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NATURAL TENSORS ON RIEMANNIAN MANIFOLDS

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Suppose that for each differentiable manifold M (without boundary) and each C^∞ metric g, we are given a C^∞ tensor field $t_{(M,g)}$ satisfying the following naturality axiom:

If $\varphi\colon (M,g) \to (N,h)$ is an isometry of M onto an open subset of N, then $\varphi_* t_{(M,g)} = t_{(N,h)} \mid \varphi M$.

In these circumstances we say that t is a *natural tensor*.

Our objective is to elucidate the nature of natural tensors. It will emerge that the situation is too complicated for there to be any hope of a complete classification. Therefore we try to find additional conditions which can be imposed on a natural tensor which will imply that it lies in a good class of natural tensors.

Our main results are as follows. In § 5 we classify all natural tensors which depend in a polynomial way of the ∞-jet of g. In § 6 we show that it is sufficient for the dependence on the ∞-jet to be a differentiable dependence (we demand C^∞-dependence in Theorem 6.2, but the proof obviously goes through with less differentiability), if in addition the tensor is homogeneous (Definition 5.1): these two conditions imply polynomial dependence. In Theorem 7.3 we prove a special result where only homogeneity is assumed and nothing whatever concerning the dependence on the ∞-jet of g. The fact that this is not trivial is shown by the existence of an example of a natural tensor depending only on the 4-jet of g, but with the dependence not even continuous (Theorem 4.1). We observe in passing that the space of germs of C^∞ Riemannian manifolds of dimension 2 can be parametrized in terms of the orbits of a linear 0(2)-action on an infinite dimensional vector space (Corollary 2.4). Finally we show that there is a unique "natural" connection ∇ for Riemannian manifolds-namely the Levi-Civita connection. In other words the fact that ∇ is torsion free and preserves the metric follows from the naturality. (For a precise statement, see Theorem 5.6.)

This works was stimulated by G. Lusztig when he asked whether the Levi-Civita connection was the unique natural connection. This was during lectures on the work of Atiyah, Bott and Patodi [1] whose treatment of Gilkey's theorem has heavily influenced this paper. In fact Gilkey's theorem deals with the problem of classifying natural q-forms. Here we relax the condition that

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the tensors be forms and consider all tensors, not only antisymmetric ones. P. Stredder [5] has taken further the characterization of the Levi-Civita connection, by considering all natural differential operators. Natural tensors have been extensively studied in the past. See for example the address by A. Nijenhuis [4] to the International Congress of Mathematicians in 1958.

Thanks are due to G. Lusztig for many stimulating conversations, and to J. N. Mather for explaining to me aspects of Whitney's extension theorem for C^∞ functions.

1. Examples

We have the following examples of natural tensors:

a) The metric $g_{\cdot} = g$ with values in $T^* \otimes T^*$.

b) Under the isomorphism $T \cong T^*$, $g_{\cdot}$ corresponds to $g^{\cdot}$ which has values in $T \otimes T$.

c) The Riemannian curvature tensor R, with values in $(T^*)^{\otimes 4}$ and its covariant derivatives $\nabla^n R$ in $(T^*)^{\otimes(n+4)}$.

d) We can tensor the above examples together, possibly repeating an example several times.

e) We can permute the factors T and T^*.

f) We can contract a copy of T against a copy of T^*.

g) We can add the above examples with constant real coefficients.

Definition 1.1. A natural tensor formed as above is called a *polynomial tensor*.

Not all natural tensors are polynomial. For example, the function

$$\Sigma 2^{-n}(1 + \|\nabla^n R\|^2)^{-1}$$

is a natural tensor of type $(0, 0)$, which depends on derivatives of g of all orders, and is therefore not polynomial.

The following theorem shows that many natural tensors are polynomial.

Theorem 1.2. *Let $r \geq 0$ and $s \geq 0$ be integers. Let F^I_J be a polynomial with real coefficients for each r-tuple $I = (i_1, \cdots, i_r)$ and s-tuple $J = (j_1, \cdots, j_s)$ of integers such that $1 \leq i_p \leq m$ and $1 \leq j_q \leq m$ for each p $(1 \leq p \leq r)$ and each q $(1 \leq q \leq s)$. Suppose that the formulas*

$$t^I_J(x) = F^I_J\left(g_{ab}(x), g^{cd}(x), \frac{\partial^\alpha}{\partial x^\alpha} g_{ij}(x)\right)$$

define a natural tensor in local coordinates. (The variables of which F^I_J is a function are given by some finite jet of g at x. The hypothesis means in particular that we get the same answer for $t(x) \in T_x{}^{\otimes r} \otimes (T_x{}^*)^{\otimes s}$ no matter which local coordinates we use for the computation.) Then t is a polynomial tensor.

Remarks. 1) If t is polynomial tensor then classical formulas for the curvature and Riemannian connection show that the converse is also true.

2) The condition here is the same as that in the exposition of Atiyah, Bott and Patodi of Gilkey's theorem, because it is equivalent to take $(\det g)^{-1}$ as an additional variable on one hand, or the variables g^{cd} on the other hand. However, the exposition of Atiyah, Bott and Patodi omits in error the square root of the determinant of the metric. The methods of § 6 of our paper are appropriate to this more general situation.

The proof of the above theorem and some corollaries will follow in § 5.

