J. DIFFERENTIAL GEOMETRY
14(1979), 369–376

A TOPOLOGICAL VERSION OF OBATA'S SPHERE THEOREM

MARK A. PINSKY

1. Introduction

Let (M, g) be a compact Riemannian manifold of nonnegative sectional curvature K. The Laplacian acting on $L^2(M, g)$ has a discrete spectrum $0 = \lambda_0 < \lambda_1 < \lambda_2 \leqslant \cdots$ with $\lim_{n\to\infty} \lambda_n = +\infty$. It is well known [1] that λ_1 can be related to bounds on the curvature. For instance if $K \geqslant k > 0$, then $\lambda_1 \geqslant nk$ with equality in case of the sphere. Furthermore the theorem of Obata [3] states that the dual relations $K \geqslant k$, $\lambda_1 = nk$ imply that (M, g) is isometric to (S^n, g_0), the standard sphere of radius $k^{-1/2}$ imbedded in R^{n+1}.

It has been conjectured that there exists $\delta > 0$ such that whenever (M, g) satisfies the dual relations $K \geqslant k$, $\lambda_1 \leqslant nk(1 + \delta)$, then M is homeomorphic to S^n. While we have not obtained this precise result, we have the following theorem.

Theorem. *Let (M, g) be a compact n-dimensional Riemannian manifold with sectional curvature K satisfying the bound*

$$K \geqslant k > 0. \tag{1}$$

Assume that there exists a C^∞-function f, not identically zero, which satisfies the inequality

$$\left|\nabla_X \nabla_X f + \frac{\lambda}{n}(X, X) f\right| \leqslant \frac{\mu\lambda}{n}(X, X)|f|, \tag{2}$$

whenever $x \in M$, $X \in T_x(M)$; λ and μ are positive constants with $0 < \mu < 1$ and

$$0 < \lambda < \frac{16nk}{9}\left[\pi - \cos^{-1}\left(\frac{1-\mu}{1+\mu}\right)\right]^2. \tag{3}$$

Then M is homeomorphic to the n-sphere S^n.

This theorem, which generalizes the Obata sphere theorem, will be proved along the lines of the pinching theorems of Rauch and Berger [2, Chapter 6]. In this context, the hypothesis (3) is a substitute for the upper bound $K < 4k$

Received May 9, 1977, and in revised form, June 30, 1978. Research supported by National Science Foundation Grant MCS78-02144.

which is assumed in the pinching theorems. A crucial step in our proof is Lemma 5, which states that if P is a critical point of f, and Q is conjugate to P along some geodesic γ, then

$$d(P, Q) \geqslant (n/\lambda)^{1/2}\left[\pi - \cos^{-1}(1-\mu)/(1+\mu)\right].$$

2. Proof of the theorem

Lemma 1. *If γ is any normal geodesic, then*

$$\frac{d^2}{dt^2}(f \circ \gamma) + \frac{\lambda}{n}(f \circ \gamma) = \delta_\gamma(t), \tag{4}$$

where $|\delta_\gamma(t)| \leqslant (\mu\lambda/n)|(f \circ \gamma)(t)|$.

Proof. Indeed $d/dt(f \circ \gamma) = (df \circ \gamma)(\gamma')$, $d^2/dt^2(f \circ \gamma) = \operatorname{Hess} f(\gamma', \gamma') + (df \circ \gamma)\nabla_{\gamma'}\gamma'$. The second term is zero since γ is a normal geodesic. From Condition (2), the first term $= -(\lambda/n)(f \circ \gamma) + \delta_\gamma(t)$ where $|\delta_\gamma| \leqslant (\mu\lambda/n)|f|$.

Lemma 2. *If γ is a normal geodesic, and $\gamma(0)$ is a critical point of f, then*

$$(f \circ \gamma)(t) = f(\gamma(0)) \cos\sqrt{\frac{\lambda}{n}}\ t + \sqrt{\frac{n}{\lambda}} \int_0^t \delta_\gamma(u) \sin\sqrt{\frac{\lambda}{n}}\ (t-u)\, du. \tag{5}$$

Proof. It is easily verified that (5) is the unique solution of (4) satisfying the initial conditions $(f \circ \gamma)'(0) = 0$, $(f \circ \gamma)(0) = f(\gamma(0))$.

