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A GENERALIZATION OF PARALLELISM IN RIEMANNIAN GEOMETRY; THE C^∞ CASE

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1. Introduction

Let $g: N^p \to M^m$ be a smooth (C^∞ or C^ω) immersion of riemannian manifolds. It is not assumed that the immersion is isometric. A smooth vector bundle map $G: T(N) \to T(M)$ between the tangent bundles will be called a *tangent bundle isometry* (T. B. I.) *along* g provided that the fibers $T(N)(n) = N_n$ are mapped isometrically by G into the fibers $T(M)(g(n)) = M_{g(n)}$. More generally, let E be a euclidean vector bundle over N, F be a euclidean vector bundle over M, and $G: E \to F$; then G will be called a *vector bundle isometry along* g if G maps the fibers $E(n)$ isometrically into the fibers $F(g(n))$. Let ∇ be the covariant derivative on M, and let $G: T(N) \to T(M)$ be a T. B. I. along g: $N^p \to M^m$. The *normal bundle to* G is the $(m - p)$-dimensional vector bundle $G^\perp$ (over N) whose fiber over $n \in N$ is the orthogonal complement, $\perp G(N_n)$, to $G(N_n)$ in $M_{g(n)}$. The *second fundamental form of* G, $II_G: G^\perp \to \operatorname{Hom}(T(N), T(N))$ is a vector bundle map defined in the following manner. If $v \in \perp G(N_n)$ and $x, y \in N_n$, extend y to a vector field Y on N in some neighborhood of n and put

$$\langle II_G(v)x, y\rangle_n = - \langle \nabla_{dg(x)} G(Y), v\rangle_{g(n)} .$$

Since ∇ is a metric connection, the definition is independent of the choice of Y. A T.B.I. G is *parallel* if $\operatorname{trace} \circ II_G: G^\perp \to R$ is the zero function.

Three pieces of evidence in support of this terminology were given in [2].

First, suppose that $\gamma: (a, b) \to M$ is a smoothly immersed curve, and let $d/dt: (a, b) \to T(a, b)$ be the standard unit vector field on (a, b). Then the formula

$$G\left(\frac{d}{dt}(t)\right) = Y(t) , \qquad t \in (a, b) ,$$

establishes a bijective correspondence between the set of T.B.I.s G along γ and the set of unit vector fields Y along γ. Under this correspondence the parallel T.B.I.s are paired with the parallel unit vector fields.

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Secondly, suppose $g: N^p \to M^m$ is a smooth isometric immersion. Then $G = dg$ is a T.B.I. along g. G is parallel if and only if (N, g) is a minimal variety (see [3]). In the particular case of a curve: $g = \gamma, N^1 = (a, b)$; $d\gamma$ is parallel if and only if γ is a geodesic.

Thirdly, the existence and uniqueness theorems for geodesics and parallel vector fields in terms of initial data can be mimiced (locally) in the C^ω case by existence and uniqueness theorems for minimal varieties and parallel T.B.I.s also in terms of appropriate initial data. The theorem for minimal varieties is well known (see [2] and [3]). The theorem for parallel T.B.I.s proved in [2] can be extended to the C^∞ case.

The main objective of the paper is to give a proof of this theorem for the C^∞ case (see § 3 for statement of theorem). Lemma 1 is also of some interest in itself. The paper is essentially self contained; it is completely independent of [2].

2. Some lemmas

Lemma 1. *Let γ: $(a, b) \to M$ be a C^∞ curve immersed in a riemannian manifold (M, ∇) with $0 \in (a, b), D$: $(a, b) \to \mathcal{G}^q(M)$ a q-dimensional C^∞ distribution along γ, and Z: $(a, b) \to T(M)$ any C^∞ vector field along γ. Then each unit vector $X_0 \in \perp D(0) \subset M_{\gamma(0)}$ extends uniquely to a C^∞ vector field X along γ such that, $X(0) = X_0$ and, for all $t \in (a, b)$, $\|X(t)\| = 1, X(t) \perp D(t)$, and*

$$[\nabla_{\dot\gamma(t)} X]^{\perp \mathrm{span}(D(t), X(t))} = [Z(t)]^{\perp \mathrm{span}(D(t), X(t))} .$$

$(v)^V$ means orthogonal projection of v into the subspace V.

Remark. If $Z \equiv 0$ and $q = 0$, then X is the unique parallel vector field along γ which extends X_0. If Z_1 and Z_2 are two C^∞ vector fields along γ which have the same projections perpendicular to D, then the corresponding solutions X_1 and X_2 are equal. If $(a, b) = (-\varepsilon, \varepsilon)$, let J: $(-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ be the map $t \to -t$. The solution along $\gamma \circ J$, which extends $-X_0$ for the data $D \circ J$ and $Z \circ J$, is $-X \circ J$.

