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HIGHER ORDER DISSECTIONS

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1. Introduction

In their paper "Sprays" [1] Ambrose, Palais and Singer introduced the concept of a dissection of the second order tangent bundle of a C^∞ manifold M and the concept of the spray of a connection on M, and proved that there is a natural bijection between the set of second order dissections and the set of sprays of M. The purpose of this paper is to investigate relations between the dissections of higher order tangent bundles of a given C^∞ manifold and related structures on its extensions, a notion due to the present writer [2]. For example, we prove that each dissection of the mth order tangent bundle of a C^∞ manifold M determines, in a natural manner, a unique dissection of the second order tangent bundle of the $(m-2)$nd extension of M for each integer $m \geq 2$. It then follows that there is a natural injection of the set of mth order dissections of M into the set of sprays on the $(m-2)$nd extension of M.

2. Preliminary remarks

Suppose that M is an n-dimensional C^∞ manifold. In a previous paper [2] the present writer has constructed a sequence

$$M = {}^0M \xleftarrow[{}^0\Pi]{} {}^1M \xleftarrow[{}^1\Pi]{} {}^2M \longleftarrow \cdots \tag{1}$$

of C^∞ manifolds and C^∞ maps which we will call the extension sequence of M, with the $(m+1)$ n-dimensional manifold ${}^mM$ called the mth extension of M. If (U, ϕ) is a coordinate chart of M with coordinate functions x^a, $a = 1, \cdots, n$, then it induces coordinates $x^{\alpha a}$, $a = 1, \cdots, n$ and $\alpha = 0, \cdots, m$, on ${}^m\Pi^{-1}(U)$ (where ${}^m\Pi = {}^0\pi \circ {}^1\pi \circ \cdots \circ {}^m\pi$), for each positive integer m, which we call the natural coordinates induced by (U, ϕ). If $f \in C^\infty(M)$, there is a lift f^m of f to ${}^mM$ (which has been called the complete lift in the case $m = 1$, e.g., see [3]) and $x^{a\alpha} = x^{\alpha a}$. In terms of natural coordinates we then have the theorem [2]

$$\frac{\partial f^m}{\partial x^{\alpha a}} = \binom{m}{\alpha} \left[\frac{\partial f}{\partial x^a} \right]^{m-\alpha} \tag{2}$$

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(no summation), and we have $f^1 = \sum_{a=1}^{n}(\partial f/\partial x^a)x^{1a}$ with the higher lifts behaving as if the lift were differentation with respect to a parameter, and thus we may formally apply the Leibnitz formula.

If (U, ϕ) and (V, ψ) are coordinate charts of M with coordinate functions x^a and y^r respectively, then we adopt the notation

$$\begin{aligned} \partial/\partial x^a &= \partial_a , & \partial^2/\partial x^a \partial x^b &= \partial^2{}_{ab}, \cdots , \\ \partial/\partial y^r &= \partial_r , & \partial^2/\partial y^r \partial y^s &= \partial^2{}_{rs}, \cdots , \\ \partial x^a/\partial y^r &= X^a_r , & \partial^2 x^a/\partial y^r \partial y^s &= X^a_{rs}, \cdots , \end{aligned}$$

and in the case of the natural coordinates induced by these charts we take

$$\partial/\partial x^{\alpha a} = \partial_{\alpha a} , \quad \partial^2/\partial y^{\rho r}\partial y^{\sigma s} = \partial^2{}_{\rho r \sigma s} , \quad \partial x^{\alpha a}/\partial y^{\rho r} = X^{\alpha a}_{\rho r} \cdots ,$$

i.e., we associate early (resp. late) letters of the alphabet with the coordinates of the chart (U, ϕ) (resp. (V, ψ)) and the natural coordinates it induces. We adopt the summation convention with regards to lower case Latin and Greek indices over the ranges from 1 to n and from 0 to m respectively. This convention will be suspended in the case of capital letters or where an alternates summation is indicated.