We now obtain some estimates on the variation of f along a geodesic. For this purpose, let

$$\begin{aligned} t_0 &= t_0(\gamma) \equiv \min\{t > 0 : (f \circ \gamma)(t) = -(f \circ \gamma)(0)\}, \\ \bar{t}_0 &= \bar{t}_0(\gamma) \equiv \min\{t > 0 : (f \circ \gamma)(t) = (f \circ \gamma)(0)\}, \\ t_0' &= t_0'(\gamma) \equiv \min\{t > 0 : (f \circ \gamma)'(t) = 0\}. \end{aligned} \tag{6}$$

Lemma 3. *If γ is a normal geodesic, $\gamma(0)$ is a critical point of f, and $(f \circ \gamma)(0) \neq 0$, then* $\min\{t_0, \bar{t}_0\} \geqslant (n/\lambda)^{1/2}[\pi - \cos^{-1}(1-\mu)/(1+\mu)]$.

Proof. Without loss of generality we may assume that $(f \circ \gamma)(0) > 0$ (otherwise the proof below may be applied to $-f$). To prove the stated lower bound, we may assume that $\min\{t_0, \bar{t}_0\} \leqslant \pi(n/\lambda)^{1/2}$, since otherwise there is nothing to prove. Now on the interval $0 \leqslant t \leqslant \min\{t_0, \bar{t}_0\}$ the sine function in (5) is positive, and we have the two-sided bound

$$\left|\sqrt{\frac{n}{\lambda}} \int_0^t \delta_\gamma(u) \sin\sqrt{\frac{\lambda}{n}}\ (t-u)\, du\right| \leqslant \mu(f \circ \gamma)(0)\left(1 - \cos\sqrt{\frac{\lambda}{n}}\ t\right). \tag{7}$$

We analyze separately two cases.

Case 1. $\bar{t}_0 < t_0$: In this case we use (7) as an upper bound for the integral in (4), with the result

$$(f \circ \gamma)(t) \leqslant (f \circ \gamma)(0) \cos\sqrt{\frac{\lambda}{n}}\; t + \mu(f \circ \gamma)(0)\left(1 - \cos\sqrt{\frac{\lambda}{n}}\; t\right),$$

$$\left(0 \leqslant t \leqslant \bar{t}_0\right).$$

Dividing by $(f \circ \gamma)(0)$ and setting $t = \bar{t}_0$ we have $1 \leqslant \cos(\lambda/n)^{1/2}\bar{t}_0$ which implies that $\bar{t}_0 \geqslant 2\pi(n/\lambda)^{1/2}$. This proves the result in this case.

Case 2. $t_0 \leqslant \bar{t}_0$: In this case we use (7) as a lower bound for the integral in (4), with the result

$$(f \circ \gamma)(t) \geqslant (f \circ \gamma)(0) \cos(\lambda/n)^{1/2}t - \mu(f \circ \gamma)(0)\left[1 - \cos(\lambda/n)^{1/2}t\right].$$

Dividing by $(f \circ \gamma)(0)$ and setting $t = t_0$, we have

$$(\mu - 1)/(\mu + 1) \geqslant \cos(\lambda/n)^{1/2}t_0,$$

which proves the result in this case.

Lemma 4. *If γ is a normal geodesic, $\gamma(0)$ is a critical point of f, and $(f \circ \gamma)(0) \neq 0$, then $t_0' \geqslant (n/\lambda)^{1/2}[\pi - \cos^{-1}(1 - \mu)/(1 + \mu)]$.*

Proof. We first differentiate (5), with the result

$$(f \circ \gamma)'(t) = -\sqrt{\frac{\lambda}{n}}\;(f \circ \gamma)(0) \sin\sqrt{\frac{\lambda}{n}}\; t$$

(8)

$$+ \int_0^t \delta_\gamma(u) \cos\sqrt{\frac{\lambda}{n}}\;(t - u)\, du.$$

Case 1. $t_0' < \pi/2(n/\lambda)^{1/2}$: In this case we see from Lemma 3 that $|(f \circ \gamma)(t)| \leqslant |(f \circ \gamma)(0)|$ for $0 \leqslant t \leqslant 7$. Therefore (8) yields

$$\sqrt{\frac{\lambda}{n}}\;|(f \circ \gamma)(0)| \sin\sqrt{\frac{\lambda}{n}}\; t_0' = \left|\int_0^{t_0'} \delta_\gamma(u) \cos\sqrt{\frac{\lambda}{n}}\;(t_0' - u)\, du\right|$$

$$\leqslant \mu\sqrt{\frac{\lambda}{n}}\;|(f \circ \gamma)(0)| \sin\sqrt{\frac{\lambda}{n}}\; t_0',$$

which contradicts $\mu < 1$. Therefore this case is impossible.