2. The space of ∞-jets of g

A natural tensor is clearly determined locally. Therefore an equivalent formulation of the problem is as follows. For each disk $D(r)$ in R^m with centre at 0 and radius r and for each Riemannian metric g on $D(r)$ we suppose we have a C^∞ tensor field $t_{(D(r),g)}$ on $D(r)$ such that if $\varphi: (D(r), g) \to (D(s), h)$ is an isometry onto an open subset of $D(s)$, then

$$\varphi_* t_{(D(r),g)} = t_{(D(s),h)} \,|\, \varphi D(r) \ . \tag{2.1}$$

For t is then consistently defined when two such coordinate neighborhoods overlap inside a general manifold (M, g).

Theorem 2.2. $t_{(D(r),g)}(0)$ *depends only on the* ∞*-jet of* g *at* 0. (*It is independent of the radius* r.)

Proof. Let g_1 and g_2 be two metrics on a disk D with the same ∞-jet at 0. Let $V_1 = \{x \in D : x_1 \geq m|x_i| \text{ for } 2 \leq i \leq m\}$ and $V_2 = \{x \in D : -x_1 \geq m|x_i| \text{ for } 2 \leq i \leq m\}$. Then $V_1 \cap V_2 = \{0\}$. By Whitney's extension theorem (see e.g. Tougeron [6, Chapter 4, Proposition 4.7]), there is a C^∞ function on $D(s)$, which is equal to g_1 on $D(s) \cap V_1$ and to g_2 on $D(s) \cap V_2$. By symmetrizing and going to a smaller neighborhood, we may assume that g is a Riemannian metric on $D(s)$. Then $t_{(D(s),g)}$ agrees with $t_{(D(r),g_1)}$ on $D(s) \cap V_1$ and with $t_{(D(r),g_2)}$ on $D(s) \cap V_2$, and so all three agree at 0. q.e.d.

The above theorem shows that to understand our problem we need to understand the space of ∞-jets of g. It is convenient to restrict t still further, which we can do without loss of generality, so that it takes its values on pairs $(D(r), g)$ where the coordinates of $D(r)$ are normal coordinates arising from an orthonormal basis at the origin. This means that the exponential map from the tangent space at the origin to $D(r)$ is regarded as the identity map. Of course we continue to insist on (2.1) for arbitrary isometries.

(2.2) Unless otherwise stated, we henceforth assume that $(D(r), g)$ has normal coordinates arising from the origin. We trivialise the tangent bundle by parallel translation along radial geodesics, and use the standard basis at the origin $(e_1, \cdots, e_m)$ with $e_i = \partial/\partial x^i$.

The following beautiful formula seems to have been better known by differential geometers 40 years ago than it is now. It was brought to the author's attention by G. Lusztig.

Theorem 2.3. *A necessary and sufficient condition for coordinates in an open disk centred at the origin, to be normal coordinates arising from an orthonormal basis at the origin, is that for each point x in the disk,*

$$g_{ij}(x)x^j = x^i .$$

Proof. It is a standard result in differential geometry (see for example Kobayashi and Nomizu [3, p. 165]), that the exponential map at a point x preserves distances along the geodesic and also sends the orthogonal subspace to a ray through the origin in the tangent space to a subspace which is orthogonal to the corresponding geodesic. This means that

$$\langle x, y\rangle_{R^n} = \langle x, y\rangle_{g(x)}$$

because y is equal to a multiple of x plus a vector orthogonal to x.

The converse can be proved by the reverse chain of reasoning. Alternatively we can use the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right) .$$

If we differentiate the formula $g_{ij}(x)\, x^j = x^i$, we obtain

$$\frac{\partial g_{ij}}{\partial x^m} x^j = \delta_{im} - g_{im} . \tag{2.4}$$

If $(y^1, \cdots, y^m)$ is fixed, then along the curve $t(y^1, \cdots, y^m) = (x^1, \cdots, x^m)$ we have

$$\begin{aligned}\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx_i}{dt}\frac{dx^j}{dt} &= \frac{\Gamma^k_{ij}}{t^2}x^i x^j \\ &= \tfrac{1}{2}t^{-2}g^{km}((\delta_{mj} - g_{mj})x^j + (\delta_{mi} - g_{mi})x^i - (\delta_{mj} - g_{mj})x^j) \\ &= \frac{g^{km}}{2t^2}(x^m - x^m) = 0 .\end{aligned}$$

Thus each ray is a geodesic. From (2.4) we see that at the origin $g_{im} = \delta_{im}$. Hence the coordinates are normal and arise from an orthonormal bases at the origin. q.e.d.

We insert here an interesting consequence of this theorem, enabling us to solve the "moduli problem" for two-dimensional Riemannian manifolds.

Let $v: R^2 \to R$ be a germ at 0 of a C^∞ function. Then we obtain a Riemannian metric g on a neighborhood of zero in R^2 by the formula

$$\varphi(v)(x, y) = g(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + v(x, y)\begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + v(x, y)\begin{bmatrix} y \\ -x \end{bmatrix}[y \quad -x] .$$

The space of such germs v has an obvious action of $0(2)$ by composing with the action of $0(2)$ on R^2.

