape operator $A(\xi_\alpha)$ for the normal vector ξ_α $(\alpha = n+1,\cdots,n+p)$. In the sequel, indices $\alpha,\beta,\cdots$ run over the range $(n+1,\cdots,n+p)$. We define a symmetric linear transformation $H = (H_{\alpha\beta})$ on NM by $H_{\alpha\beta} = \operatorname{Tr}(A_\alpha A_\beta)$. Using (2.4) we shall verify the following.

Lemma 2.1. *If the shape operator A of NM into* $\mathrm{End}(TM, TM)$ *satisfies* $\operatorname{Tr} A_\xi A_{\sigma(X,Y)} = L\langle A_\xi(X), Y\rangle$ *for any vector fields X, Y and ξ, where L is a constant, then M is locally symmetric.*

Proof. In terms of the symmetric linear transformation H, it is easily seen that the condition given in Lemma 2.1 is equivalent to $H(\sigma(X, Y)) = L\sigma(X, Y)$. Since the matrix $(H_{\alpha\beta})$ is positive semi-definite, from the condition it follows that L is nonnegative.

Suppose that $L = 0$. Then it is easily seen that the immersion f is totally geodesic. Since it is a contradiction to the nonzero constant isotropic immersion, L must be a positive constant. Taking account of this property and (2.4), we see $A_{(\nabla'\sigma)(e_i, e_j, e_k)}(e_l) = 0$, which means $\langle(\nabla'\sigma)(e_i, e_j, e_k), \sigma(e_l, e_m)\rangle = 0$ for any indices $i, \cdots, m$. Because σ and $\nabla'\sigma$ are linear forms, we have

$$\langle(\nabla'\sigma)(u_1, u_2, u_3), \sigma(u_4, u_5)\rangle = 0$$

for any unit vectors $u_i(1 = 1, \cdots, 5)$ at any fixed point x, from which we can show that M is locally symmetric. This concludes our proof.

3. Second fundamental forms

In this section let M be a complete n-dimensional Riemannian manifold, and f a nonzero constant isotropic minimal immersion of M into $\overline{M} = M^{n+p}(c)$. To begin with let us analyze the assumption concerning the second fundamental form, which is given by

$$(3.1)\qquad (\nabla'^2\sigma)(X, X, X, X) + l^2\|X\|^2\sigma(X, X) = 0$$

for any vector field X on M, where l is a nonnegative constant. Now let φ be an m-form on a vector space V. For any vectors $v_1, \cdots, v_m$ and any permutation τ in a symmetric group of order m, we define an m-form $\tau\phi$ by $(\tau\phi)(v_1, \cdots, v_m) = \phi(v_{\tau(1)}, \cdots, v_{\tau(m)})$, and the symmetrizer $\mathfrak{S}_m$ by $\mathfrak{S}_m\phi(v_1, \cdots, v_m) = \Sigma(\tau\phi)(v_1, \cdots, v_m)$, where the summation runs over all permutations τ. Because of the multi-linearity of σ and $\nabla'^2\sigma$, the equation $\mathfrak{S}_4\{(\nabla'^2\sigma)(X_1, X_2, X_3, X_4) + l^2\langle X_1, X_2\rangle\sigma(X_3, X_4)\} = 0$ follows from (3.1). By taking account of the fact that the 4-form $\nabla'^2\sigma$ is symmetric with respect to the first three elements, and that σ and $\langle\, ,\rangle$ are also symmetric, the above equation is reduced to

$$\begin{aligned}&3!\mathfrak{S}_4(\nabla'^2\sigma)(X_1, X_2, X_3, X_4)\\ &\quad + (2!)^2 l^2\mathfrak{S}_3\{\langle X_1, X_2\rangle\sigma(X_3, X_4) + \langle X_4, X_1\rangle\sigma(X_2, X_3)\} = 0\end{aligned}$$

for any vector fields, which together with the Ricci identity implies

$$
\begin{aligned}
6(\nabla'^2\sigma)(X_1, X_2, X_3, Y) = \mathfrak{S}_3\{(-3c + 3k^2 - l^2)\langle X_1, X_2\rangle\sigma(X_3, Y) \\
+ (3c + 3k^2 - l^2)\langle Y, X_1\rangle\sigma(X_2, X_3) \\
- 6\Sigma_\alpha\langle A_\alpha(X_1), X_2\rangle\sigma(X_3, A_\alpha(Y))\}.
\end{aligned} \tag{3.2}
$$