Case 2. $\pi > t_0'(\lambda/n)^{1/2} \geqslant \pi/2$. If $t_0'(\lambda/n)^{1/2} \geqslant [\pi - \cos^{-1}(1 - \mu)/(1 + \mu)]$, there is nothing to prove. Otherwise we may again apply (8) and Lemma

3, in the form

$$\sqrt{\frac{\lambda}{n}}\,|(f\circ\gamma)(0)|\sin\sqrt{\frac{\lambda}{n}}\,t_0' = \left|\int_0^{t_0'}\delta_\gamma(u)\cos\sqrt{\frac{\lambda}{n}}\,(t_0'-u)\,du\right|$$

$$\leqslant \mu\frac{\lambda}{n}|(f\circ\gamma)(0)|\int_0^{t_0'}\left|\cos\sqrt{\frac{\lambda}{n}}\,u\right|du$$

$$= \mu\sqrt{\frac{\lambda}{n}}\,|(f\circ\gamma)(0)|\left(2-\sin\sqrt{\frac{\lambda}{n}}\,t_0'\right).$$

Therefore $(1+\mu)\sin(\lambda/n)^{1/2}t_0' \leqslant 2\mu$, which is rewritten in the form $\sin t_0' \leqslant 2\mu/(1+\mu) \leqslant 2\mu^{1/2}/(1+\mu)$. Hence

$$\sqrt{\frac{\lambda}{n}}\,t_0' \geqslant \pi - \sin^{-1}\frac{2\sqrt{\mu}}{1+\mu} = \pi - \cos^{-1}\frac{1-\mu}{1+\mu}.$$

This proves the required estimate in this case.

Case 3. $t_0'(\lambda/n)^{1/2} \geqslant \pi$: In this case there is nothing to prove.

Lemma 5. *Let γ be a normal geodesic where $(f\circ\gamma)(0) \neq 0$, and $\gamma(0)$ is a critical point of f. If $\gamma(t)$ is conjugate to $\gamma(0)$ along γ, then* $t \geqslant (n/\lambda)(\pi - \cos^{-1}(1-\mu)/(1+\mu))$.

Proof. Assume that γ is free of conjugate points for $0 \leqslant t < t_c$, and that $\gamma(t_c)$ is a conjugate point. Let $\{Y(\,),\ 0 \leqslant t \leqslant t_c\}$ be a Jacobi field along γ with $Y(0) = 0 = Y(t_c)$. $Y(t)$ can be realized as the infinitesimal variation of the geodesic γ through the formula

$$Y(t) = \frac{\partial}{\partial s}\gamma_s(t)\Big|_{s=0},$$

where $Y_s(t) = \exp_{\gamma(0)} tV_s$, $V_s = \{sY'(0) + \gamma'(0)\}(1 + s^2|Y'(0)|)^{-1/2}$. Now by the second variation formula [1, p. 135], we have

$$\frac{\langle Y'(t), Y(t)\rangle}{\langle Y(t), Y(t)\rangle} = \frac{d^2L_t}{ds^2}\Big|_{s=0}, \tag{10}$$

where $L_t(s)$ is the length of the geodesic segment $\{\gamma_s(\tau),\ 0 \leqslant \tau \leqslant t\}$. To compute the right-hand member of (10), we apply Lemma 2 to γ_s. Thus

$$(f\circ\gamma_s)(t) = (f\circ\gamma)(0)\cos\sqrt{\frac{\lambda}{n}}\,L_t(s)$$

$$+\sqrt{\frac{n}{\lambda}}\int_0^{L_t(s)}\delta_{\gamma_s}(u)\sin\sqrt{\frac{\lambda}{n}}\,(L_t(s)-u)\,du.$$