Theorem 2.4. *The map φ defines a map*

$\bar{\varphi}$: {*Germs at* 0 *of C^∞ functions $R^2 \to R$*} $\to$ {*Isomorphism classes of germs at a point x of C^∞ Riemannian manifolds of dimension* 2}

which is onto, and the inverse image of an isomorphism class is a single orbit under $0(2)$.

Proof. An easy computation shows that if $A \in 0(2)$, then A gives an isometry of the manifold with metric $\varphi(vA)$ onto the manifold with metric $\varphi(v)$. Since $\varphi(v)$ satisfies the conditions of Theorem 2.3, the coordinates are normal. If $\bar{\varphi}(v) = \bar{\varphi}(w)$, then the two sets of normal coordinates must differ by at most an element of $0(2)$. So this shows that $\bar{\varphi}$ is $1 - 1$.

In order to show that $\bar{\varphi}$ is onto, we take normal coordinates and apply Theorem 2.3. Then

$$g_{11}x + g_{12}y = x , \qquad g_{21}x + g_{22}y = y .$$

Therefore $g_{11} - 1 = yh$ for some C^∞ function h and $g_{12} = g_{21} = -xh$. It follows from the second equation that $h = yv$ for some C^∞ function v. Then

$$g_{11} = 1 + vy^2 , \quad g_{12} = g_{21} = -xyv , \quad g_{22} = 1 + vx^2 .$$

So we have proved that the space of germs of two-dimensional Riemannian manifolds is the quotient of an infinite dimensional representation space for $0(2)$ by the action of $0(2)$.

Theorem 2.3 also enables us to obtain explicit information about the Taylor series for $g_{ij}(x)$ in normal coordinates. We write the Taylor series (which is a formal power series) in the form

$$\delta_{ij} + \sum_{r \geq 1} g_{iji_1 \cdots i_r} x^{i_1} \cdots x^{i_r} ,$$

where the coefficients $g_{iji_1 \cdots i_r}$ satisfy the following conditions:

(2.5.1) We have symmetry in the first two indices.

(2.5.2) We have symmetry in the last r indices.

(2.5.3) Given $\alpha(1), \cdots, \alpha(r+1)$ with each $\alpha(s)$ satisfying $1 \leq \alpha(s) \leq m$, we have

$$\sum_{\sigma \in S(r+1)} g_{i,\alpha\sigma(1),\cdots,\alpha\sigma(r+1)} = 0 \ .$$

The third condition follows immediately from Theorem 2.3. Since $g_{iji_1\cdots i_r} = \dfrac{1}{r!}\dfrac{\partial^\alpha}{\partial x^\alpha} g_{ij}(0) = \dfrac{1}{r!}\dfrac{\partial^\alpha}{\partial x^\alpha} g\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\right)$ where $\alpha = (i_1, \cdots, i_r)$ and different normal coordinates are related by an element of $0(m)$, we see that $g_{iji_1\cdots r_r}$ defines without ambiguity a tensor of type $(0, r+2)$. In fact we have a natural tensor by taking normal coordinates at each point in the manifold. We define $f_r \in (V^*)^{\otimes(r+2)}$, where V is the tangent space at 0, by

$$f_r(e_i, e_j, e_{i_1}, \cdots, e_{i_r}) = g_{iji_1\cdots i_r} \ .$$

Theorem 2.6. a) *The set of elements of* $(V^*)^{\otimes(r+2)}$ *satisfying the symmetry conditions corresponding to* (2.5.1), (2.5.2) *and* (2.5.3) *is an irreducible GLV-module* Y_r *with Young diagram having r squares in the first row and* 2 *squares in the second row, except that if* $r = 1$ *the only element satisfying these conditions is zero.*

b) *If* $f_r \in Y_r$ $(2 \leq r < \infty)$ *is an arbitrary sequence of elements then there is a Riemannian metric whose Taylor series gives us the elements* f_r. (*This is a well-known result, first told to the author by R. Penrose.*)

Proof. If $r = 1$, we have $g_{ijk} = g_{jik}$ and $g_{ijk} + g_{ikj} = 0$. Hence

$$g_{ijk} = g_{jik} = -g_{jki} = -g_{kji} = g_{kij} = g_{ikj} = -g_{ijk} \ ,$$

and so $g_{ijk} = 0$.

If $r \geq 2$, we define for each $(r-2)$-tuple I

(2.6.1) $$S_{ijklI} = g_{ilkjI} - g_{ikljI} + g_{jkliI} - g_{jlkiI} \ .$$

Clearly $S_{ijklI} = -S_{jiklI} = -S_{ijlkI}$ and S_{ijklI} is unchanged by permuting the entries of I. Moreover the two Bianchi identities

$$S_{ijklI} + S_{ikljI} + S_{iljkI} = 0 \ , \quad S_{ijklrJ} + S_{ijrklJ} + S_{ijlrkJ} = 0 \ ,$$

where J is an $(r-3)$-tuple, follow from (2.5.1) and (2.5.2) by an easy computation. But these are precisely the conditions which ensure that S is in the subspace of $(V^*)^{\otimes(r+2)}$ with the Young diagram described in the theorem.