Proof of Lemma 1. The conclusion of the lemma is easily seen to be equivalent to the following: there is a unique C^∞ vector field X along γ such that:

$$* \qquad \begin{cases} X(0) = X_0 \text{ and, for all } t \in (a, b), X(t) \perp D(t) \text{ and} \\ [\nabla_{\dot\gamma(t)} X]^{\perp \mathrm{span}(D(t))} = [Z(t)]^{\perp (\mathrm{span}(D(t), X(t))} . \end{cases}$$

Choose a C^∞ frame field $W_1, \cdots, W_m$ along γ so that $D(t) = \mathrm{span}\,(W_1(t), \cdots, W_q(t))$ and $\perp D(t)) = \mathrm{span}\,(W_{q+1}(t), \cdots, W_m(t))$ and $W_{q+1}(0) = X_0$. If $Y_1, \cdots, Y_m$ form a parallel frame field along γ, there is a C^∞ curve $A = (A_{ij})$: $(a, b) \to 0(m, R)$ where $A_{ij}(t) = \langle W_i(t), Y_j(t) \rangle_{\gamma(t)}$ [here $0(m, R)$ is the group of $m \times m$

orthogonal matrices, its Lie algebra $o(m, R)$ is the set of $m \times m$ skew symmetric matrices]. Let A^* be the transpose curve of A and $(A^*)'$ its derivative, then $t \to A(t) \cdot (A^*)'(t)$ is easily seen to be a C^∞ curve in $o(m, R)$. Let $B: (a, b) \to o(m - q, R)$ be the C^∞ curve where $B(t)$ is the lower right hand $(m - q) \times (m - q)$ submatrix of $A(t) \cdot (A^*)'(t)$. The vector field Z can be written $Z = \sum_{i=1}^{m} z_i W_i$ where $z_1, \cdots, z_m$ are C^∞ functions on (a, b). A C^∞ vector field $X = \sum_{i=q+1}^{m} x_i W_i$ along γ is a solution of $*$ if and only if the curve $\bar{X} = (x_{q+1}, \cdots, x_m)$: $(a, b) \to R^{m-q}$ is a C^∞ solution on (a, b) of the first order non-linear system

$$** \qquad \bar{X}' = \bar{Z} - \frac{\langle \bar{Z}, \bar{X} \rangle \bar{X}}{\langle \bar{X}, \bar{X} \rangle} - B(\bar{X})$$

subject to the initial data

$$\bar{X}(0) = (1, 0, \cdots, 0) .$$

Here $\langle \, , \rangle$ means euclidean inner product and $\bar{Z} = (z_{q+1}, \cdots, z_m)$: $(a, b) \to R^{m-q}$, while B is considered as acting on R^{m-q}. Solutions of $**$ have constant euclidean length since $B(t) \in o(m - q, R)$. In particular, the theory of ordinary differential equations shows that there is a unique C^∞ solution $\bar{X} = (x_{q+1}, \cdots, x_m)$ defined on (a, b) for the system $**$ satisfying the initial data. This completes the proof of the lemma.

The theory of ordinary differential equations also says that if the data in Lemma 1 depends differentiably on a parameter $\tilde{n}$ which runs over a differentiable manifold N^{p-1}, then the solutions also depend differentiably on $\tilde{n}$. This information is incorporated into Lemma 2 for later use.

Let $\varepsilon: N^{p-1} \to R$ be a positive C^∞ function. Then

$$V = \{(\tilde{n}, t) \,|\, |t| < \varepsilon(\tilde{n})\}$$

is an open submanifold of $N^{p-1} \times R$. Let $i: N^{p-1} \to V$ be the inclusion and denote by ∂ the C^∞ vector field defined on V by $\partial(n) = \partial(\tilde{n}, t) = (0, \partial/\partial t|_t)$; ∂ is the tangent vector at time t to the curve $t \to (\tilde{n}, t)$.

Lemma 2. *Suppose $f: V \to M$ is a C^∞ map into a riemannian manifold (M, ∇), $D: V \to \mathscr{G}^q(M)$ is a C^∞ q-dimensional distribution along $f, Z: V \to T(M)$ is a C^∞ vector field along f, and $X_0: N^{p-1} \to T(M)$ is a C^∞ unit vector field along $f \circ i$ orthogonal to $D \circ i$. Then there is a unique C^∞ vector field along f such that $X \circ i = X_0$ and, for all $n \in V$, $\|X(n)\| = 1, X(n) \perp D(n)$,*

$$[\nabla_{df(\partial(n))} X]^{\perp \operatorname{span}(D(n), X(n))} = [Z(n)]^{\perp \operatorname{span}(D(n), X(n))} .$$