We fix t and differentiate the equation with respect to s; thus

$$\frac{d}{ds}(f \circ \gamma_s) = -(f \circ \gamma)(0)\sqrt{\frac{\lambda}{n}}\, \frac{dL_t}{ds} \sin\sqrt{\frac{\lambda}{n}}\, L_t(s)$$

$$(11) \qquad + \frac{dL_t}{ds} \int_0^{L_t(s)} \delta_{\gamma_s}(u) \cos\sqrt{\frac{\lambda}{n}}\, (L_t(s) - u)\, du$$

$$+ \int_0^{L_t(s)} \frac{d}{ds} \delta_{\gamma_s}(u) \sin\sqrt{\frac{\lambda}{n}}\, (L_t(s) - u)\, du.$$

Upon taking second derivatives and setting $s = 0$, we see that

$$\frac{d^2}{ds^2}(f \circ \gamma_s)\bigg|_{s=0} = \frac{d^2 L_t}{ds^2}\bigg|_{s=0} \times \left\{ -(f \circ \gamma)(0)\sqrt{\frac{\lambda}{n}}\, \sin\sqrt{\frac{\lambda}{n}}\, t \right.$$

$$(12) \qquad \left. + \int_0^t \delta_\gamma(u) \cos\sqrt{\frac{\lambda}{n}}\, (t - u)\, du \right\}$$

$$+ \int_0^t \frac{d^2}{ds^2} \delta_{\gamma_s}(u) \sin\sqrt{\frac{\lambda}{n}}\, (t - u)\, du.$$

On the other hand, from (8), (10) we can write the above equation in the form

$$(13) \qquad \frac{\langle Y'(t), Y(t) \rangle}{\langle Y(t), Y(t) \rangle} = \frac{G(t)}{D(t)},$$

where $D(t) = (f \circ \gamma)'(t)$ and

$$(14) \qquad G(t) = \frac{d^2}{ds^2}(f \circ \gamma_s)\bigg|_{s=0} - \int_0^t \frac{d^2}{ds^2} \delta_{\gamma_s}(u) \sin\sqrt{\frac{\lambda}{n}}\, (t - u)\, du.$$

$G(t)$ is a continuous function for $0 \leqslant t \leqslant t_c$. From Lemma 4, $D(t)$ is nonzero for $0 \leqslant t \leqslant (n/\lambda)^{1/2}[\pi - \cos^{-1}(1 - \mu)/(1 + \mu)]$. Now $Y(t) \neq 0$ for $0 < t < t_0$ and hence $Y(r) \neq 0$ for sme $r > 0$ for some $r > 0$. Integrating (13) on $[r, t]$ we see that

$$(15) \qquad |Y(t)| = |Y(r)| \exp\left\{ 2 \int_r^t \frac{G(u)}{D(u)}\, du \right\}, \quad (r \leqslant t < t_c).$$

To complete the proof of the lemma, assume that
$t_c < (n/\lambda)^{1/2}[\pi - \cos^{-1}(1 - \mu)/(1 + \mu)]$.
Let $t \to t_c$ in (15), with the conclusion

$$0 = |Y(r)| \exp\left\{ 2 \int_t^t \frac{G(u)}{D(u)}\, du \right\} \neq 0.$$

Hence $t_c \geqslant (n/\lambda)^{1/2}[\pi - \cos^{-1}(1 - \mu)/(1 + \mu)]$, which was to be proved.

Lemma 6. *Let P be a critical point of f, $f(P) \neq 0$. Then the geodesic ball of radius $(n/\lambda)^{1/2}(\pi - \cos^{-1}(1 - \mu)/(1 + \mu))$ is within the cut-locus of P.*

