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On the C^{α}-convergence of the Solution of the Chern-Ricci Flow on Elliptic Surfaces

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Abstract. We will study the Chern-Ricci flow on non-Kähler properly elliptic surfaces. These surfaces are compact complex surfaces whose first Betti number is odd, Kodaira dimension is equal to 1 and admit an elliptic fibration to a smooth compact curve. We will show that a solution of the Chern-Ricci flow converges in C^{α}-topology on these elliptic surfaces by choosing a special initial metric.

1. Introduction

The Chern-Ricci flow is an analogue of the Ricci flow and the Kähler Ricci flow for Hermitian metrics on compact complex manifolds. It was firstly investigated by Gill [5] and he showed that a suitably normalized solution of the flow converges to Hermitian metrics with vanishing Chern-Ricci form in the C^{∞}-topology on a compact Hermitian manifold with its first Bott-Chern class is equal to zero. After that, Tosatti and Weinkove started to study the Chern-Ricci flow on some complex surfaces such as properly elliptic surfaces, Hopf surfaces and Inoue surfaces [10]. In this paper, we especially focus on the convergence of a solution of the normalized Chern-Ricci flow on minimal non-Kähler properly elliptic surfaces.

It has been investigated the behavior of the Kähler-Ricci flow in the case of a product elliptic surface $M = E \times S$, where E is an elliptic curve and S is a Riemann surface with $c_1(S) < 0$ by Song and Weinkove [8]. In this case, the solution of the normalized Kähler-Ricci flow on $M = E \times S$ converges to a Kähler-Einstein metric on S in C^{α}-topology for any $0 < \alpha < 1$. In the case of (unnormalized) Chern-Ricci flow on minimal non-Kähler properly elliptic surfaces, a smooth solution of the flow divided by t converges to an orbifold Kähler-Einstein metric smoothly as t goes to infinity [10]. And, it has been shown that the solution of the normalized Chern-Ricci flow converges to a Kähler-Einstein metric exponentially fast in C^0-topology on minimal non-Kähler properly elliptic surfaces [11].

Our aim is to show that the convergence is possible in C^{α}-topology with the initial metric in the $\partial\bar{\partial}$-class of the Vaisman metric [12]. This study is the one which was stimulated by the investigation of the normalized Chern-Ricci flow on Inoue surfaces [4].

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The normalized Chern-Ricci flow is given by

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t)\,, \\ \omega(t)|_{t=0} = \omega_0\,, \end{cases}$$

where $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is a starting Hermitian metric and

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det(g)$$

is the Chern-Ricci form of ω.

A non-Kähler properly elliptic surface M is a compact complex surface with its first Betti number $b_1(M) =$ odd and the Kodaira dimension $\mathrm{Kod}(M) = 1$ which admits an elliptic fibration $\pi : M \to S$ to a smooth compact curve S. The Kodaira-Enriques classification tells us that properly elliptic surfaces are the only one case for minimal non-Kähler complex surfaces with $\mathrm{Kod} = 1$ (cf. [1, p. 244]).

We assume that M is minimal, that is, there is no (-1)-curve on M. It has been shown that the universal cover of M is $\mathbf{C} \times H$ [6, Theorem 28], where H is the upper half plane. Also, it is known that there is a finite unramified covering $p : M' \to M$ which is a minimal properly elliptic surface $\pi' : M' \to S'$ and π' is an elliptic fiber bundle over a compact Riemann surface S' of genus at least 2, with fiber an elliptic curve E (cf. [3, Lemma 1, 2]). So we firstly assume that $\pi : M \to S$ is an elliptic bundle with fiber E with $g(S) \geq 2$, with M minimal, non-Kähler and $\mathrm{Kod}(M) = 1$. It will be more convenient for us to work with $\mathbf{C}^* \times H$, where $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$. We define

$$h : \mathbf{C} \times H \to \mathbf{C}^* \times H, \quad h(z, z') = (e^{-\frac{z}{2}}, z')\,,$$

which is a holomorphic covering map. We will write (z_1, z_2) for the coordinates on $\mathbf{C}^* \times H$ and $z_i = x_i + \sqrt{-1}y_i$, $x_i, y_i \in \mathbf{R}$ for $i = 1, 2$, which means that we have $y_2 > 0$.