To prove existence in Lemma 2, it is only necessary to note that for $\tilde{n} \in N^{p-1}$, the curves $\gamma_{\tilde{n}}: (-\varepsilon(\tilde{n}), \varepsilon(\tilde{n})) \to M$ where $\gamma_{\tilde{n}}(t) = f(\tilde{n}, t)$ and the data $D_{\tilde{n}} =$

$D\circ(\tilde{n},\ ), Z_{\tilde{n}} = Z\circ(\tilde{n},\ ), X_{0,\tilde{n}} = X_0(\tilde{n})$, fulfill the hypotheses of Lemma 1 and depend differentiably on $\tilde{n}$. Thus the solutions $X_{\tilde{n}}$ ($\tilde{n} \in N^{p-1}$) fit together to form a C^∞ solution X, $X(n) = X(\tilde{n}, t) = X_{\tilde{n}}(t)$, for Lemma 2. Uniqueness follows from the fact that any solution Y for Lemma 2 restricts to solutions $Y_{\tilde{n}} = Y\circ(\tilde{n},\ )$, $\tilde{n} \in N^{p-1}$, for Lemma 1 where uniqueness is known.

3. The theorem

If $i: X \to Y$, and E is a vector bundle over Y, then $i_*: i^*E \to$ is the induced map of the induced bundle. A distribution on N and the subbundle of $T(N)$ which it defines will be denoted by the same letter.

Theorem. *Let $g: N^p \to M^m$ be an (not necessarily isometric) immersion of riemannian manifolds, H be a $(p-1)$-dimensional distribution on N^p, and (N^{p-1}, i) be a homeomorphically embedded integral manifold of H. Suppose there is given as initial data*

(1) *$G^{p-1}: H \to T(M)$, a vector bundle isometry along g,*

(2) *$G^p: i^*T(N^p) \to T(M)$ a vector bundle isometry along $g\circ i$,*

where it is assumed that G^{p-1} and G^p agree where they are both defined:

$$G^p|_{i^*H} = G^{p-1}\circ i_*: i^*H \to T(M) \ .$$

Then, assuming that the data is all C^∞, there is a neighborhood U of N^{p-1} in N^p and a unique parallel T.B.I. *$G: T(U) \to T(M)$ which extends the initial data:*

$$G|_H = G^{p-1}: H \to T(M) \quad \textit{and} \quad G\circ i_* = G^p: i^*T(N^p) \to T(M) \ .$$

Proof of the theorem. If $\tilde{n} \in N^{p-1}$, then there is a special coordinate system $u = (u_1, \cdots u_p): \mathcal{U} \to R^p$ about $\tilde{n}$ in N^p which satisfies the following conditions.

a. The slices $u_1 =$ constant, $\cdots$, $u_{p-1} =$ constant are integral manifolds of the distribution $\perp H$.

b. $\mathcal{U} \cap N^{p-1}$ is the slice $u_p = 0$.

c. For $n \in \mathcal{U}$, $|u_p(n)|$ is the arc length measured along the integral manifold of $\perp H$ from $u(u_1(n), \cdots, u_{p-1}(n), 0) \in N^{p-1}$ to n.

d. The system is centered at $\tilde{n}$: $u_i(\tilde{n}) = 0, i = 1, \cdots, p$.

e. The system has breadth δ: $p \in u(\mathcal{U})$ if and only if $r_i(p) < \delta, i = 1, \cdots, p$. ($r_1, \cdots, r_p$ are the coordinates on R^p).

Such a coordinate system may be obtained by choosing a unit vector field which spans $\perp H$ in a neighborhood of $\tilde{n}$ in N^p. By [1, p. 89], there is a coordinate system $v = (v_1, \cdots, v_p): \mathcal{U} \to R^p$ in which this vector field is $\partial/\partial v_p$. This system satisfies *a*. Since N^{p-1} has the relative topology and the integral manifolds of $\perp H$ cross N^{p-1} transversally it may be assumed that $v(N^{p-1} \cap \mathcal{U})$ appears as the graph of a C^∞ function k of the first $p-1$ coordinates $r_1, \cdots, r_{p-1}$ in $v(\mathcal{U}) \subset R^p$. The new coordinate system $u = (u_1, \cdots, u_p): \mathcal{U} \to R^p$ where

$u_i = v_i, i = 1, \cdots, p-1$, and $u_p = v_p - k\circ(v_1, \cdots, v_{p-1})$ satisfies a, b and c. It is easily adjusted to satisfy d and e also.