3. Higher order dissections

Suppose that $(TM)^m$ denotes the mth order (C^∞) tangent bundle of M. If (U, ϕ) is a coordinate chart of M, then any mth order vector field t on M may be expressed locally in the form

$$t = a^a\partial_a + a^{ab}\partial^2_{ab} + \cdots + a^{a_1\cdots a_m}\partial^m_{a_1\cdots a_m} ,$$

where the a's are symmetric. If $TM = (TM)^1$ denotes the tangent bundle of M, then TM_p is the subspace of $(TM)^m_p$ spanned by the set $\{\partial_a\}$ at $p \in U$. We see that TM_p is determined by any coordinate chart at p, but has no natural complement. This may be seen by noting that if the a^a's vanish in a given coordinate system, they do not necessarily do so in a second.

Definition. A dissection of $(TM)^m$ is a choice of a complement M^c_p to TM_p at each $p \in M$ such that $\bigcup_{p\in M} M^c_p = M^c$ is a C^∞ distribution (of $(TM)^m$) on M.

Lemma 1. *If A and B are C^∞ distributions of a given vector bundle structure, with a finite dimensional fiber, on M, and $\dim(A \cap B)_p = r$ for each $p \in M$, then $A \cap B$ is an r-dimensional C^∞ distribution on M.*

Proof. This may be proved, for example, using systems of linear equations, hence we omit it.

Theorem 1. *Suppose that M^c is a dissection of $(TM)^m$. If $p \in M$ and (U, ϕ) is a coordinate chart at p, then there exists an open neighborhood N of p such* *that at each point $q \in N$ there is a coordinate chart $(V, \psi)_q$ such that the partial derivatives of order $2 \leq k \leq m$ at q with respect to the coordinates of $(V, \psi)_q$ span M^c_q. Moreover, these partial derivatives are related in a C^∞ manner, determined by partial derivatives of order $\leq m$, to those of (U, ϕ). Such a chart, $(V, \psi)_q$ is called a spanning chart of M^c at q.*

Proof. If $p \in M$, choose a chart (U, ϕ) at p. If $\langle \partial_{ab} \rangle$ denotes the subspace of $(TM)^2$ spanned by ∂_{ab} at each point of U, and M^c is a dissection of $(TM)^2$, then, since

$$\dim (M^c_p \cap TM_p \oplus \langle \partial_{ab} \rangle_p) = \dim M^c_p + \dim (TM_p \oplus \langle \partial_{ab} \rangle_p) \\ - \dim (M^c_p + TM_p \oplus \langle \partial_{ab} \rangle_p) ,$$

we have

$$\dim (M^c_p \cap TM_p \oplus \langle \partial_{ab} \rangle_p) = 1$$

at each $p \in U$, and thus, by Lemma 1, $M^c \cap TM \oplus \langle \partial_{ab} \rangle$ is a 1-dimensional C^∞ distribution on U for each pair of indices a, b. We may thus choose a C^∞ vector field A_{ab} on some open neighborhood N of p, which spans the intersection and has the property that its projection on the second factor of $TM \oplus \langle \partial_{ab} \rangle$ is ∂_{ab}. Let R^c_{ab} be the symmetric components such that

$$A_{ab} = \partial^2_{ab} + R^c_{ab} \partial_c$$

at each point $q \in N$. The R^c_{ab}'s are thus C^∞ on N. For each $q \in N$ we may take a second coordinate chart $(V, \psi)_q$ such that

$$X^a_r(q) = \delta^a_r , \quad X^a_{rs}(q) = R^a_{rs}(q) , \\ \partial^2_{rs}(q) = \partial^2_{ab}(q) \delta^a_r \delta^b_s + R^a_{rs}(q) \partial_a(q) , \tag{3}$$

by the inverse function theorem. Thus we conclude that the second order partial derivatives with respect to the second system span M^c_q at each $q \in N$, and that they are related in the required C^∞ manner to those of (U, ϕ). Consequently, the theorem is true for $m = 2$.