Taking account of the minimality of the immersion, from (3.2) we obtain that

$$12\Sigma_\alpha\sigma(A_\alpha(X), A_\alpha(Y)) = \{-3nc + (n+4)(3k^2 - l^2)\}\sigma(X, Y)$$

for any vector fields X and Y, where k is the constant isotropy. By using the relation $\Sigma_\alpha\sigma(A_\alpha(X), A_\alpha(Y)) = \Sigma_{i,j}\langle\sigma(X, e_i), \sigma(Y, e_j)\rangle\sigma(e_i, e_j)$, the condition of the isotropic immersion, and the left-hand side of the above equation we can easily show that the condition of Lemma 2.1 is satisfied, where $L = \frac{1}{6}\{3nc - 3nk^2 + (n+4)l^2\}$. Thus we arrive at

Lemma 3.1. *M is locally symmetric.*

Since the sectional curvature is continuous on $S_p \times S_p$ for any point p, where S_p is a unit sphere in M_p, there exist orthonormal vectors u and v at p in such a way that the sectional curvature $K(u, v)$ of the linear space spanned by u and v attains the minimal value δ_0. We define a linear transformation K_u of M_p into itself by $K_u w = R(w, u)u$ for any vector w at p, and then v becomes an eigenvector of K_u with an eigenvalue δ_0, because δ_0 is the minimal value of the sectional curvatures at p. Therefore by the Gauss equation we get

$$3\langle\sigma(u, v), \sigma(u, w)\rangle = \big(c + k^2 - \delta_0\big)\langle v, w\rangle \tag{3.3}$$

for any vector w at p.

Proposition 3.2. *A complete n-dimensional Riemannian manifold M cannot be minimally and nonzero constant isotropically immersed in* $\mathbf{R}^{n+p}$ *or* $H^{n+p}(c)$, *so that condition* (3.1) *is satisfied.*

Proof. For any point x in M and any unit vector w at x, let γ_w be a geodesic parametrized by the arc length and passing through $x = \gamma_w(0)$ with the initial vector w. Now we set $W = \gamma'_w$. Differentiating (3.2) in the direction of W and taking account of the definition of $\nabla'^3\sigma$, we obtain

$$
\begin{aligned}
&6(\nabla'^3\sigma)(X_1, X_2, X_3, Y, W) \\
&\quad = \mathfrak{S}_3\{(-3c + 3k^2 - l^2)\langle X_1, X_2\rangle(\nabla'\sigma)(X_3, Y, W) \\
&\quad\quad + (3c + 3k^2 - l^2)\langle Y, X_1\rangle(\nabla'\sigma)(X_2, X_3, W) \\
&\quad\quad - 6\Sigma_\alpha\langle A_\alpha(X_1), X_2\rangle(\nabla'\sigma)(X_3, A_\alpha(Y), W)\}.
\end{aligned}
$$

From this equation and the Ricci identity with respect to $(\nabla'^3\sigma)(X, Y, Z, V, W)$ it follows that

$$\begin{aligned}(3.4)\qquad &\mathfrak{S}_3\Big[\tfrac{1}{6}(9c + 3k^2 - l^2)\{\langle W, X_1\rangle(\nabla'\sigma)(X_2, X_3, Y)\\ &\qquad -\langle Y, X_1\rangle(\nabla'\sigma)(X_2, X_3, W) \cdot +\Sigma_\alpha\langle A_\alpha(X_1), X_2\rangle\}\\ &\qquad \cdot \{(\nabla'\sigma)(X_3, A_\alpha(Y), W) - (\nabla'\sigma)(X_3, A_\alpha(W), Y)\}\\ &\qquad +\Sigma_\alpha\{\langle A_\alpha(W), X_1\rangle(\nabla'\sigma)(X_2, X_3, A_\alpha(Y))\\ &\qquad\qquad -\langle A_\alpha(Y), X_1\rangle(\nabla'\sigma)(X_2, X_3, A_\alpha(W))\}\Big] = 0,\end{aligned}$$

in which we set $X_i = Y (i = 1, 2, 3)$, and suppose that Y and W are orthonormal. We then have

$$\begin{aligned}(3.5)\qquad &(9c + 9k^2 - l^2)(\nabla'\sigma)(Y, Y, W)\\ &\qquad -30\Sigma_\alpha\langle A_\alpha(Y), W\rangle(\nabla'\sigma)(Y, Y, A_\alpha(Y)) = 0.\end{aligned}$$

Now for any orthonormal basis $\{e_1, \cdots, e_n\}$ of M_x, the vector $\Sigma_\alpha\langle A_\alpha(Y), W\rangle A_\alpha(Y)$ at x is expressed as $\Sigma_i\langle\sigma(Y, W), \sigma(Y, e_i)\rangle e_i$. At the given point p, combining (3.5) and the above relation together with (3.3), we obtain $(10\delta_0 - c - k^2 - l^2)(\nabla'\sigma)(u, u, v) = 0$. Since the immersion is k-isotropic, (3.3) yields $3\langle\sigma(u, u), \sigma(v, w)\rangle = (-2c + k^2 + 2\delta_0)\langle v, w\rangle$. Taking account of (3.4) at the point p, setting $X_1 = Y = u$ and $X_2 = X_3 = W = v$, and combining the equation together with (3.5) and the above equation, we have

$$(c + k^2 + l^2 - 10\delta_0)(\nabla'\sigma)(v, v, v) = 0$$

by a direct calculation.