Given a function S of $(r+2)$-tuples of integers i with $1 \leq i \leq m$, which is antisymmetric in the first two variables, in the third and fourth, which is symmetric in the lest $(r-2)$ variables, and which satisfies the two Bianchi identities (so that S lies in space of the Young diagram) we define

$$g_{ij\alpha(1)\cdots\alpha(r)} = -\frac{r-1}{(r+1)!}\Sigma_{\sigma\in S(r)}S_{i,\alpha\sigma(1),j,\alpha\sigma(2),\alpha\sigma(3),\cdots,\alpha\sigma(r)} \ . \tag{2.6.2}$$

A computation shows that (2.6.1) and (2.6.2) set up an isomorphism between the vector subspace corresponding to the Young diagram on the one hand, and the elements satisfying (2.5.1), (2.5.2) and (2.5.3). This proves part a) of Theorem 2.6.

In order to prove b), let $f_r \in Y_r$ for $2 \leq r < \infty$, where we take $V = R^m$. We then have coefficients $g_{iji_1\cdots i_r} \in R$. Let $g_{ij} = g_{ji}$ be a C^∞ function with these coefficients giving the derivative at the origin by the formula

$$\frac{\partial^I g_{ij}}{\partial x^I} = r!g_{ijI}$$

for any r-tuple $I = (i_1, \cdots, i_r)$ $(r \geq 2)$. We also insist that $g_{ij}(0) = \delta_{ij}$ and $\dfrac{\partial g_{ij}}{\partial x^k}(0) = 0$. This defines a Riemannian metric in a neighborhood of 0. However the coordinates may not be normal.

We now show that if we change to normal coordinates for g then the ∞-jet of the change of coordinates is the identity at 0. The geodesic through the origin with tangent at the origin $a^k\partial/\partial x^k$ has an ∞-jet at the origin which can be found recursively from the equation

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \ .$$

Since the ∞-jet of $g_{ij}(x)x^j - x^i$ is zero at the origin, we know that the ∞-jet of $\dfrac{\partial g_{ij}}{\partial x^k}x^j + g_{ik} - \delta_{ik}$ is zero at the origin. We assume by induction that the solution has the same N-jet as $t \to t\,(a^1, \cdots, a^m)$. The $(N-1)$-jet of $\Gamma^k_{ij}\dfrac{dx^i}{dt}\dfrac{dx^j}{dt}t^2$ is then equal to that of

$$\frac{1}{2}g^{km}\left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right)x^ix^j = 0 \ .$$

Thus the $(N-1)$-jet of $t^2\dfrac{d^2x^k}{dt^2}$ is zero at the origin. It follows that the $(N+1)$-jet of x^k is equal to that of a^kt. This completes the induction. Hence with respect to normal coordinates the coefficients $g_{iji_1\cdots i_r}$ are unaltered, which proves the theorem.

Theorem 2.6 means that in a certain sense we can regard $\prod_{r\geq 2} Y_r$ as the space of ∞-jets of Riemannian metrics at a point.

3. Another description of a natural tensor

By trivalising the tangent bundle as described in (2.2), we have a map for each C^∞ metric g on a disk $D(s)$ such that the coordinates are normal at the origin:

$$j(g) : D(s) \to \prod_{r\geq 2} Y_r \ .$$

Here Y_r is the representation space obtained from $V = T_0(D(r)) = R^m$ as described in Theorem 2.6. The composition $\pi_r \circ j(g) : D(s) \to Y_r$ is clearly C^∞ for each r.

Suppose we have a natural tensor t of type (p, q) — that is, taking its values in sections of $T^{\otimes p} \otimes (T^*)^q$. Then according to Theorem 2.2, t defines a map

$$\hat{t} : \prod_{r\geq 2} Y_r \to V^{\otimes p} \otimes (V^*)^{\otimes q} \ .$$

We have

$$\hat{t} \circ j(g) = t_g \ . \tag{3.1}$$

Remark. We could identify V with V^* by using the Riemannian metric g. However we will not bother to do this.

Proposition 3.2. *$\hat{t}$ is invariant under* $0(m)$. (*We are identifying R^m with V.*)

Proof. Let g be a Riemannian metric on $D(s)$, whose jet at the origin is $j(g)(0)$, and suppose the coordinates of $D(s)$ are normal with respect to the origin. Let $A \in 0(m)$. Then

$$j(A_* g)(0) = A_* j(g)(0) \ ,$$

so that

$$\begin{aligned} A_* \hat{t} j(g)(0) &= A_* t_g(0) = t_{A_* g}(0) \qquad \text{by } (2.1) \\ &= \hat{t} j(A_* g)(0) = \hat{t} A_* j(g)(0) \ . \end{aligned}$$

Since $j(g)(0)$ is an arbitrary element of $\prod_{r\geq 2} Y_r$, the result follows.

In fact it is clear that if we have an $0(m)$-invariant map $\hat{t} : \prod_{r\geq 2} Y_r \to V^{\otimes p} \otimes (V^*)^{\otimes q}$ such that $\hat{t} \circ j(g)$ is C^∞ for each C^∞ metric g on a disk $D(s)$ with normal coordinates from the origin, then we have a natural tensor. Note that there is no requirement for $\hat{t}$ to be continuous.

4. A counterexample

In this section we present a natural function (a tensor of type $(0, 0)$) on 2-dimensional manifolds such that t depends only on 4-jet of g (in $Y_2 \times Y_3 \times Y_4$) and such that $\hat{t}$ is discontinuous.