Proof. Let Q realize the minimum distance from P to its cut-locus, and assume $t_1 = d(P, Q) < (n/\lambda)^{1/2}[\pi - \cos^{-1}(1 - \mu)/(1 + \mu)]$. Then by a known result [2, Lemma 5.6, p. 95] either there is a minimal geodesic from P to Q along which Q is conjugate to P, or there are precisely two minimal geodesics γ, σ from P to Q such that $\gamma'(P) = -\sigma'(P)$. From Lemma 5 the first case is impossible. Therefore we may apply the lemma again to Q to conclude that we can define a smooth closed geodesic $\gamma(t)$ with $\gamma(0) = P$, $\gamma(t_1) = Q$, $\gamma(2t_1) = P$. Let $\bar{f}(t) = (f \circ \gamma)(t)$ for $0 \leqslant t \leqslant 2t_1$. Without loss of generality we may assume that $f(P) > 0$. Thus $\bar{f}''(t) > 0$ for small t and hence $\bar{f}'(t) \leqslant 0$ for small t. Applying Lemma 4 we see that $\bar{f}'(t) < 0$ for $0 < t < t_1 = d(P, Q)$. On the other hand by reversing the time along γ, we must have $\bar{f}'(t) > 0$ for $t_1 < t < 2t_1$. Therefore $\bar{f}'(t_1) = 0$ which contradicts Lemma 4.

We now let $P_{\max}$ (resp. $P_{\min}$) be the location of the maximum (resp. minimum) of f on M. It follows from hypothesis (2) that $f(P_{\max}) > 0 > f(P_{\min})$. Indeed, by taking the trace, we see that

$$\lambda \int_M f = \int_M \Delta f + \lambda f \leqslant \mu\lambda \int_M f.$$

If for instance $f(P_{\min}) \geqslant 0$, then $f \geqslant 0$ on all of M, which contradicts $\int_M f \leqslant \mu \int_M f$.

Let R realize the maximum distance from $P_{\max} = P$.

Lemma 7. *Given $v \in T_P(M)$, there exists a minimal geodesic γ from P to R such that* $(\gamma'(0), v) \leqslant \pi/2$.

The statement of Lemma 7 is essentially the same as that of Lemma 6.2 in [2], and the proof of Lemma 7 is therefore omitted.

Lemma 8. $M = B(P_{\max}; \pi/2k^{1/2}) \cup B(R, \pi/2k^{1/2})$.

Proof. Let $d(P_{\max}; x) > \pi/2k^{1/2}$. Let γ_2 be a minimal geodesic from $P_{\max}$ to x, and by Lemma 7 choose a minimal geodesic γ_1 from $P_{\max}$ to R such that $(\gamma'(0), \gamma_2'(0)) \leqslant \pi/2$. Thus the geodesic triangle formed by $(\gamma_1, \gamma_2, \sphericalangle(\gamma_1'(0), \gamma_2'(0)))$ satisfies the hypotheses of Toponogov's theorem. Therefore we can compare with a geodesic triangle of opening $\pi/2$ in a sphere of curvature $= k$. Following the steps of [2, Lemma 6.3] we see that $d(R, x) < \pi/2k^{1/2}$.

Lemma 9. *If* $(n/\lambda)^{1/2}[\pi - \cos^{-1}(1 - \mu)/(1 + \mu)] > 3\pi/4k^{1/2}$, *then*

$$M = B\left(P_{\max}; 3\pi/4k^{1/2}\right) \cup B\left(P_{\min}; 3\pi/4k^{1/2}\right). \tag{16}$$

Proof. Note that by Meyers' theorem, (1) implies $\operatorname{diam}(M) \leqslant \pi/k^{1/2}$. Hence $d(P_{\max}, P_{\min}) \leqslant d(P_{\max}, R) \leqslant \pi/k^{1/2}$. On the other hand from Lemma

5, we have $d(P_{\max}, P_{\min}) > (n/\lambda)^{1/2}[\pi - \cos^{-1}(1-\mu)/(1+\mu)] > 3\pi/4k^{1/2}$. Repeating the reasoning of Lemma 8, we apply Toponogov's theorem to the geodesic triangle $(\gamma_1, \gamma_2 \measuredangle(\gamma_1'(0), \gamma_2'(0)))$ where γ_2 is a minimal geodesic from $P_{\max}$ to $P_{\min}$, and γ_1 is a minimal geodesic from $P_{\max}$ to R such that $(\gamma_1'(0), \gamma_0') \leqslant \pi/2$. This shows that $d(R, P_{\min}) \leqslant \pi/4k^{1/2}$. Now if $d(x, P_{\max}) > 3\pi/4k^{1/2}$, we have $d(x, R) < \pi/2k^{1/2}$ (from Lemma 8) and hence $d(x, P_{\min}) < d(x, R) + d(R, P_{\min}) < \pi/4k^{1/2} = 3\pi/4k^{1/2}$.