It has been shown that there exists a discrete subgroup $\Gamma \subset \mathrm{SL}(2, \mathbf{R})$ with $H/\Gamma = S$, together with $\lambda \in \mathbf{C}^*$ with $|\lambda| \neq 1$ and $\mathbf{C}^*/\langle\lambda\rangle = E$ and with a character $\chi : \Gamma \to \mathbf{C}^*$ such that M is biholomorphic to the quotient of $\mathbf{C}^* \times H$ by the $\Gamma \times \mathbf{Z}$-action defined by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, n\right) \cdot (z_1, z_2) = \left((cz_2 + d) \cdot z_1 \cdot \lambda^n \cdot \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \frac{az_2 + b}{cz_2 + d}\right),$$

and the map $\pi : M \to S$ is induced by the projection $\mathbf{C}^* \times H \to H$ (cf. [2, Proposition 2], [13, Theorem 7.4]). We define two forms on $\mathbf{C}^* \times H$ below:

$$\alpha := \frac{\sqrt{-1}}{2y_2^2}dz_2 \wedge d\bar{z}_2\,, \quad \gamma := \sqrt{-1}\left(-\frac{2}{z_1}dz_1 + \frac{\sqrt{-1}}{y_2}dz_2\right) \wedge \left(-\frac{2}{\bar{z}_1}d\bar{z}_1 - \frac{\sqrt{-1}}{y_2}d\bar{z}_2\right).$$

We will denote $\omega_\infty := \alpha$. Since we can check that the forms on $\mathbf{C}^* \times H$; $\frac{\sqrt{-1}}{y_2^2}dz_2 \wedge d\bar{z}_2$ and $-\frac{2}{z_1}dz_1 + \frac{\sqrt{-1}}{y_2}dz_2$ are $\Gamma \times \mathbf{Z}$-invariant, these forms α and γ are invariant under the $\Gamma \times \mathbf{Z}$-action. Hence they descend to M and we define a Hermitian metric called a Vaisman metric

$\omega_V = 2\alpha + \gamma$, which is Gauduchon, i.e., ω_V is a Hermitian metric satisfying $\partial\bar{\partial}\omega_V = 0$. Note that we may work in a single compact fundamental domain for M in $\mathbf{C}^* \times H$ using z_1, z_2 as local coordinates and we may assume that z_1, z_2 are uniformly bounded and that y_2 is uniformly bounded below away from zero.

Our main result is as follows:

THEOREM 1.1. *Let M be a minimal non-Kähler properly elliptic surface and let $\omega(t)$ be the solution of the normalized Chern-Ricci flow starting at a Hermitian metric of the form*

$$\omega_0 = \omega_V + \sqrt{-1}\partial\bar{\partial}\psi > 0\,.$$

Then the metrics $\omega(t)$ are uniformly bounded in the C^1-topology, and as $t \to \infty$,

$$\omega(t) \to \omega_\infty\,,$$

in the C^α-topology, for every $0 < \alpha < 1$.

2. PROOF OF THEOREM 1.1

We define reference metrics

$$\tilde{\omega} := e^{-t}\omega_V + (1 - e^{-t})\alpha = e^{-t}\gamma + (1 + e^{-t})\alpha\,,$$

which are Hermitian for any $t \geq 0$.

We define a volume form Ω by

$$\Omega = 2\alpha \wedge \gamma\,.$$

Then we have

$$\mathrm{Ric}(\Omega) = -\sqrt{-1}\partial\bar{\partial}\log\Omega = -\alpha \in c_1^{BC}(M)\,.$$

It follows that the normalized Chern-Ricci flow is equivalent to the equation

$$(\dagger) \qquad \frac{\partial}{\partial t}\varphi = \log\frac{e^t(\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} - \varphi\,, \quad \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi > 0\,, \quad \varphi(0) = \psi\,.$$

If $\varphi = \varphi(t)$ solves $(\dagger)$, then $\omega(t) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ is the solution of the normalized Chern-Ricci flow. On the other hand, given a solution $\omega(t)$ of the normalized Chern-Ricci flow, we can find a solution $\varphi = \varphi(t)$ of the equation $(\dagger)$ with $\omega(t) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$.

Here we let $\varphi = \varphi(t)$ solves the equation above and we will write

$$\omega = \omega(t) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi\,.$$

We have the following lemma (cf. [4, Lemma 2.2], [11, Lemma 3.4]):

LEMMA 2.1. *There exists a uniform constant $C > 0$ such that for all $t \geq 0$,*

(1) $|\varphi| \leq C(1 + t)e^{-t}$
(2) $|\dot{\varphi}| \leq C$

(3) $C^{-1}\tilde{\omega}^2 \le \omega^2 \le C\tilde{\omega}^2$.

We can show the desired result by computing directly as in [4] and the following estimates play the most important role in our argument.

LEMMA 2.2. *There exists a uniform constant $C > 0$ such that*

(1) $|\tilde{T}|_{\tilde{g}} \le C$.

(2) $|\bar{\partial}\tilde{T}|_{\tilde{g}} + |\tilde{\nabla}\tilde{T}|_{\tilde{g}} + |\tilde{R}|_{\tilde{g}} \le C$.