Suppose that l is positive. Then we have

$$(3.6)\qquad c + k^2 + l^2 - 10\delta_0 = 0,$$

unless it follows from the last equation that $(\nabla'\sigma)(v, v, v) = 0$, which shows that $k^2 l^2 = 0$, because of the assumption $(\nabla'^2\sigma)(V, V, V, V) = -l^2\sigma(V, V)$. Since (3.3) implies that $c + k^2 - \delta_0 \geqslant 0$, (3.6) means $9\delta_0 \geqslant l^2$. Thus M must be of positive curvature and is compact by Myers' theorem. This together with the property of minimal immersions shows that c is positive.

In the case where $l = 0$, it is easy to see that c is positive. Thus the proof is complete.

As a direct consequence of the process of the proof, Lemma 3.1, and the well known properties about symmetric spaces we have the following lemma.

Lemma 3.3. *If l is positive, then the universal covering manifold of M is a compact symmetric space of rank one. If $l = 0$, then f is parallel.*

Remark 1. We give here an example of submanifolds satisfying $\nabla'\sigma \not\equiv 0$ and $\nabla'^2\sigma \equiv 0$. Let M be a surface in a Euclidean 4-space $\mathbf{R}^4$ defined by

$$x^1 = \int_0^r \cos t^2 dt,\ x^2 = \int_0^r \sin t^2 dt,\ x^3 = \int_0^s \cos t^2 dt,\ x^4 = \int_0^s \sin t^2 dt.$$

This immersion satisfies above properties, but its surface is neither minimal nor isotropic.

Remark 2. Parallel immersions of M into a real space form and the submanifolds have been completely classified by Takeuchi [10]. By means of the above example it seems that isometric immersions of M into $M^{n+p}(c)$ satisfying $\nabla'\sigma \not\equiv 0$ and $\nabla'^2\sigma \equiv 0$ and submanifolds cannot be classified in the form similar to the beautiful one in [10].

Remark 3. The example in Remark 1 can be generalized to a complete hypersurface in $\mathbf{R}^{n+1}$, which preserves above properties for the immersion.

4. Determination of M and f

In this section let M be a complete simply connected $n(\geqslant 3)$-dimensional Riemannian manifold, and let f be a full isotropic minimal immersion of M into $S^{n+p}(c)$, where the constant isotropy k is positive. The isometric immersion $f\colon M \to S^{n+p}(c)$ is said to be *full*, if $f(M)$ is not contained in any totally geodesic hypersurface of $S^{n+p}(c)$. In particular, the condition (0, 1) for the second fundamental form is assumed. This section is devoted to determining completely the submanifold M and the immersion f. Now we consider the decomposition of the normal space with respect to the immersion f. For any point x in M, the normal space N_x is given by $N_x = (df_x(M_x))^N$, superscript N means the orthogonal complement into the tangent space $\overline{M}_{f(x)}$ of the ambient space. The second fundamental form σ_x at x is a linear symmetric map of $M_x \times M_x$ into N_x, and satisfies $\sigma_x(X, Y) = (\overline{\nabla}_X Y)^N$, where X and Y are vector fields on a neighborhood of x in M. For convenience' sake, we put $\sigma_{2x} = \sigma_x$, and so σ_{2x} can be regarded as a linear map of a symmetric square $S^2(M_x)$ for M_x into the normal space N_x. Set $N_x^1 = \sigma_{2x}(S^2(M_x))$, which is called a *first normal space* of the f at x. Thus N_x^1 is the linear subspace of N_x spanned by normal vectors $\sigma_x(u, v)$ for any vectors u and v at x, so we see that $\dim N_x^1 \leqslant \frac{1}{2}(n + 2)(n - 1)$, because f is minimal. A point x in M is said to be *2-regular* if N_x^1 is of maximal dimension with respect to the basic points. Here we calculate the dimension of the first normal space. Denote by K_0 (resp. k_0) the maximum (resp. minimum) of the sectional curvature of M at x, and choose an orthonormal basis $\{e_1, \cdots, e_n\}$ in M_x in such a way that $K(e_1, e_2) = K$, where $K = K_0$ or k_0. Then $K_{e_1}e_2 = Ke_2$ and $K_{e_2}e_1 = Ke_1$ for the curvature transformation K_u, which together with the Gauss formula implies that $\langle\sigma_{11}, \sigma_{2r}\rangle - \langle\sigma_{12}, \sigma_{1r}\rangle = 0$ for $r \geqslant 3$, where $\sigma(e_i, e_j)$ is simply expressed as σ_{ij}. Because f is k-isotropic, we have

$$\langle\sigma_{11}, \sigma_{2r}\rangle = \langle\sigma_{12}, \sigma_{1r}\rangle = 0, \quad r \geqslant 3. \tag{4.1}$$