Lemma 10. *Let* $(n/\lambda)^{1/2}[\pi - \cos^{-1}(1-\mu)/(1+\mu)] > 3\pi/4k^{1/2}$, *and let* γ *be a normal geodesic with* $\gamma(0) = P_{\max}$. *Then there is a unique point* x *on* γ *such that* $d(P_{\max}, x) = d(P_{\min}, x) \leqslant 3\pi/4k^{1/2}$.

Proof. Let $\psi(t) = d(P_{\max}, \gamma(t)) - d(P_{\min}, \gamma(t))$, $0 \leqslant t \leqslant 3\pi/4k^{1/2}$. *Clearly* $\psi(0) < 0$ *and* $\psi(3\pi/4k^{1/2}) = 3\pi/4k^{1/2} - d(P_{\min}, \gamma(3\pi/4k^{1/2})) > 0$ by Lemma 9. Therefore by the intermediate value theorem there is a $\bar{t} \in (0, 3\pi/4k^{1/2})$ such that $\psi(\bar{t}) = 0$. If $\bar{t}_1$, $\bar{t}_2$ are two such values, suppose $\bar{t}_1 < \bar{t}_2$. Then $d(P_{\min}, \gamma(\bar{t}_2)) = d(P_{\max}, \gamma(\bar{t}_2)) = d(P_{\max}, \gamma(\bar{t}_1)) + d(\gamma(\bar{t}_1), \gamma(\bar{t}_2)) = d(P_{\min}, \gamma(\bar{t}_1)) + d(\gamma(\bar{t}_1), \gamma(\bar{t}_2))$. Therefore the path from $\gamma(\bar{t}_2)$ to $P_{\min}$ via $\gamma(\bar{t}_2)$ has the same length as the minimal geodesic from $P_{\min}$ to $\gamma(\bar{t}_2)$. Hence this path must be a smooth geodesic and hence must pass through $P_{\max}$, which contradicts $P_{\max} \neq P_{\min}$.

Proof of the theorem. Let S^n denote the unit sphere in R^{n+1}, and P_1, P_2 a pair of antipodal points. Let

$$I\colon T_{P_1}(S^n) \to T_{P_{\max}}(M) \tag{17}$$

be an isometry of the tangent spaces at the indicated points. For each unit vector $v \in T_{P_{\max}}(M)$, define $\varphi = t_0 v$ by letting $\exp \varphi(v)$ be the point along the geodesic $t \to \exp_{P_{\max}} tv$ which is equidistant from $P_{\max}$ and $P_{\min}$. Lemma 10 implies the existence and uniqueness of $t_0 \in (0, 3\pi/4k^{1/2})$. Let $\Phi(x) = \exp_{P_{\max}}(\varphi(I(\exp_{P_1}^{-1}(x))))$. Define $h\colon S^n \to M$ by the rule

$$h(x) = \begin{cases} P_{\max}, & x = P_1, \\ \exp_{P_{\max}}(2d(x, P_1)) \exp_{P_{\max}}^{-1}(\Phi(x)), & 0 < d(x, P_1) < \pi/2, \\ \exp_{P_{\min}}(2d(x, P_2)) \exp_{P_{\min}}^{-1}(\Phi(x)), & 0 < d(x, P_2) < \pi/2, \\ P_{\min}, & x = P_2. \end{cases} \tag{18}$$

Repeating step-by-step the proof of Theorem 6.1 in [2], we see that H is continuous, injective, and surjective from S^n to M. Therefore M is a homeomorphism, and the proof is complete.

Added in Proof. Recently some new results on the above problem were obtained by S. Gallot, *Un théorème de pincement et une estimation sur la* *première valeur propre du laplacien d'une variété riemannienne*, C. R. Acad. Sci. Paris **289** (1979) 441–444.

References

[1] M. Berger, P. Gauduchon & E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math. Vol. 194, Springer, Berlin, 1971.

[2] J. Cheeger & D. G. Ebin, *Comparison theorems in Riemannian geometry*, North Holland, Amsterdam, 1975.

[3] M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations*, Proc. United States-Japan Sem. Differential Geometry, Kyoto, 1965, Nippon Hyoransha, Tokyo, 1966, 101–114.

NORTHWESTERN UNIVERSITY