(3) $|\tilde{\nabla}\tilde{R}|_{\tilde{g}} + |\overline{\tilde{\nabla}}\tilde{\nabla}\tilde{T}|_{\tilde{g}} + |\tilde{\nabla}\overline{\tilde{\nabla}}\tilde{T}|_{\tilde{g}} \le C$,

where $\tilde{T}$ is the torsion tensor of $\tilde{g}$, written locally as $\tilde{T}^k_{ij} = \tilde{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ji}$, $\tilde{T}_{ij\bar{l}} = \tilde{T}^k_{ij} g_{k\bar{l}}$, $\tilde{R}$ is the Chern curvature tensor of $\tilde{g}$, locally written as $\tilde{R}_{i\bar{j}k}{}^{l} = -\partial_{\bar{j}}\tilde{\Gamma}^l_{ik}$ and $\tilde{\nabla}$ is the Chern connection associated to $\tilde{g}$.

PROOF. Using the local coordinates (z_1, z_2) as in the previous section, we will write $\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$, $\tilde{\omega} = \sqrt{-1}\tilde{g}_{i\bar{j}}dz^i \wedge d\bar{z}^j$ and we have

$$\tilde{g}_{1\bar{1}} = \frac{4}{|z_1|^2}e^{-t}, \quad \tilde{g}_{1\bar{2}} = \sqrt{-1}\frac{2}{z_1 y_2}e^{-t}, \quad \tilde{g}_{2\bar{1}} = -\sqrt{-1}\frac{2}{\bar{z}_1 y_2}e^{-t},$$

$$\tilde{g}_{2\bar{2}} = \frac{1+e^{-t}}{2y_2^2} + \frac{e^{-t}}{y_2^2}, \quad \det\tilde{g} = \frac{2e^{-t}(1+e^{-t})}{|z_1|^2 y_2^2},$$

and

$$\tilde{g}^{1\bar{1}} = \frac{e^t(1+3e^{-t})|z_1|^2}{4(1+e^{-t})}, \quad \tilde{g}^{1\bar{2}} = \sqrt{-1}\frac{z_1 y_2}{1+e^{-t}}, \quad \tilde{g}^{2\bar{1}} = -\sqrt{-1}\frac{\bar{z}_1 y_2}{1+e^{-t}}, \quad \tilde{g}^{2\bar{2}} = \frac{2y_2^2}{1+e^{-t}}.$$

The Christoffel symbols $\tilde{\Gamma}^k_{ij}$ of the Chern connection of $\tilde{g}$ are as follows:

$$\tilde{\Gamma}^2_{11} = \tilde{\Gamma}^1_{12} = \tilde{\Gamma}^2_{12} = 0,$$

$$\tilde{\Gamma}^1_{21} = -\sqrt{-1}\frac{1}{y_2(1+e^t)}, \quad \tilde{\Gamma}^2_{21} = -\frac{2}{z_1(1+e^t)},$$

$$\tilde{\Gamma}^1_{22} = -\frac{z_1(1+3e^{-t})}{4y_2^2(1+e^{-t})}, \quad \tilde{\Gamma}^1_{11} = -\frac{1+3e^{-t}}{z_1(1+e^{-t})} + \frac{2}{z_1(1+e^t)},$$

and

$$\tilde{\Gamma}^2_{22} = -\sqrt{-1}\frac{1}{y_2(1+e^t)} + \sqrt{-1}\frac{1+3e^{-t}}{y_2(1+e^{-t})}.$$

Hence the torsion tensor $\tilde{T}$ of $\tilde{g}$ can be given by

$$\tilde{T}^1_{21} = -\sqrt{-1}\frac{1}{y_2(1+e^t)}, \quad \tilde{T}^2_{12} = \frac{2}{z_1(1+e^t)}.$$

The Chern curvature tensor $\tilde{R}$ of $\tilde{g}$ can be computed in the following way:

$$\tilde{R}_{2\bar{2}1}{}^{1} = \frac{1}{2y_2^2(1+e^t)}\,, \quad \tilde{R}_{2\bar{2}2}{}^{2} = -\frac{2+e^t}{2y_2^2(1+e^t)}\,,$$

$$\tilde{R}_{2\bar{2}2}{}^{1} = -\sqrt{-1}\frac{z_1(1+3e^{-t})}{4y_2^3(1+e^{-t})}$$

and other components of the tensor $\tilde{R}$

$$\tilde{R}_{2\bar{1}2}{}^{1}, \tilde{R}_{2\bar{1}1}{}^{1}, \tilde{R}_{1\bar{2}1}{}^{2}, \tilde{R}_{1\bar{1}1}{}^{2}, \tilde{R}_{1\bar{1}1}{}^{1}, \tilde{R}_{2\bar{1}2}{}^{2}, \tilde{R}_{1\bar{2}1}{}^{1},$$
$$\tilde{R}_{1\bar{1}2}{}^{2}, \tilde{R}_{1\bar{2}2}{}^{2}, \tilde{R}_{2\bar{1}1}{}^{2}, \tilde{R}_{2\bar{2}1}{}^{2}, \tilde{R}_{1\bar{1}2}{}^{1}, \tilde{R}_{1\bar{2}2}{}^{1}$$

are all equal to zero.