By Lemma 3.3, the submanifold may be considered as an n-dimensional symmetric space of rank one, which consists of a sphere S^n, a complex projective space $P\mathbf{C}^n$ ($n = 2m \geqslant 4$), a quaternion projective space PQ^n ($n = 4m \geqslant 8$) and a Cayley projective space PCa. When the curvature transformation K_u with respect to the vector u at x is regarded as the linear transformation of the orthogonal complement to the vector u in M_x, we suppose that the maximal eigenvalue of K_u has multiplicity $s - 1$. We divide the range $I = \{1,\cdots,n\}$; $n = ms$, into m parts $I_1,\cdots,I_m$, where $I_p = \{(p-1)s+1,\cdots,ps\}$, and indices $p, q, \cdots$ run over the range $1,\cdots,m$. Then for any point x there exists an orthonormal basis $\{e_1,\cdots,e_m\}$ of M_x such that $K_{ij} = K(e_i, e_j) = K_0 = 4k_0$ for $i, j \in I_p$ or k_0 for $i \in I_p, j \in I_q, p \neq q$. Denote $\langle R(e_i, e_j)e_k, e_1\rangle$ by R_{ijkl}. From this value of the sectional curvatures and (4.1) it follows

$$R_{ijki} = 0, \quad \langle\sigma_{ii}, \sigma_{jk}\rangle = \langle\sigma_{ij}, \sigma_{ik}\rangle = 0 \tag{4.2}$$

for mutually distinct induces i, j and k. By a direct calculation and applying the first equation of (4.2), we obtain the following relations for any constants a, b, c and d such that $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$, and mutually distinct indices i, j, k and l:

$$\begin{aligned} K(ae_i + be_j, ce_k + de_l) = {} & a^2c^2K_{ik} + a^2d^2K_{il} + b^2c^2K_{jk} \\ & + b^2d^2K_{jl} + 2abcd(R_{iklj} + R_{ilkj}). \end{aligned} \tag{4.3}$$

On the other hand, the isotropicness and the Gauss formula imply

$$3\langle\sigma_{ij}, \sigma_{kl}\rangle = R_{iklj} + R_{ilkj} \tag{4.4}$$

for mutually distinct indices. The right-hand side of the above equation shall be shown to be equal to 0. In order to prove the fact, the following four cases are considered:

(i) each index is contained in a different range,

(ii) two of four indices are contained in a certain range; for example, $i, j \in I_p$,

(iii) three of four indices are contained in a certain range; for example, $i, j, k \in I_p$,

(iv) four indices are all contained in a certain range; for example, $i, j, k, l \in I_p$.

For case (i) it is obvious that $K_{ij} = K_{ik} = K_{il} = K_{jk} = K_{jl} = K_{kl} = k_0$, which imply $K_{ae_j+be_j}e_k = k_0 e_k$ and $K_{ae_i+be_j}e_l = k_0 e_l$. These show that $K(ae_i + be_j, ce_k + de_l) = k_0$, from which it follows that the necessary conclusion is given. In case (iv) we can come to the conclusion after the discussion similar to that of case (i).

Case (ii) is also easily verified. Lastly we shall investigate the remaining case (iii). Under the assumption we get $K_{e_l}e_i = k_0 e_i$, $K_{e_l}e_j = k_0 e_j$ and $K_{e_l}e_k = k_0 e_k$, which imply $K(ae_i + be_j + ce_k, e_1) = k_0$ for any constants a, b and c such that $a^2 + b^2 + c^2 = 1$. Accordingly we see that $K_{ae_i+be_j+ce_k}e_l = k_0 e_l$, so that $R_{iklj} + R_{ilkj} = 0$ by taking the inner product of the left-hand side of this equation with the vector e_k. Using the first Bianchi's formula we come to the conclusion. Thus

$$\langle \sigma_{ij}, \sigma_{kl} \rangle = 0 \tag{4.5}$$

for mutually distinct indices follows from (4.4). This will lead the following lemma.