Let $\psi : R^2 \to R$ be a function, which is C^∞ except at 0, is equal to 1 on

$\{(0, b) \mid b \neq 0\}$, and is equal to zero on the two circles of radius one which are tangent at the origin to the b-axis.

The natural function is defined by slightly modifying the formula:

$$\psi(R_{1212;11}{}^4, R_{1212}{}^2 + (R_{1212;1} - 6)^2 + R_{1212;22}{}^4 + (R_{1212;12} - 10)^4) .$$

As it stands, this is not well defined, because we have used the coordinates of R, ∇R and $\nabla^2 R$ with respect to a basis. So we must describe the basis in an intrinsic way.

Now $\Lambda^2 T^*$ is one-dimensional and has an inner product on a Riemannian manifold. Hence $\Lambda^2 T^*$ is canonically isomorphic to R, up to a sign (the orientation of T^*) and $\Lambda^2 T^* \otimes \Lambda^2 T^*$ is canonically isomorphic to R, without ambiguity. We consider the image $\alpha(\nabla R)$ in T^* of $\nabla R \in (T^*)^{\otimes 5}$ under the composition

$$\alpha : (T^*)^{\otimes 5} \to \Lambda^2 T^* \otimes \Lambda^2 T^* \otimes T^* \cong T^* .$$

Now ∇R is function of the 3-jet g. (In fact the tensors s_{ijkl} and s_{ijklm} of Theorem 2.6 are equal to the curvature R and the derivative ∇R of the curvature.)

Let $X_1 = \{(y_2, y_3, y_4) \in Y_2 \times Y_3 \times Y_4 : \alpha(\nabla R(y_2, y_3)) = 0\}$. This is a closed subset of $Y_2 \times Y_3 \times Y_4$. Outside X_1 we define e^1 to be the unit vector which is a positive multiple of $\alpha(\nabla R)$.

Let $\beta : (T^*)^{\otimes 6} \to \Lambda^2 T^* \otimes \Lambda^2 T^* \otimes T^* \otimes T^* \cong T^* \otimes T^*$. We have two choices for e^2, depending on the orientation. Let

$$X_2 = \{(y_2, y_3, y_4) : \beta(\nabla^2 R(y_2, y_3, y_4))(e_1 \otimes e_2) = 0 \quad \text{or} \quad (y_2, y_3, y_4) \in X_1\} ,$$

where (e_1, e_2) is the dual basis to (e^1, e^2). The definition of X_2 is independent of the choice of e^2. Outside X_2, we define e^2 so that

$$\beta(\nabla^2 R)(e_1 \otimes e_2) > 0 .$$

Let $\varphi : Y_2 \times Y_3 \times Y_4 \to R$ be a C^∞ function which is one outside a small neighborhood of X_2 and zero on a smaller neighborhood. We define our natural function as the product $\varphi \cdot \psi$ outside X_2 and zero otherwise.

Theorem 4.1. *On any fixed C^∞ Riemannian manifold of dimension two, the above function is C^∞. However there is a C^∞ 1-parameter family of Riemannian metrics on a disk, such that the function is a discontinuous function of the parameter (so that the function $\hat{t}$ of § 3 is discontinuous).*

Proof. We prove the first statement by contradiction. Let $x_0 \in M$ be a point which has no neighborhood on which the function is C^∞. Then at x_0, we are at $(0, 0)$ the point where ψ is not C^∞. It follows that

$$R_{1212} = R_{1212;1} - 6 = R_{1212;11} = R_{1212;22} = R_{1212;12} - 10 = 0 .$$

Moreover $R_{1212;2} = 0$ by our definition of α and e^1. It is clear that this point is not in X_2 and so the value of φ is 1 in a neighborhood of x_0. Moreover the basis e_1, e_2 chosen by the process described above gives us the *same* basis as that with respect to which we have already expressed R, ∇R and $\nabla^2 R$. From the Ricci identity and $R_{1212} = 0$ we see that $R_{1212;12} = R_{1212;21} = 10$ at x_0.

Now there is an elegant exposition by A. Gray [2] which enables us to compute in normal coordinates near x_0. We denote by (x, y) the normal coordinates arising from the basis (e_1, e_2) dual to (e_1, e_2) at x_0. Then, with $r^2 = x^2 + y^2$, we have in some neighborhood of x_0

$$\begin{aligned} g_{11} &= 1 - xy^2 - xy^3 + 0(r^5) , & g_{22} &= 1 - x^3 - x^3y + 0(r^5) , \\ g_{12} = g_{21} &= x^2y + x^2y^2 + 0(r^5) , & R_{1212} &= 6x + 10xy + 0(r^3) , \\ R_{1212;1} &= 6 + 10y + 0(r^2) , & R_{1212;2} &= 10x + 0(r^2) . \end{aligned}$$

Let

$$\begin{aligned} u = {} & dx \otimes dy \otimes dx \otimes dy + dy \otimes dx \otimes dy \otimes dx \\ & - dx \otimes dy \otimes dy \otimes dx - dy \otimes dx \otimes dx \otimes dy . \end{aligned}$$