We compute

$$\partial_{\bar{1}}\tilde{T}_{12}^2 = \partial_{\bar{2}}\tilde{T}_{12}^2 = 0\,, \quad \partial_{\bar{1}}\tilde{T}_{21}^1 = 0\,, \quad \partial_{\bar{2}}\tilde{T}_{21}^1 = -\frac{1}{2y_2^2(1+e^t)}$$

and

$$\tilde{\nabla}_1\tilde{T}_{12}^2 = \partial_1\tilde{T}_{12}^2 - \tilde{\Gamma}_{11}^1\tilde{T}_{12}^2 = O(e^{-t})\,, \quad \tilde{\nabla}_1\tilde{T}_{21}^1 = 0,$$

$$\tilde{\nabla}_2\tilde{T}_{21}^1 = \partial_2\tilde{T}_{21}^1 - \tilde{\Gamma}_{22}^2\tilde{T}_{21}^1 + \tilde{\Gamma}_{22}^1\tilde{T}_{21}^2 = O(e^{-t})\,, \quad \tilde{\nabla}_2\tilde{T}_{12}^2 = -\tilde{\Gamma}_{21}^1\tilde{T}_{12}^2 + \tilde{\Gamma}_{21}^2\tilde{T}_{12}^1 = O(e^{-2t})\,.$$

By direct calculation, we have

$$\tilde{\nabla}_1\tilde{\nabla}_{\bar{2}}\tilde{T}_{21}^1 = 0\,, \quad \tilde{\nabla}_2\tilde{\nabla}_{\bar{2}}\tilde{T}_{21}^1 = \partial_2\partial_{\bar{2}}\tilde{T}_{21}^1 + (\tilde{\Gamma}_{22}^1 - \tilde{\Gamma}_{22}^2)\partial_{\bar{2}}\tilde{T}_{21}^1 = O(e^{-t})\,,$$

$$\tilde{\nabla}_{\bar{2}}\tilde{\nabla}_{\bar{1}}\tilde{T}_{21}^1 = -\overline{\tilde{\Gamma}_{21}^2}\partial_{\bar{2}}\tilde{T}_{21}^1 = O(e^{-2t})\,, \quad \tilde{\nabla}_{\bar{2}}\tilde{\nabla}_{\bar{2}}\tilde{T}_{21}^1 = \partial_{\bar{2}}\partial_{\bar{2}}\tilde{T}_{21}^1 - \overline{\tilde{\Gamma}_{22}^2}\partial_{\bar{2}}\tilde{T}_{21}^1 = O(e^{-t})$$

and

$$\tilde{\nabla}_2\tilde{\nabla}_{\bar{2}}\tilde{T}_{12}^2 = \tilde{\Gamma}_{21}^2\partial_{\bar{2}}\tilde{T}_{12}^1 = O(e^{-2t})\,, \quad \tilde{\nabla}_2\tilde{\nabla}_{\bar{1}}\tilde{T}_{12}^2 = 0\,, \quad \tilde{\nabla}_1\tilde{\nabla}_{\bar{1}}\tilde{T}_{12}^2 = 0\,,$$

$$\tilde{\nabla}_1\tilde{\nabla}_{\bar{2}}\tilde{T}_{12}^2 = \partial_1\partial_{\bar{2}}\tilde{T}_{12}^2 - \tilde{\Gamma}_{11}^1\partial_{\bar{2}}\tilde{T}_{12}^2 + \tilde{\Gamma}_{11}^2\partial_{\bar{2}}\tilde{T}_{12}^1 = 0\,.$$

For any $i, j = 1, 2$, we have

$$\tilde{\nabla}_i\tilde{\nabla}_{\bar{1}}\tilde{T}_{21}^1 = 0\,, \quad \tilde{\nabla}_{\bar{1}}\tilde{\nabla}_{\bar{j}}\tilde{T}_{21}^1 = 0$$

and

$$\tilde{\nabla}_{\bar{i}}\tilde{\nabla}_{\bar{j}}\tilde{T}_{12}^2 = 0\,.$$

We can also check that

$$\tilde{\nabla}_2\tilde{R}_{2\bar{2}2}{}^{2}, \tilde{\nabla}_1\tilde{R}_{2\bar{2}2}{}^{1}, \tilde{\nabla}_2\tilde{R}_{2\bar{2}2}{}^{1}$$