Lemma 4.1. *The dimension of the first normal space is constant on M, and any point on M is 2-regular.*

Proof. For the orthonormal basis $\{e_1, \cdots, e_n\}$ of M_x chosen as above suitably, the Gauss formula implies that the square of the length of σ_{ij} for distinct indices i and j is equal to $\frac{1}{3}(c + k^2 - K_0) = K_1$ or $\frac{1}{3}(c + k^2 - k_0) = k_1$, and moreover $\langle \sigma_{ij}, \sigma_{jj} \rangle = k^2 - 2K_1$ or $k^2 - 2k_1$. Now we suppose that $K_0 = c + k^2$. We then have $\|\sigma_{ij}\|^2 = 0$ (resp. k_0) for $i, j \in I_p$ (resp. $i \in I_p$, $j \in I_q$, $p \neq q$). On the other hand, since the immersion is k-isotropic, $2\sum_{j=1}^n \|\sigma_{lj}\|^2 = (n + 2)k^2$ holds and therefore $(n - s)c = (n + s)k^2$. For any point x in M and the matrix $(H_{\alpha\beta})$ of order p defined by $H_{\alpha\beta} = \operatorname{Tr} A_\alpha A_\beta$, it is seen in [7] that

$$\Sigma_\beta H_{\alpha\beta} A_\beta = L A_\alpha, \quad L = \tfrac{1}{6}\{3nc - 3nk^2 + (n + 4)l^2\}, \tag{4.6}$$

and the rank of the matrix is bounded below from $n(n + 2)/4$ and above from $(n + 2)(n - 1)/2$; in particular, the rank is equal to the upper bound if and only if M is of constant curvature $c - S_2/[n(n - 1)]$, where S_2 is the square of the length of σ. Since $rL = \operatorname{Tr} H = S_2$, the rank r satisfies $r < (n + 2)(n - s)/(2s)$, which contradicts to the lower bound of r. Thus it is possible to assert that K_1 is positive, so that $K_0 < c + k^2$. This yields that each normal vector σ_{ij} is nonzero. In order to prove the lemma it is sufficient to show that the normal vectors σ_{ii}, $1 \leq i \leq n - 1$, and σ_{ij}, $1 \leq i < j \leq n$, are linearly independent. Then $\dim N_x^1 = (n + 2)(n - 1)/2$. Suppose that $\Sigma_{i=1}^{n-1} a_i \sigma_{ii} + \Sigma_{i<j} a_{ij} \sigma_{ij} = 0$ for any constants a_i and a_{ij}. The inner product of this relation with $\sigma_{kl} (k < l)$ implies $a_{kl} = 0$, because of (4.2) and (4.5). Thus

$\Sigma_{i=1}^{n-1} a_i\sigma_{ii} = 0$, and

$$
(4.7)\qquad
\begin{aligned}
\left\langle \Sigma_{i=1}^{n-1} a_i\sigma_{ii}, \sigma_{11} \right\rangle &= a_1k^2 + (a_2 + \cdots + a_s)\left(k^2 - 2K_1\right) \\
&\quad + (a_{s+1} + \cdots + a_{n-1})\left(k^2 - 2k_1\right) \\
&= 0, \\
\left\langle \Sigma_{i=1}^{n-1} a_i\sigma_{ii}, \sigma_{22} \right\rangle &= a_1\left(k^2 - 2K_1\right) + (a_3 + \cdots + a_s)\left(k^2 - 2K_1\right) \\
&\quad + a_2k^2 + (a_{s+1} + \cdots + a_{n-1})\left(k^2 - 2k_1\right) \\
&= 0,
\end{aligned}
$$

from which it follows that $K_1(a_1 - a_2) = 0$, so that $a_1 = a_2$ since K_1 is positive. Similarly we have $a_1 = \cdots = a_s$, $a_{s+1} = \cdots = a_{2s}, \cdots, a_{(m-1),s+1} = \cdots = a_{n-1}$. Accordingly

$$
\begin{aligned}
a_s\left(\sigma_{11} + \cdots + \sigma_{ss}\right) + a_{2s}\left(\sigma_{s+1,s+1} + \cdots + \sigma_{2s,2s}\right) + \cdots \\
+ a_{n-1}\left(\cdots + \sigma_{n-1,n-1}\right) = 0.
\end{aligned}
$$

Substituting the inner product of this equation with σ_{11} for that with σ_{s+1s+1}, we obtain $(a_s - a_{2s})\{k^2 + (s-1)(k^2 - 2K_1) - s(k^2 - 2k_1)\} = 0$. From the property $K_0 < c + k^2$ it follows that $a_s = a_{2s}$. Similarly we have $a_s = a_{2s} = \cdots = a_{(m-1)s}$. Thus we have

$$
a_1\left(\sigma_{11} + \cdots + \sigma_{rr}\right) + a_{r+1}\left(\sigma_{r+1,r+1} + \cdots + \sigma_{n-1,n-1}\right) = 0,
$$

where $r = s(m-1)$. Repeating the similar process one gets that $K_1a_{r+1} = 0$, so that $a_{r+1} = 0$, which asserts that normal vectors σ_{ii} and σ_{ij} are linearly independent, because of (4.7). q.e.d.