Proceeding by induction we assume that $m - 1$ is the greatest integer for which the theorem holds. Noting that the transformation law for partial derivatives of order m has the form

$$\partial^m_{r_1, \cdots, r_m} = \sum_{a_i=1}^{n} \sum_{s=2}^{m} P^{a_1, \cdots, a_s}_{r_1, \cdots, r_m} \partial^s_{a_1, \cdots, a_s} + X^a_{r_1, \cdots, r_m} \partial_a , \tag{4}$$

where $P^{a_1, \cdots, a_s}_{r_1, \cdots, r_m}$ consists of a sum of products of partial derivatives of the coordinates of order at most $m - 1$. From dimensional considerations and Lemma 1 we see that if M^c is a dissection of $(TM)^m$, then $M^{c\prime} = M^c \cap (TM)^{m-1}$ is a dissection of $(TM)^{m-1}$. We apply the induction hypothesis and require that the second coordinate chart in (4) be a spanning chart of $M_q^{c\prime}$ at each q in some open neighborhood N' of p. In this case,

$$B_{r_1,\cdots,r_m} = \sum_{a_i=1}^{n} \sum_{s=2}^{m} P_{r_1,\cdots,r_m}^{a_1,\cdots,a_s} \partial_{a_1,\cdots,a_s}^s$$

is an mth order C^∞ vector field on N' determined by $M^{c\prime}$. By Lemma 1,

$$TM \oplus \langle B_{r_1,\cdots,r_m} \rangle \cap M^c$$

is a 1-dimensional C^∞ distribution on N' for each set of indices $r_1, \cdots, r_m$. Take $A_{r_1,\cdots,r_m}$ to the vector field such that it spans the intersection, is C^∞, and its projection on the second factor of $TM \oplus \langle B_{r_1,\cdots,r_m} \rangle$ is $B_{r_1,\cdots,r_m}$ on some open neighborhood N of p. Let $R_{r_1,\cdots,r_m}^a$ be the symmetric C^∞ components such that

$$A_{r_1,\cdots,r_m} = B_{r_1,\cdots,r_m} + R_{r_1,\cdots,r_m}^a \partial_a .$$

If we place the additional condition on the coordinates that $X_{r_1,\cdots,r_m}^a(q) = R_{r_1,\cdots,r_m}^a(q)$, then we see that their partial derivatives of order $2 \leq k \leq m$ span M^c on N and are related in the required C^∞ manner to those of (U, ϕ).

If $(U, \phi)_q$ is a spanning chart of M^c at q, then the partial derivatives of a given order at q with respect to the coordinates of this chart determine a subspace of M_q^c, and this subspace is independent of the spanning chart chosen. For, if $(V, \psi)_q$ is a second spanning chart at q, then we have

$$\partial_{rs}^2(q) = \partial_{ab}^2(q) X_r^a(q) X_s^b(q) + \partial_a(q) X_{rs}^a(q) ,$$

and since $\partial_{rs}^2(q) \in M_q^c$, we see that $X_{rs}^a(q) = 0$ so that

$$\partial_{rs}^2(q) = \partial_{ab}^2(q) X_r^a(q) X_s^b(q) ,$$

and thus $\{\partial_{rs}^2(q)\}$ and $\{\partial_{ab}^2(q)\}$ span the same subspace. Similarly, we see that $X_{r_1,\cdots,r_k}^a(q) = 0$ and that $\partial_{r_1,\cdots,r_k}^k(q)$ and $\partial_{a_1,\cdots,a_k}^k(q)$ span the same subspace for each $1 \leq k \leq m$.

Theorem 2. *Each dissection of the mth order tangent bundle of M determines in a natural manner a unique dissection of the second order tangent bundle of the* $(m-2)$*nd extension* ${}^{m-2}M$ *of M for each integer* $m \geq 2$.