are of order $O(1)$,

$$\tilde{\nabla}_2 \tilde{R}_{2\bar{2}1}{}^{1}$$

is of order $O(e^{-t})$, and other components

$$\tilde{\nabla}_1 \tilde{R}_{2\bar{1}2}{}^{2}, \tilde{\nabla}_2 \tilde{R}_{2\bar{1}2}{}^{2}, \tilde{\nabla}_1 \tilde{R}_{2\bar{2}2}{}^{2}, \tilde{\nabla}_1 \tilde{R}_{1\bar{1}1}{}^{1}, \tilde{\nabla}_2 \tilde{R}_{1\bar{1}1}{}^{1}, \tilde{\nabla}_1 \tilde{R}_{1\bar{1}1}{}^{2},$$
$$\tilde{\nabla}_2 \tilde{R}_{1\bar{1}1}{}^{2}, \tilde{\nabla}_1 \tilde{R}_{1\bar{2}1}{}^{2}, \tilde{\nabla}_2 \tilde{R}_{1\bar{2}1}{}^{2},$$
$$\tilde{\nabla}_1 \tilde{R}_{2\bar{1}1}{}^{1}, \tilde{\nabla}_2 \tilde{R}_{2\bar{1}1}{}^{1}, \tilde{\nabla}_1 \tilde{R}_{2\bar{2}1}{}^{1}, \tilde{\nabla}_1 \tilde{R}_{2\bar{1}2}{}^{1}, \tilde{\nabla}_2 \tilde{R}_{2\bar{1}2}{}^{1}$$

are all equal to zero. □

Using the estimates in Lemma 2.1, we can obtain the following estimates (cf. [4, Theorem 2.4], [11, Section 5, 6, 7]):

LEMMA 2.3. *For $\varphi = \varphi(t)$ solving (†) on M, the estimates below hold.*

(1) *There exists a uniform constant $C > 0$ such that*

$$\frac{1}{C}\tilde{\omega} \le \omega(t) \le C\tilde{\omega}.$$

(2) *There exists a uniform constant $C > 0$ such that the Chern scalar curvature $Scal_{g(t)}$ of $g(t)$ satisfies the bound*

$$-C \le Scal_{g(t)} \le C.$$

(3) *For any η, σ with $0 < \eta < \frac{1}{2}$ and $0 < \sigma < \frac{1}{2}$, there exists a constant $C_{\eta,\sigma} > 0$ such that*

$$-C_{\eta,\sigma} e^{-\eta t} \le \dot{\varphi}(t) \le C_{\eta,\sigma} e^{-\sigma t}.$$

(4) *For any ε with $0 < \varepsilon < \frac{1}{4}$, there exists a constant $C_\varepsilon > 0$ such that*

$$(1 - C_\varepsilon e^{-\varepsilon t})\tilde{\omega} \le \omega(t) \le (1 + C_\varepsilon e^{-\varepsilon t})\tilde{\omega}.$$

Denote by $\Psi_{ij}^k := \Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$, the difference of the Christoffel symbols of g and $\tilde{g}$, which satisfies $\mathcal{S} := |\tilde{\nabla} g|_g^2 = |\Psi|_g^2$. The quantity $|\tilde{\nabla} g|_g^2$ is equivalent to $|\tilde{\nabla} g|_{\tilde{g}}^2$ from the result (1) in Lemma 2.2. Note that we will write locally $\alpha = \sqrt{-1}\alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

Then we compute the evolution of $\mathcal{S}$ (cf. [7]):

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{S} = {} & \mathcal{S} - |\overline{\nabla}\Psi|_g^2 - |\nabla\Psi|_g^2 \\ & + g^{i\bar{j}} g^{r\bar{s}} g^{a\bar{b}} \left(\nabla_r \overline{T_{bj\bar{a}}} + \nabla_{\bar{b}} T_{ar\bar{j}}\right) \Psi_{ip}^k \overline{\Psi_{sq}^l} g^{p\bar{q}} g_{k\bar{l}} \\ & + g^{i\bar{j}} g^{r\bar{s}} g^{a\bar{b}} \left(\nabla_r \overline{T_{bj\bar{a}}} + \nabla_{\bar{b}} T_{ar\bar{j}}\right) \Psi_{pi}^k \overline{\Psi_{qs}^l} g^{p\bar{q}} g_{k\bar{l}}\end{aligned}$$