The property (4.6) mentioned in the proof above implies that a normal vector σ_{ij} for any indices is an eigenvector of the linear transformation H of the normal space N_x. Since σ_{ij} are linearly independent, the rank r of H is not less than $(n+2)(n-1)/2$, in consequence of the proof of Lemma 4.2. On the other hand, it is already seen that $(n+2)(n-1)/2$ is the upper bound for the value of the rank r. Hence $r = (n+2)(n-1)/2$, and M is of constant curvature. Using this we shall prove

Proposition 4.2. *M is isometric to the sphere of constant curvature* $\dfrac{nc}{3(n+2)}$.

Proof. Since M is of constant curvature, say c_0, it it easily seen that the Gauss equation and the isotropicness imply that the square S_2 of the length of the second fundamental form σ satisfies

$$
(4.8)\qquad 2(n-1)(c - c_0) = (n+2)k^2, \quad S_2 = n(n-1)(c - c_0).
$$

For the square S_3 of the length of $\nabla'\sigma$, the formula of Simons' type in this situation yields

$$(4.9) \qquad S_3 = \frac{n^2}{n+2}\left(c - \frac{2(n+1)}{n}c_0\right)S_2,$$

which implies that S_3 is a nonnegative constant. On the other hand, from the assumption (0. 1) it follows that for any vector field X, $\langle(\nabla'\sigma)(X, X, X), (\nabla'\sigma)(X, X, X)\rangle = k^2 l^2 \| X \|^6$, which implies

$$\begin{aligned}&\mathfrak{S}_6\langle(\nabla'\sigma)(X_1, X_2, X_3), (\nabla'\sigma)(X_4, X_5, X_6)\rangle \\ &\qquad = k^2 l^2 \mathfrak{S}_6\langle X_1, X_2\rangle\langle X_3, X_4\rangle\langle X_5, X_6\rangle\end{aligned}$$

for the symmetrizer $\mathfrak{S}_6$, because of the linearity of $\nabla'\sigma$ and $\langle\, ,\rangle$. Taking account of the minimality of the immersion, we get $6\, S_3 = n(n+2)(n+4)k^2 l^2$ from the above equation, and hence $l^2 = 3S_3/[(n+4)S_2]$ in consequence of (4.8). Thus S_3 is a positive constant, because of the assumption that l is a positive constant. Combining (4.8), (4.9) and this relation, and using that fact M is of constant curvature, we obtain

$$(4.10) \qquad \begin{aligned}(\nabla'^2\sigma)(X_1, X_2, X_3, Y) = \frac{S_3}{n(n+4)S_2}\mathfrak{S}_3\{&2\langle X_1, X_2\rangle\sigma(X_3, Y) \\ &- (n+2)\langle Y, X_1\rangle\sigma(X_2, X_3)\},\end{aligned}$$

which implies that the square S_4 of the length of $\nabla'^2\sigma$ is equal to $3(n+2)S_3^2[n(n+4)S_2]$. On the other hand, the formula of Simons' type for the tensor field $\nabla'^3\sigma$ yields $S_4 = \{nc - 3(n+1)c_0\}S_3$. Thus we have

$$\left\{c - \frac{3(n+2)}{n}c_0\right\}S_3 = 0.$$

Since S_3 is positive, the assertion is therefore proved.

In the remainder of this section, we shall investigate the structure of the immersion f. By Lemma 4.1 each point in M is 2-regular. Now in general we consider the decomposition of the orthogonal complement of the first normal space concerning with the isometric immersion f of M into $\overline{M} = M^{n+p}(c)$. Denote by M_2 the set consisting of all 2-regular points in M. M_2 is open in M. For any point x in M_2, we set $O_x^2 = df_x(M_x) \oplus N_x^1$, which is called a *second osculating space* of f at x. For the 2-regular point x we define a trilinear map σ_{3x} of $M_x \times M_x \times M_x$ into $(O_x^2)^N$ by $\sigma_{3x}(X, Y, Z) = (\overline{\nabla}_Z(\sigma_{2x}(X, Y)))^{N_2}$ for any vector fields X, Y and Z, where the superscript N_2 denotes the orthogonal projection into $(O_x^2)^N$. Thus σ_{3x} is well defined and symmetric, and induces a linear mapping σ_{3x} of the symmetric third power $S^3(M_x)$ of M_x into $(O_x^2)^N$. σ_{3x} is called a *third fundamental form* of f at x, and a linear subspace N_x^2 defined by

$N_x^2 = \sigma_{3x}(S^3(M_x))$ is called a *second normal space* of f at x. The second normal space at x is the orthogonal complement in $(O_x^2)^N$ of the linear subspace spanned by $(\nabla'\sigma)(u, v, w)$ for any vectors u, v and w at x, so $\dim N_x^2 \leqslant n(n+1)(n+2)/6$. The point x in M_2 is said to be 3-*regular* if N_x^2 is of maximal dimension with respect to basic points.