Then $\nabla R = (6 + 10y + 0(r^2))u \otimes dx + (10x + 0(r^2))u \otimes dy$. Since the image of u in $\Lambda^2 T^* \otimes \Lambda^2 T^*$ has length 4 det $g = 4 + 0(r^2)$,

$$\tfrac{1}{4}\alpha(\nabla R) = (6 + 10y + 0(r^2))dx + (10x + 0(r^2))dy ,$$

and therefore

$$\tfrac{1}{16}\|\alpha(\nabla R)\|^2 = 36 + 120y + 0(r^2) ,$$

or

$$\tfrac{1}{4}\|\alpha(\nabla R)\| = 6 + 10y + 0(r^2) .$$

Hence

$$\begin{aligned} e^1 &= (1 + 0(r^2))dx + (\tfrac{5}{3}x + 0(r^2))dy , \\ e^2 &= (\tfrac{5}{3}x + 0(r^2))dx + (1 + 0(r^2))dy , \end{aligned}$$

the sign of e^2 being determined by continuity, since we know $e^2 = dy$ at x_0.

We write S_{1212} or $S_{1212;1}$ etc. when we wish to express ∇R with respect to the basis (e^1, e^2). Since

$$S_{1212} = 6x + 0(r^2) , \quad \text{and} \quad S_{1212;1} = 6 + 10y + 0(r^2) ,$$

we can use as coordinates the functions $\alpha = S_{1212}$ and $\beta = S_{1212;1} - 6$ on M near x_0. So near x_0 where $\alpha = \beta = 0$ our function is equal to

$$\psi(\rho(\alpha, \beta), \alpha^2 + \beta^2 + \sigma(\alpha, \beta))$$

where ρ and σ are C^∞ functions depending on M, each with 3-jet zero. So in some neighborhood of x_0 the function is identically zero and hence C^∞. This contradiction proves the first part of the theorem.

Now let us consider the 1-parameter family g_t of metrics defined by

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - (x + xy + tx^2)\begin{bmatrix} y^2 & -xy \\ -xy & x \end{bmatrix}$$

on the disk. By Theorem 2.3 we know the coordinates are normal at the origin for each t.

From the expansion of g_{ij} in normal coordinates [2], we can read off at the origin

$$\begin{aligned} &R_{1212} = 0\ , \\ &R_{1212;1} = 6\ , \quad R_{1212;2} = 0\ , \\ &R_{1212;11} = 20t\ , \quad R_{1212;22} = 0\ , \quad R_{1212;12} = R_{1212;21} = 10\ . \end{aligned}$$

This gives $e^1 = dx$ and $e^2 = dy$ at the origin $(0, 0)$, for each t.

The natural function is therefore equal to

$$\begin{aligned} \psi((20t)^4, 0)) &= 1 \qquad \text{if } t \neq 0 \\ &= 0 \qquad \text{if } t = 0\ . \end{aligned}$$

This completes the proof of the properties of the counter example.

5. Polynomial tensors

Definition 5.1. We say a natural tensor t is homogeneous of weight k if $t_{(M,\lambda^2 g)} = \lambda^k t_{(M,g)}$ for each real $\lambda > 0$. Clearly every polynomial tensor is the sum of homogeneous tensors, (and the weights are always even). We see that $g.$ and $\nabla^n R$ are of weight two and $g^{\cdot}$ is of weight minus two.

We now prove Theorem 1.2, restating it slightly.

Theorem 5.2. *Let t be a natural tensor such that $\hat{t} : \prod_{r\geq 2} Y_r \to V^{\otimes p} \otimes (V^*)^{\otimes q}$ is the composition of the projection onto a finite product $\prod_{2\leq r\leq N} Y_r$ followed by a polynomial map. Then t is a polynomial tensor* (*in the sense of* § 1).

Proof. Let the polynomial referred to in the hypothesis be denoted $F: \prod_{r=2}^{N} Y_r \to V^{\otimes p} \otimes (V^*)^{\otimes q}$. By Proposition 3.2, F is $0(m)$-invariant. We can write F as a sum of polynomials, each $0(m)$-invariant, and each homogeneous in the coordinates of Y_r for each r. The process of polarization allows us to see that such a polynomial induced from an $0(m)$-invariant linear map

$$Y_2^{\otimes r_2} \otimes \cdots \otimes Y_N^{\otimes r_N} \to V^{\otimes p} \otimes (V^*)^{\otimes q}$$

by composition with the diagonal map. Now Y_i is a GLV direct summand of $(V^*)^{\otimes(i+2)}$ and so our map F is induced by a diagonal map followed by an $0(m)$-invariant map

$$(V^*)^{\otimes K} \to V^{\otimes p} \otimes (V^*)^{\otimes q} .$$

Hermann Weyl's results on $0(m)$-invariants (see Atiyah, Bott, Patodi [1]) can now be applied. We see that if we compute at a point x_0 at which we have taken normal coordinates, then $t(x_0)^{k_1\cdots k_p}_{j_1\cdots j_q}$ is given by taking real linear combinations of expressions formed by multiplying the tensors $g_{iji_1\cdots i_r}$ in various ways, multiplying by tensors such as g_{ab} or g^{cd} (each equal to the Kronecker symbol at x_0), and contracting an upper index against a lower one.