$$
\begin{aligned}
&-g^{i\bar{j}}g^{r\bar{s}}g^{a\bar{b}}\Big(\nabla_k\overline{T_{bs\bar{a}}}+\nabla_{\bar{b}}T_{ak\bar{s}}\Big)\Psi_{ip}^{k}\overline{\Psi_{jq}^{m}}g^{p\bar{q}}g_{r\bar{m}}\\
&-2\mathrm{Re}\Big[[g^{r\bar{s}}(\nabla_i\nabla_p\overline{T_{sl\bar{r}}}+\nabla_i\nabla_{\bar{s}}T_{rp\bar{l}}\\
&-T_{ir}^{a}R_{a\bar{s}p\bar{l}}+g_{k\bar{l}}\nabla_r\tilde{R}_{i\bar{s}p}{}^{k})+g_{k\bar{l}}\tilde{g}^{k\bar{s}}\tilde{\nabla}_i\alpha_{p\bar{s}}]\overline{\Psi_{jq}^{l}}g^{i\bar{j}}g^{p\bar{q}}\Big],
\end{aligned}
$$

where ∇, Δ are the Chern connection and the Laplacian with respect to g, and in this computation, we used especially that $\frac{\partial}{\partial t}\tilde{g}_{k\bar{l}}=-\tilde{g}_{k\bar{l}}+\alpha_{k\bar{l}}$ and $\frac{\partial}{\partial t}\tilde{\Gamma}_{ip}^{k}=\tilde{g}^{k\bar{\delta}}\tilde{\nabla}_i\alpha_{p\bar{\delta}}$.

With using $\tilde{T}_{ij\bar{k}}=\tilde{T}_{ij}^{k}\tilde{g}_{k\bar{l}}=T_{ij}^{k}g_{k\bar{l}}=T_{ij\bar{k}}$, we can compute as follows:

$$
\nabla_{\bar{b}}T_{ar\bar{j}}=\tilde{\nabla}_{\bar{b}}\tilde{T}_{ar\bar{j}}-\overline{\Psi_{bj}^{s}}\tilde{T}_{ar\bar{s}}\,,
$$

$$
\begin{aligned}
\nabla_i\nabla_{\bar{s}}T_{rp}^{k}=g^{\bar{l}k}\Big(&\tilde{\nabla}_i\tilde{\nabla}_{\bar{s}}\tilde{T}_{rp\bar{l}}-\Psi_{ir}^{a}\tilde{\nabla}_{\bar{s}}\tilde{T}_{ap\bar{l}}-\Psi_{ip}^{a}\tilde{\nabla}_{\bar{s}}\tilde{T}_{ra\bar{l}}\\
&-(\tilde{\nabla}_i\overline{\Psi_{sl}^{q}})\tilde{T}_{rp\bar{q}}-\overline{\Psi_{sl}^{q}}(\tilde{\nabla}_i\tilde{T}_{rp\bar{q}}-\Psi_{ir}^{a}\tilde{T}_{ap\bar{q}}-\Psi_{ip}^{a}\tilde{T}_{ra\bar{q}})\Big)
\end{aligned}
$$

$$
g^{r\bar{s}}T_{ir}^{a}R_{a\bar{s}p\bar{l}}=g^{r\bar{s}}g^{a\bar{b}}\tilde{T}_{ir\bar{b}}\Big(\tilde{R}_{a\bar{s}p}{}^{\delta}g_{\delta\bar{l}}-\nabla_{\bar{s}}\Psi_{ap}^{\delta}g_{\delta\bar{l}}\Big)\,.
$$

And we also can easily compute

$$
\nabla_r\tilde{R}_{i\bar{s}p}{}^{k}=\tilde{\nabla}_r\tilde{R}_{i\bar{s}p}{}^{k}-\Psi_{ri}^{a}\tilde{R}_{a\bar{s}p}{}^{k}-\Psi_{rp}^{a}\tilde{R}_{i\bar{s}a}{}^{k}+\Psi_{ra}^{k}\tilde{R}_{i\bar{s}p}{}^{a}\,,
$$

and

$$
|\tilde{\nabla}\alpha|_{\tilde{g}}\le C
$$

since the only nonzero component of α is $\alpha_{2\bar{2}}=\frac{1}{2y_2^2}$ (cf. [4]).

Therefore, with using the estimate in Lemma 2.1 and (1) in Lemma 2.3, we obtain

$$
\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathcal{S}\le C(\mathcal{S}^{\frac{3}{2}}+1)-\frac{1}{2}(|\overline{\nabla}\Psi|_g^2+|\nabla\Psi|_g^2)\,.
$$

We also have the evolution of $\mathrm{tr}_{\tilde{g}}\,g$ (cf. [7]):