Coming back to the situation where we discuss at present, we see $\sigma_{3x}(X, Y, Z) = (\nabla'\sigma)(X, Y, Z)$, which means that the second normal space at x is the linear subspace spanned only by vectors $(\nabla'\sigma)(u, v, w)$ for any vectors u, v and w at x, because $\nabla'\sigma$ is orthogonal to σ. Now let us take up the dimension of the second normal space at each point x in M. Combining $\langle(\nabla'\sigma)(X, Y, Z), \sigma(U, V)\rangle = 0$ together with (4.9) and (4.10) we obtain, by a straightforward calculation,

$$\begin{aligned}&\langle(\nabla'\sigma)(X_1, X_2, X_3), (\nabla'\sigma)(X, Y, Z)\rangle\\ &\quad= -\frac{2n^2(n+3)}{9(n+2)^4}c^2\mathfrak{S}_3[2\langle X_1, X_2\rangle(\langle X_3, X\rangle\langle Y, Z\rangle + \langle X_3, Y\rangle\langle Z, X\rangle\\ &\qquad+\langle X_3, Z\rangle\langle X, Y\rangle) - (n+2)\langle X_1, Z\rangle(\langle X_2, X\rangle\langle X_3, Y\rangle\\ &\qquad\qquad+\langle X_2, Y\rangle\langle X_3, X\rangle)].\end{aligned} \tag{4.11}$$

For a suitably chosen orthonormal basis $\{e_1, \cdots, e_n\}$ of M_x, we denote simply by σ_{ijk} a normal vector $(\nabla'\sigma)(e_i, e_j, e_k)$. Since the immersion is minimal, $\Sigma_j\sigma_{ijj} = 0$ for each index i holds. At each point x, since it is already seen that $\langle\sigma(u_1, u_2), (\nabla'\sigma)(u_3, u_4, u_5)\rangle = 0$ for any vectors in M_x, the second normal space N_x^2 is a linear subspace spanned by vectors $\sigma_{ijk}(i, j, k = 1, \cdots, n)$, and therefore $\dim N_x^2 \leqslant n(n+4)(n-1)/6$.

In general, let M_3 be the set consisting of all 3-regular points in M_2. Then M_3 is open in M_2. For any 3-regular point x, we denote by O_x^3 the direct sum of O_x^2 and N_x^2, which is called a *third osculating space* of f at x. We now proceed inductively and suppose that the $(j-1)$-osculating space O_x^{j-1} of f at the $(j-1)$-regular point x is defined. Then it is possible to define a linear mapping σ_{jx} of the symmetric jth power $S^j(M_x)$ of M_x into $(O_x^{j-1})^N$ by $\sigma_{jx}(X_1, \cdots, X_j) = (\overline{\nabla}_{X_1}(\sigma_{j-1x}(X_2, \cdots, X_j)))^{N_{j-1}}$ for any vector fields, where the superscript N_{j-1} denotes the orthogonal projection into $(O_x^{j-1})^N$. Then σ_{jx} is called a *jth fundamental form* of f at x, and $N_x^{j-1} = \sigma_{jx}(S^j(M_x))$ (resp. $O_x^j = O_x^{j-1} \oplus N_x^{j-1}$) is called a *jth normal* (resp. *osculating*) *space* at x. Clearly, the process must be eventually stopped, because of $\dim O_x^j \leqslant \dim \overline{M}_x$. Thus there exists a first integer q for which $\sigma_j \equiv 0$ for $j > q$ and σ_q does not vanish identically. Then q is called a *degree* of f, and the set M_q is open in M.