Proof. At each point $p \in M$ choose a coordinate chart which spans M_p^c. The natural coordinate system induced on ${}^{m-2}M$ then determines a complement to $T^{m-2}M$ in $(T^{m-2}M)^2$ at each point of ${}^{m-2}\Pi^{-1}(p)$. That these complements are independent of the charts chosen may be seen as follows. Suppose that $(U, \phi)_p$ and $(V, \psi)_p$ both span M_p^c; then in terms of the natural coordinates they induce on ${}^{m-2}M$ we have

$$\partial_{\rho r \sigma s}^2 = \partial_{\alpha a \beta b}^2 X_{\rho r}^{\alpha a} X_{\sigma s}^{\beta b} + \partial_{\alpha a} X_{\rho r \sigma s}^{\alpha a} .$$

From (2) we see that

$$X^{\alpha a}_{\rho r \sigma s} = \binom{A}{P}\binom{A-P}{\Sigma}[X^{a}_{rs}]^{A-P-\Sigma},$$

and since $\alpha \leq m-2$ only partial derivatives of order $2 \leq k \leq m$ appear, and these are zero at p, we have $X^{\alpha a}_{\rho r \sigma s} = 0$ on ${}^{m-2}\Pi^{-1}(p)$ so that

$$\partial^2_{\rho r \sigma s} = \partial^2_{\alpha a \beta b} X^{\alpha a}_{\rho r} X^{\beta b}_{\sigma s}$$

on ${}^{m-2}\Pi^{-1}(p)$. That this choice of a complement to $T^{m-2}M$ in $(T^{m-2}M)^2$ is a C^∞ distribution may be seen as follows. If $x \in {}^{m-2}M$, $p = {}^{m-2}\Pi(x)$, (U, ϕ) is a chart at p, and in

$$\partial^2{}_{\rho r \sigma t} = \partial^2_{\alpha a \beta b} X^{\alpha a}_{\rho r} X^{\beta b}_{\sigma s} + X^{\alpha a}_{\rho r \sigma s} \partial_{\alpha a}$$

the second chart at each q of some open neighborhood N of p is a spanning chart of Theorem 1, then the partial derivatives $\partial^2_{\rho r \sigma s}$ span the complement and are C^∞ on the open neighborhood ${}^{m-2}\Pi^{-1}(N)$ of x, since each of $X^{\alpha a}_{\rho r}$ and $X^{\alpha a}_{\rho r \sigma s}$ is C^∞ on this neighborhood by (2) and Theorem 1.

We note that this correspondence of a dissection of $(TM)^m$ to a dissection of $(T^{m-2}M)^2$ is also injective, for if $(U, \phi)_p$ and $(V, \psi)_p$ span dissections of $(TM)^m$ at p such that, in the natural coordinates they induce, $\partial^2{}_{\alpha a \beta b}$ and $\partial^2{}_{\rho r \sigma s}$ span the same dissection of $({}^{m-2}M)^2$ on ${}^{m-2}\Pi^{-1}(p)$, then $X^{\alpha a}_{\rho r \sigma s} = 0$ on ${}^{m-2}\Pi^{-1}(p)$ which implies that $X^{a}_{r_1, \cdots, r_k}(p) = 0$ for each $2 \leq k \leq m$ and thus that $(U, \phi)_p$ and $(V, \psi)_p$ span the same dissection of $(TM)^m$. That this correspondence is not surjective may be seen by noting that in the case $m = 3$, $X^{\alpha a}_{1r1s} = 0$ and hence that the subspace spanned by $\{\partial_{1r1s}\}$ must be contained in any dissection of $(T^1M)^2$ determined by a dissection of $(TM)^3$. Thus we have the theorem, using the natural bijection between the dissections of $(T^{m-2}M)^2$ and the sprays on ${}^{m-2}M$ of [1].

Theorem 3. *There exists a natural injection of the set of dissections of $(TM)^m$ into the set of sprays on ${}^{m-2}M$.*

References

[1] W. Ambrose, R. S. Palais & I. M. Singer, *Sprays*, An. Acad. Brasil. Ci. **32** (1960) 163–178.
[2] R. H. Bowman, *On differentiable extensions*, Tensor (N.S.) **21** (1970) 139–150.
[3] K. Yano & S. Kobayashi, *Prolongations of tensor fields and connections to tangent bundles*. I, J. Math. Soc. Japan **18** (1966) 194–210.

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