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathrm{tr}_{\tilde{g}}\,g=&-g^{\bar{j}p}g^{\bar{q}i}\tilde{g}^{\bar{l}k}\tilde{\nabla}_k g_{i\bar{j}}\tilde{\nabla}_{\bar{l}}g_{p\bar{q}}-2\mathrm{Re}\Big(g^{\bar{j}i}\tilde{g}^{\bar{l}k}\tilde{T}_{ki}^{p}\tilde{\nabla}_{\bar{l}}g_{p\bar{j}}\Big)\\
&-g^{\bar{j}i}\Big(\tilde{\nabla}_i\overline{\tilde{T}_{jp}^{p}}+\tilde{g}^{\bar{l}k}\tilde{\nabla}_{\bar{l}}\tilde{T}_{ik\bar{j}}\Big)+g^{\bar{j}i}\tilde{g}^{\bar{l}k}\Big(\tilde{\nabla}_i\overline{\tilde{T}_{jl}^{q}}-\tilde{R}_{i\bar{l}p}{}^{s}\tilde{g}_{s\bar{j}}\tilde{g}^{\bar{q}p}\Big)g_{k\bar{q}}\\
&+g^{\bar{j}i}\tilde{g}^{\bar{l}k}\tilde{T}_{ik}^{p}\overline{\tilde{T}_{jl}^{q}}(\tilde{g}-g)_{p\bar{q}}-\tilde{g}^{k\bar{2}}\tilde{g}^{2\bar{l}}g_{k\bar{l}}\alpha_{2\bar{2}}\,.
\end{aligned}
$$

We use the fact that g and $\tilde{g}$ are uniformly equivalent in Lemma 2.3 (1). We compute that

$$
\tilde{g}^{k\bar{2}}\tilde{g}^{2\bar{l}}g_{k\bar{l}}\alpha_{2\bar{2}}\le C\tilde{g}^{k\bar{2}}\tilde{g}^{2\bar{l}}\tilde{g}_{k\bar{l}}\alpha_{2\bar{2}}=C\tilde{g}^{2\bar{2}}\alpha_{2\bar{2}}=\frac{C}{1+e^{-t}}\le C
$$

for some constant $C > 0$ independent of t, then we again use the result in Lemma 2.1 and can obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_{\tilde{g}} g \le -\frac{1}{C}\mathcal{S} + C(\mathcal{S}^{\frac{1}{2}} + 1)\,,$$

for a uniform constant $C > 0$. Then we apply the way in [7, Section 3] and we have the uniform estimate $\mathcal{S} \le C$.

Note that we write locally $\omega_V = \sqrt{-1}(g_V)_{i\bar{j}} dz^i \wedge d\bar{z}^j$. As we confirmed in Lemma 2.1, since all components of the Christoffel symbols of $\tilde{g}$ are uniformly bounded as t approaches infinity, we have that

$$|\tilde{\Gamma} - \Gamma_V|_{g_V} \le C\,,$$

where Γ_V are the Christoffel symbols of g_V. Together with the fact that $\tilde{g} \le C g_V$, we finally obtain

$$|\nabla_V g|_{g_V} \le |\tilde{\nabla} g|_{g_V} + C \le C|\tilde{\nabla} g|_{\tilde{g}} + C \le C\,.$$

Then it only suffices to apply the same way in the proof of [11, Corollary 1.2] and the result holds also on a minimal non-Kähler properly elliptic surface: Recall that there exists a finite unramified covering $p : M' \to M$ with deck transformation group Γ, where M' is a minimal properly elliptic surface, $\pi' : M' \to S'$ is an elliptic fiber bundle over a compact Riemann surface S' of genus at least 2 (π' is Γ-equivalent) and M is a non-Kähler minimal properly elliptic surface which admits an elliptic fibration $\pi : M \to S$ to a smooth compact curve S. The curve S' is a finite cover of S ramified at the images of the multiple fibers of π, with quotient $S = S'/\Gamma$ and the quotient map $q : S' \to S$ satisfies $q \circ \pi' = \pi \circ p$. We denote by α the form on S associated to the form α' on S' (which is the form $\alpha = \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2$ defined in the previous section). $(\pi')^*\alpha'$ is a smooth real $(1,1)$-form on M', which equals to $p^*\pi^*\alpha$.

Note that when $\pi : M \to S$ is not a fiber bundle, π has no singular fibers, but it might have multiple fibers. Let $D \subset M$ be the set of all multiple fibers of π, so that $\pi(D)$ consists of finitely many points. Then we have that M is a quotient of $\mathbf{C}^* \times H$ by a discrete subgroup Γ' of $\mathrm{SL}(2, \mathbf{R}) \times \mathbf{C}^*$, which acts by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t\right) \cdot (z_1, z_2) = \left((cz_2 + d) \cdot z_1 \cdot t, \frac{az_2 + b}{cz_2 + d}\right),$$

and the map $\pi : M \to S$ is induced by the projection $\mathbf{C}^* \times H \to H$. The case we were considering in the previous section can be obtained by mapping

$$\mathrm{SL}(2, \mathbf{R}) \times \mathbf{Z} \ni (A, n) \mapsto (A, \lambda^n \chi(A)) \in \mathrm{SL}(2, \mathbf{R}) \times \mathbf{C}^*\,,$$

where $\lambda \in \mathbf{C}^*$ with $|\lambda| \ne 1$ and $\mathbf{C}^*/\langle\lambda\rangle = E$ and with a character $\chi : \mathrm{SL}(2, \mathbf{R}) \to \mathbf{C}^*$. If we consider the projection Γ'' of Γ' to $\mathrm{SL}(2, \mathbf{R})$, the Γ''-action on H is generally not free.