Let $\{(\mathscr{U}^\alpha, u^\alpha) \mid \alpha \in J\}$ be a locally finite cover of N^{p-1} by such special coordinate systems, and the breadth of $(\mathscr{U}^\alpha, u^\alpha)$ be δ_α. For $\tilde{n} \in N^{p-1}$ let $\mathscr{I}(\tilde{n})$ be the maximal integral manifold of $\perp H$ through $\tilde{n}$. If $n \in \mathscr{I}(\tilde{n})$, let $L(\tilde{n}, n)$ be the arc length measured along $\mathscr{I}(\tilde{n})$ between $\tilde{n}$ and n. Choose a positive C^∞ function $\delta\colon N^{p-1} \to R$ such that for each $\tilde{n} \in N^{p-1}$, $\{n \in \mathscr{I}(\tilde{n}) \mid L(\tilde{n}, n) < 2\delta(\tilde{n})\} \cap N^{p-1} = \{\tilde{n}\}$. Finally, choose a positive C^∞ function $\varepsilon\colon N^{p-1} \to R$ so that $\varepsilon(\tilde{n}) < \min(\delta(\tilde{n}), \delta_{\alpha_1}, \cdots, \delta_{\alpha_k})$ where $\mathscr{U}^{\alpha_1}, \cdots, \mathscr{U}^{\alpha_k} (\alpha_j \in J)$ are the coordinate neighborhoods which contain $\tilde{n}$. Then

$$U = \bigcup_{\tilde{n} \in N^{p-1}} \{n \in \mathscr{I}(\tilde{n}) \mid L(\tilde{n}, n) < \varepsilon(\tilde{n})\}$$

is a neighborhood of N^{p-1} in N^p.

If $\tilde{n} \in N^{p-1}$, choose $(\mathscr{U}^\alpha, u^\alpha)$ so that $\tilde{n} \in \mathscr{U}^\alpha$ and put $V^\alpha = U \cap \mathscr{U}^\alpha$. Choose C^∞ orthonormal frames $F_1, \cdots, F_p$ on V^α adapted to $H|_{V^\alpha}$ [thus span $(F_1(n), \cdots, F_{p-1}(n)) = H(n)$] and such that $F_p(n) = \partial/\partial u_p^\alpha(n)$]thus span $F_p(n) = \perp H(n))$ for all $n \in V^\alpha$.

The initial data (1) determines both the vector field Z defined along $g|_{V^\alpha}$ by

$$Z(n) = -\sum_{i=1}^{p-1} \nabla_{dg(F_i(n))} G^{p-1}(F_i) \ , \qquad n \in V^\alpha \ ,$$

and the $(p-1)$-dimensional distribution D defined along $g|_{V^\alpha}$ by

$$D(n) = G^{p-1}(H(n)) \ , \qquad n \in V^\alpha \ .$$

Also, the initial data (2) determines the vector field X_0 defined along $g\circ i|_{N^{p-1}\cap V^\alpha}$ by

$$X_0(n) = G^p(F_p(n)) \ , \qquad n \in N^{p-1} \cap V^\alpha \ ;$$

X_0 is orthogonal to $D\circ i|_{N^{p-1}\cap V^\alpha}$.

The formulas:

$$\left.\begin{aligned} G(F_i(n)) &= G^{p-1}(F_i(n)) \ , \quad i = 1, \cdots, p-1 \ , \\ G(F_p(n)) &= X(n) \ , \end{aligned}\right\} n \in V^\alpha \ ,$$

place the set of parallel T.B.I.s G along $g|_{V^\alpha}$, which extend the initial data (1) and (2), in one-to-one correspondence with the set of C^∞ vector fields X along $g|_{V^\alpha}$ which satisfy: $X_0 = X\circ i|_{N^{p-1}\cap V^\alpha}$ and, for all $n \in V^\alpha$, $\|X(n)\| = 1$, $X(n) \perp D(n)$,

$$[\nabla_{dg((\partial/\partial u_p^\alpha)(n))} X]^{\perp \mathrm{span}(D(n), X(n))} = [Z(n)]^{\perp \mathrm{span}(D(n), X(n))} \ .$$

By Lemma 2, there is exactly one parallel T.B.I. G^{α} along $g|_{V^{\alpha}}$ which extends the initial data.

In order to show that the locally defined G^{α} patch together into a T.B.I. G along $g|_{U}$ (where $G(n) = G^{\alpha}(n)$ if $n \in V^{\alpha}$) it is enough to show that on overlaps $V^{\alpha} \cap V^{\beta}$, $G^{\alpha}|_{V^{\alpha} \cap V^{\beta}} = G^{\beta}|_{V^{\alpha} \cap V^{\beta}}$. Lemma 2 applies to $V^{\alpha} \cap V^{\beta}$ (with the coordinates u^{α}) and yields a unique parallel T.B.I. $G^{\alpha,\beta}$ along $g|_{V^{\alpha} \cap V^{\beta}}$ which extends the initial data there. Thus $G^{\alpha}|_{V^{\alpha} \cap V^{\beta}} = G^{\alpha,\beta} = G^{\beta}|_{V^{\alpha} \cap V^{\beta}}$. The resulting T.B.I. G is both parallel and a unique extension of the initial data since it locally has these properties.

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