To complete the proof we need to know that the tensors f_r formed from the $g_{iji_1\cdots i_r}$ as explained in § 2 are in fact natural polynomial tensors. This can be easily deduced from the formulas in Gray's paper [2]. For example, in the notation of Theorem 2.6,

$$S_{ijkl;i_1\cdots i_r} = R_{ijkl;i_1\cdots i_r} + p_r .$$

Here p_r is a polynomial with rational coefficients formed from terms like $R_{abcd;j_1\cdots j_s}$ $(s < r)$ by multiplication and contraction. Of course in each monomial, the indices left over after contraction are precisely $i, j, k, l, i_1, \cdots, i_r$, in some order.

Now contraction can be effected by multiplying a g^{ab} and then summing (since $g^{ab} = \delta_{ab}$ in normal coordinates). It follows that f_r is a natural polynomial tensor in the language of § 1 (we have equality of f_r with a polynomial tensor in normal coordinates and hence in any coordinates). Similarly t is itself a polynomial tensor. This completes the proof.

Remark 5.3. It fact f_r is of weight two. This is seen by computing the effect on $\dfrac{\partial^r}{\partial x^{i_1} \cdots \partial x^{i_r}} g_{ij} = r g_{iji_1\cdots i_r}$ of multiplying g by λ^2. Let normal coordinates with respect to $\lambda^2 g = h$ be denoted by $(y^1, \cdots, y^m)$. Since the affine connection is unchanged, the geodesics are unchanged. Hence the only change is in the orthonormal basis. Since $\delta_{ij} = h(\partial/\partial y^i, \partial/\partial y^j) = \lambda^2 g(\partial/\partial y^i, \partial/\partial y^j)$ we have $(\lambda\partial/\partial y^i = \partial/\partial x^i)$, and so $\lambda dx^i = dy^i$ and $\lambda x^i = y^i$. With an obvious notation, we have

$$h_{iji_1\cdots i_r} = \frac{1}{r!} \frac{\partial^r}{\partial y^{i_1} \cdots \partial y^{i_r}} h\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \lambda^{-r} g_{iji_1\cdots i_r} . \tag{5.3.1}$$

Hence

$$\begin{aligned} f_r(\lambda^2 g) &= h_{iji_1\cdots i_r} dy^i \otimes dy^j \otimes dy^{i_1} \otimes \cdots \otimes dy^{i_r} \\ &= \lambda^2 g_{iji_1\cdots i_r} dx^i \otimes dx^j \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_r} = \lambda^2 f_r(g) . \end{aligned}$$

Corollary 5.4. *Let t be a natural polynomial tensor of type (u, l) (i.e., $t \in T^{\otimes u} \otimes (T^*)^{\otimes l}$). Suppose t is homogeneous of weight w and is not zero. Then $l - u \geq w$. Moreover $l - u = w$ if and only if t is formed from g. and*

$g^{\cdot}$ without making use of the curvature and its derivatives (in the description in § 1).

Proof. Let a typical monomial appearing in t be constructed using A copies of $g_{\cdot}$, B copies $g^{\cdot}$ and the tensors $\nabla^{n_i}R$ $(i = 1, \cdots, k)$, followed by contracting C upper indices against C lower indices. Then

$$l = 2A + 4k + \Sigma n_i - C \,, \quad u = 2B - C \,, \quad w = 2(A + k - B) \,.$$

Now $l - u = 2(A + k - B) + 2k + \Sigma n_i \geq w$. If $l - u = w$ then $k = 0$, and conversely.

Definition 5.5. By a natural connection we mean a connection $\nabla_{(M,g)}$ on the tangent bundle of M for each C^∞ Riemannian manifold (M, g), such that the following naturality conditions are satisfied:

1) If $\varphi\colon (M, g) \to (N, h)$ is an isometry of M onto N, then $\nabla' = \nabla_{(M,g)}$ and $\nabla'' = \nabla_{(N,h)}$ correspond under φ_* — that is, $\varphi_*(\nabla'_X Y) = \nabla''_{\varphi_* X}\varphi_* Y$ if X and Y are vector fields on M.

2) If U is an open subset of M, then $\nabla_{(U,g)}$ is the restriction of $\nabla_{(M,g)}$ to U.

We say the natural connection is *polynomial* if Γ^k_{ij} is a polynomial in some finite jet of g, where Γ^k_{ij} is as usual defined by

$$\nabla_{\partial/\partial x^i}(\partial/\partial x^j) = \Gamma^k_{ij}(\partial/\partial x^k) \,.$$

It also makes sense to talk of a natural connection being homogeneous. In this case the weight must be zero since all connections have the same symbol.

Theorem 5.6. *Let ∇ be a natural connection which is polynomial and homogeneous (of weight zero). Then ∇ is the Riemannian connection (sometimes called the Levi-Civita connection).*

Later we will strengthen this theorem so that the Γ^k_{ij} need only depend in a C^∞ rather than a polynomial way on the ∞-jet of g.

Proof. The difference between ∇ and the Levi-Civita connection is a natural polynomial tensor of type (2, 1), which is homogeneous of weight zero. Then Corollary 5.4 shows that this tensor is zero.

6. C^∞ dependence of the ∞-jet of g

Definition 6.1. We say that a function defined on $\prod_{r\geq 2} Y_r$ with values in a finite dimensional vector space is C^∞ if:

1) it is continuous with respect to the product topology,

2) it is C^∞ on each finite dimensional subspace Y^N $(N = 2, 3, 4, \cdots)$ where Y^N is the set of elements of $\prod_{r\geq 2} Y_r$ with projection to Y_r zero for each $r > N$.