Hence the quotient $S = H/\Gamma''$ is an orbifold and the finitely many orbifolds of S are precisely equal to $\pi(D)$. Since the two forms α and γ on $\mathbf{C}^* \times H$ are still invariant under the Γ'-action, they descend to M. We can then define ω_0 as in the previous section. Notice that α is now an orbifold Kähler-Einstein metric on S.

Given any initial metric ω_0 in the $\partial\bar{\partial}$-class of the Vaisman metric on M, we denote $\omega_0' = p^*\omega_0$, which is a Γ-invariant Gauduchon metric in the $\partial\bar{\partial}$-class of the Vaisman metric on M'. Then, let $\omega(t)$, $\omega'(t)$ be solutions of the normalized Chern-Ricci flow on each surface M' and M starting at ω_0, ω_0' respectively. Note that $\omega'(t)$ is equal to $p^*\omega(t)$.

For a sufficiently small open set $U \subset M$ so that $p^{-1}(U)$ is a disjoint union of finitely many copies U_j of U. Then $p : U_j \to U$ is a biholomorphism for each j and the Γ-action on $p^{-1}(U)$ permutes the U_j's. Hence for each j, the map $p : U_j \to U$ gives an isometry between $\omega'(t)|_{U_j}$ and $\omega(t)|_U$ and also between $(\pi'^*\alpha')|_{U_j}$ and $(\pi^*\alpha)|_U$. We apply the argument we discussed above to the elliptic bundle $\pi' : M' \to S'$. Then it follows that $\omega'(t)$ converges to α' in C^{α}-topology for any $\alpha \in (0, 1)$ as $t \to \infty$. Hence the solution of the normalized Chern-Ricci flow $\omega(t)$ on M starting at ω_0 also converges to ω_∞ in C^{α}-topology for any $\alpha \in (0, 1)$ as $t \to \infty$.

References

[1] BARTH, W. P., K. HULEK, C. A. M. PETERS and A. VAN DE VEN, *Compact Complex Surfaces*, Springer-Verlag, Berlin, 2004.

[2] BELGUN, F. A., On the metric structure of non-Kähler complex surfaces, Math. Ann. **317** (2000), no. 1, 1–40.

[3] BRÎNZĂNESCU, V., Néron-Severi group for nonalgebraic elliptic surfaces. II. Non-Kählerian case, Manuscripta Math. **84** (1994), no. 3–4, 415–420.

[4] FANG, S., V. TOSATTI, B. WEINKOVE and T. ZHENG, Inoue surfaces and the Chern-Ricci flow, preprint, arXiv: 1501.07578v1.

[5] GILL, M., Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds, Communications in Analysis and Geometry **19** (2011), no. 2, 277–304.

[6] KODAIRA, K., On the structure of compact complex analytic surfaces, II, Amer. J. Math. **88** (1966), no. 3, 682–721.

[7] SHERMAN, M. and B. WEINKOVE, Local Calabi and curvature estimates for the Chern-Ricci flow, New York J. Math. **19** (2013), 565–582.

[8] SONG, J. and B. WEINKOVE, Lecture notes on the Kähler-Ricci flow, preprint, arXiv: 1212.3653.

[9] TOSATTI, V. and B. WEINKOVE, On the evolution of a Hermitian metric by its Chern-Ricci form, J. Differential Geom. **99** (2015), no. 1, 125–163.

[10] TOSATTI, V. and B. WEINKOVE, The Chern-Ricci flow on complex surfaces, Compos. Math. **149** (2013), no. 12, 2101–2138.

[11] TOSATTI, V., B. WEINKOVE and X. YANG, Collapsing of the Chern-Ricci flow on elliptic surfaces, preprint, arXiv: 1302.6545v1, to appear in Math. Ann.

[12] VAISMAN, I., Non-Kähler metrics on geometric complex surfaces, Rend. Sem. Mat. Univ. Politec. Torino **45** (1987), no. 3, 117–123.

[13] WALL, C. T. C., Geometric structure on compact complex analytic surfaces, Topology **25** (1986), no. 2, 119–153.

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