Concerning the regularity of points in M we have

Lemma 4.3. *The dimension of the second normal space is constant on M, and each point in M is 3-regular.*

Proof. In order to verify these assertions, it is sufficient to show that the first statement is valid. Namely, by means of (4.11) a number of linearly independent normal vectors σ_{ijk} may be calculated, because N_x^2 is spanned by σ_{ijk}, $1 \leqslant i, j, k \leqslant n$. We put $A = \frac{2}{9}n^2(n+3)c^2/(n+2)^4$. Then the square of the lengths of the vectors σ_{111}, σ_{112} and σ_{123} are given as follows: $\|\sigma_{111}\|^2 = 6(n-1)A$, $\|\sigma_{112}\|^2 = 2(n+1)A$ and $\|\sigma_{123}\|^2 = (n+2)A$, which mean that any vector σ_{ijk} for any indices is not zero. Next, normal vectors which are not mutually orthogonal are limited to the following two types, except for $\langle \sigma_{ijk}, \sigma_{ijk} \rangle \neq 0$; $\langle \sigma_{iii}, \sigma_{ijj} \rangle = -6A$, $\langle \sigma_{iij}, \sigma_{jkk} \rangle = -2A$ for any mutually distinct indices. The linear combination $\Sigma_{i \leqslant j \leqslant k}\, a_{ijk}\sigma_{ijk}$ are considered, where $a_{inn} = 0$ for $i = 1, \cdots, n$. The inner product of σ_{111} and σ_{122} with $\Sigma_{i \leqslant j \leqslant k}\, a_{ijk}\sigma_{ijk} = 0$ are reduced to $(n-1)a_{111} - a_{122} - \cdots - a_{1mm} = 0$, where $m = n-1$, and $-3a_{111} + (n+1)a_{122} - \cdots - a_{1mm} = 0$, which imply $a_{111} = a_{122}$. Similarly we have $a_{111} = a_{122} = \cdots = a_{1mm} = 0$, which together with the above equations means that the coefficients a_{ijk} except for mutually distinct indices are equal to 0. It is almost obvious that for mutually distinct indices, $a_{ijk} = 0$, which implies that the normal vectors belonging to the second normal space are linearly independent, except for σ_{1nn} for $i = 1, \cdots, n$. Thus $\dim N_x^2 \geqslant {}_nH_3 - n$. Therefore the dimension of N_x^2 is equal to $n(n+4)(n-1)/6$ for any point x in M, and hence is constant on M. By means of the definition of 3-regularity, this means that any point on M is 3-regular. q.e.d.

On the other hand, an isometric immersion f of M into $S^{n+q} \subset R^{n+q+1}$ is said to be *linearly rigid*, if there is a linear transformation g of $\mathbf{R}^{n+q+1}$ with the following property: if $g(f(M)) \subset S^{n+q}$, and $g \circ f\colon M \to \mathbf{R}^{n+q+1}$ is also an isometric immersion, then g is an orthogonal transformation. Linear rigidity is a weaker notion than rigidity, and it is seen in [11] that the rigidity for minimal immersions induces the linear rigidity, and in particular for isotropy irreducible symmetric spaces of compact types, rigidity and linear rigidity are essentially the same notion. It is also seen in [11] that if M is analytic, f is full, and the degree of f is not greater than 3, then f is linearly rigid.

We come back to the proof of the main theorem. Suppose that $n \geqslant 3$. By Lemma 4.3, each point in M is 3-regular. Moreover, the assumption (0.1) implies that the 4th fundamental form vanishes identically on M, so that the degree of f is equal to 3. In particular, it implies that the direct sum of N_x^1 and N_x^2 is invariant under the parallelism of the normal bundle, since it follows from Lemmas 4.1 and 4.3 that the dimension of $N_x^1 \oplus N_x^2$ is constant, say q, on M. Then a theorem due to Erbacher [2] yields that M is contained in an

$(n+q)$-dimensional great sphere in $\overline{M} = S^{n+p}(c)$. Thus we have $q = p$, and therefore $n + p = N(3)$, since the immersion f is full. We note that f is analytic, because $M = S^n(c_0)$ is analytic and f is minimal. By the theorem due to Wallach and the relation between the rigidity and the linear rigidity, f: $S^n(c_0) \to S^{n+p}(c)$ is rigid, and there exists an orthogonal transformation g of $\mathbf{R}^{n+p+1}$ such that $g \circ f = f_3$.

Suppose that $n = 2$. In the proof of the case where $n \geqslant 3$, the restriction of the dimension is not necessarily essential except for Lemma 4.1. In this situation, Lemma 3.3. shows that M is isometric to a sphere, and by means of Lemma 4.2 the constant curvature c_0 is equal to $c/6$. Thus the dimension of N_x^1 is equal to 2 for any point x in M, and Lemma 4.1 is true for $n = 2$.

This concludes the proof of the theorem stated in the introduction.

Remark. Let M be an n-dimensional Riemannian manifold, and let f be a minimal immersion of M into $M = M^{n+p}(c)$ satisfying the condition that $(\nabla'^2\sigma)(u, u, u, u) + l^2\sigma(u, u) = 0$ for any unit vector u, where 1 is a nonnegative constant. Then it seems of interest to classify such (M, f).

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