Theorem 6.2. *Let t be a natural tensor of type (p, q) which is homogeneous of weight w and nonzero. If $\hat{t}$ is a C^∞ function on $\prod_{r\geq 2} Y_r$, then t is a polynomial tensor (and so w is an even integer).*

Proof. By writing out everything in coordinates with respect to the standard orthonormal basis, using (5.3.1), $dy^i = \lambda dx^i$, $\lambda\partial/\partial y^i = \partial/\partial x^i$, and $t(\lambda^2 g) = \lambda^w t(g)$, we see that

$$(6.2.1) \qquad \hat{t}^{k_1\cdots k_p}_{r_1\cdots r_q}(\cdots, \lambda^{-r} g_{iji_1\cdots i_r}, \cdots) = \lambda^{w+p-q} \hat{t}^{k_1\cdots k_p}_{r_1\cdots r_q}(\cdots, g_{iji_1\cdots j_r}, \cdots) \ .$$

It follows that the Taylor series expansion of $\hat{t} \,|\, Y^N$, taken at the origin of Y^N, has only a finite number of nonzero terms. Moreover the coefficient of

$$g_{I(1)} \cdots g_{I(k)}$$

(where each $I(i)$ is an $(r_i + 2)$-tuple of integers each equal to 1 or 2 or $\cdots$ m) is nonzero only if

$$-[r_1 + \cdots + r_k] = w - q + p \ ,$$

which is really the same formula as that appearing in Corollary 5.4, with $q = l$, $p = u$, $r_i = n_i + 2$. So the Taylor series of $\hat{t} \,|\, Y^N$ is unaltered by increasing N once $N > q - p - w$, and we can talk of the "Taylor series of $\hat{t}$". The Taylor series of $\hat{t}$ satisfies the same kind of equation as (6.2.1).

The difference δ between $\hat{t}^{k_1\cdots k_p}_{r_1\cdots r_q}$ and its Taylor series has all derivatives zero at the origin and

$$(6.2.2) \qquad \delta(\cdots, \lambda^{-r} g_{iji_1\cdots i_r}, \cdots) = \lambda^a \delta(\cdots, g_{iji_1\cdots i_r}) \ ,$$

where $a = w + p - q$. Now as $\lambda \to +\infty$ the left hand side is small compared with λ^{-s} for any value of s (if we restrict to Y^N). Hence $\delta \,|\, Y^N = 0$ for each N. By continuity δ is identically zero. This proves that $\hat{t}$ satisfies the hypothesis of Theorem 5.2.

Corollary 6.3. *If ∇ is a natural connection such that the Γ^k_{ij} are C^∞ functions on $\prod_{r>2} Y_r$. Then ∇ is the Levi-Civita connection.*

7. Conjecture

The obvious problem is whether every natural tensor which is homogeneous is also polynomial. The author has not been able to settle this. While there may well be a counterexample, it seems to the author likely that if the following condition is added then no counterexample should exist.

7.1. Axiom of analyticity. Suppose t is a natural tensor such that $t_{(M,g)}$ is analytic whenever (M, g) is an analytic Riemannian manifold. Then we say t is analytic. Note that this is entirely different from the assumption that $\hat{t}$ is analytic.

7.2. Conjecture. Let t be a natural analytic homogeneous tensor. Then t is a polynomial tensor.

The only result we have been able to obtain in this direction is the following unsatisfactory theorem.

Theorem 7.3. *Let t be a natural tensor of type (p, q), which is homogeneous of weight w and is nonzero. Then $w + p - q \leq 0$. Moreover if $w + p = q$, then t is polynomial and we are in the situation of Corollary* 5.4.

Proof. Let $y \in \prod_{r\geq 2} Y_r$ be a fixed point with coordinates $(g_{iji_1\cdots i_r})$. Let $x_s = (1/s, 0, \cdots, 0) \in R^m$ $(s = 1, 2, 3, \cdots)$. At x_s we place the ∞-jet of g corresponding to $(e^{-rs}g_{iji_1\cdots i_r})$, and at $0 \in R^m$ we have the ∞-jet of g corresponding to (0) — namely the ∞-jet of the flat metric. We now check that this set of ∞-jets satisfies the hypothesis of Whitney's extension theorem [6, p. 77], and so there is a Riemannian metric g on a neighborhood of 0 with the stated ∞-jet at x_s for each sufficiently large s and at 0.

Then $t_{(g)}$ is a C^∞ tensor on this neighborhood. By (6.2.1) (whose proof used only homogeneity), we see that

$$t^{k_1\cdots k_p}_{r_1\cdots r_q}(x_s) = e^{s(w+p-q)}\hat{t}^{k_1\cdots k_p}_{r_1\cdots r_q}(y) .$$

We let $s \to \infty$ and see that either $(w + p - q) \leq 0$ or $\hat{t}(y) = 0$. It $w + p - q = 0$, then letting $s \to \infty$ shows that $\hat{t}(y)$ is equal to t applied to the flat metric. Therefore $\hat{t}(y)$ is a constant, independent of y, in $V^{\otimes p} \otimes (V^*)^{\otimes q}$. This shows that t satisfies the hypothesis of Theorem 5.2.

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