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A Refined Subsolution Estimate of Weak Subsolutions to Second Order Linear Elliptic Equations with a Singular Vector Field

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Abstract. We consider second order linear elliptic equations $-\mathrm{div}(A(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u = 0$ with a singular vector field $\mathbf{b}$. We prove a refined subsolution estimate, which contains a precise dependence of the quantities of $\mathbf{b}$, for weak subsolutions and a weak Harnack inequality for weak supersolutions under certain assumptions on $\mathbf{b}$.

1. Introduction and main results

We consider second order linear elliptic equations of divergence type:

$$\begin{aligned} &-\mathrm{div}(A(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u \\ &= -\sum_{i,j=1}^{n} \partial_j (a_{ij}(x)\partial_i u) + \sum_{i=1}^{n} b_i(x)\partial_i u = 0 \quad \text{in } \Omega, \end{aligned} \tag{DE}$$

where Ω is a domain in $\mathbb{R}^n$ $(n \geq 3)$. Throughout this paper, we assume that $A(x) = (a_{ij}(x))_{1\leq i,j\leq n}$ is measurable and satisfies the uniform ellipticity condition: there exist positive constants $0 < \nu \leq L < \infty$ such that

$$|a_{ij}(x)| \leq L, \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \Omega. \tag{A}$$

We also assume that a vector field $\mathbf{b}(x) = (b_i(x))_{1\leq i\leq n}$ belongs to $L^2_{\mathrm{loc}}(\Omega)$. We say that u is a weak subsolution (supersolution) to (DE) in Ω if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies

$$\int_{\Omega} (A\nabla u) \cdot \nabla\phi + \mathbf{b} \cdot \nabla u \phi \, dx \leq (\geq)\, 0 \tag{3}$$

for all $\phi \in C_c^{\infty}(\Omega)$ and $\phi \geq 0$. Here, $W^{1,2}_{\mathrm{loc}}(\Omega)$ is the standard Sobolev space. We say that u is a weak solution to (DE) in Ω if u is a weak subsolution and a weak supersolution. If $\mathbf{b} \in$

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$L^p(\Omega)$ with $p > n$, then Hölder continuity and Harnack's inequality of weak solutions are well-known (see e.g., [17, 14, 8]). Stampacchia ([23]) proved the same properties when $\mathbf{b} \in L^n(\Omega)$. Furthermore, he proved Liouville type theorem in the case where $\mathbf{b} \in L^n(\mathbb{R}^n)$ under the smallness condition on $L^n(\mathbb{R}^n)$ norm of $\mathbf{b}$. When $\mathbf{b} \in L^p(\Omega)$ with $p < n$, in general a weak solution u loses its local boundedness (see Remark 3 and [5]). Recently, motivated by applications for the equation of fluid mechanics, parabolic equations corresponding to (DE) under the assumption $\mathrm{div}\mathbf{b} = 0$ has been studied extensively ([19, 15, 24, 21, 6, 22, 18, 5]). Friedlander and Vicol ([6]) proved Hölder continuity of weak solutions under the conditions $\mathrm{div}\mathbf{b} = 0$ and $\mathbf{b} \in L^\infty_t BMO^{-1}_x$. Independently, Seregin et al. ([22]) proved parabolic Harnack inequality in the same conditions. Nazarov and Uraltseva ([18]) proved parabolic Harnack inequality when $\mathrm{div}\mathbf{b} \le 0$ and $\mathbf{b}$ belongs to a suitable space-time Morrey space. Furthermore, they also improved the Harnack inequality due to Stampacchia for the case $\mathbf{b} \in L^n(\Omega)$ by using Safonov's idea ([20]) for elliptic equation (DE) (see Corollary 1). Inspired by these works, in this paper we assume the following conditions for the vector field $\mathbf{b}$.

CONDITION (B) A vector field $\mathbf{b} \in L^2_{\mathrm{loc}}(\Omega)$ can be represented as $\mathbf{b} = \mathbf{b}^{(1)} + \mathbf{b}^{(2)} + \mathbf{b}^{(3)} + \mathbf{b}^{(4)}$ and each $\mathbf{b}^{(i)} \in L^2_{\mathrm{loc}}(\Omega)$ satisfies the following conditions:

1. $\mathbf{b}^{(1)}$ belong to some Lorentz space $L^{n,q}(\Omega)$ with $n \le q < \infty$. (See Section 2 for the definition of Lorentz spaces and basic properties.)
2. $\mathbf{b}^{(2)}$ is small relative to the lower bound ν of (A) in the following sense: there exists a constant $\mathcal{B}_2 = \mathcal{B}_2(\Omega) < \nu$ such that
$$\int_\Omega |\mathbf{b}^{(2)}|^2 \phi^2 \, dx \le (\mathcal{B}_2)^2 \int_\Omega |\nabla \phi|^2 \, dx, \quad \forall \phi \in C^\infty_c(\Omega). \tag{4}$$
3. $\mathbf{b}^{(3)}$ satisfies the form boundedness condition and $\mathrm{div}\mathbf{b}^{(3)} \le 0$ in the distribution sense: there exists a constant $\mathcal{B}_3 = \mathcal{B}_3(\Omega) < \infty$ such that
$$\begin{aligned} &\int_\Omega |\mathbf{b}^{(3)}|^2 \phi^2 \, dx \le (\mathcal{B}_3)^2 \int_\Omega |\nabla \phi|^2 \, dx, \quad \forall \phi \in C^\infty_c(\Omega), \\ &\int_\Omega \mathbf{b}^{(3)} \cdot \nabla \phi \, dx \ge 0, \quad \forall \phi \in C^\infty_c(\Omega), \ \phi \ge 0. \end{aligned} \tag{5}$$
4. $\mathbf{b}^{(4)} = \left(b^{(4)}_i\right)_{1 \le i \le n}$ can be written in the form $b^{(4)}_i = \sum_{j=1}^n \partial_j V_{ij}$ in the distribution sense, where $V = (V_{ij})$ satisfies $V_{ij} = -V_{ji}$ and $V_{ij} \in BMO(\Omega)$ $(1 \le i, j \le n)$. (See Section 2 for the definition of $BMO(\Omega)$.) We define $\|V\|_{BMO(\Omega)} = \sum_{i,j} \|V_{ij}\|_{BMO(\Omega)}$.

REMARK 1. It is easy to see $\mathrm{div}\mathbf{b}^{(4)} = 0$ in the distribution sense. We do not impose the form boundedness of $|\mathbf{b}^{(4)}|^2$.

Main results of this paper are as follows. We assume the conditions (A) and (B) on $A(x)$ and $\mathbf{b}(x)$ respectively in the following statements in Ω.

THEOREM 1 (subsolution estimate). *Let $B_{2R}(x_0) \subset \Omega$. Suppose u is a weak subsolution of* (DE) *in $B_R(x_0)$. Let $0 < \rho < R$. Then for any $p > 0$ there is a constant C depending only on n, L, q, and p such that*

$$\operatorname*{ess\,sup}_{B_\rho(x_0)} u_+ \le C(n, L, q, p) \left\{ K_1^{n+1} K_2^{qn} \right\}^{\frac{1}{p}} \left(\frac{1}{(R-\rho)^n} \int_{B_R(x_0)} u_+^p \, dx \right)^{\frac{1}{p}},$$

where $K_1 = \frac{1+\mathcal{B}_3+\|V\|_{BMO(\Omega)}}{\nu-\mathcal{B}_2}$ and $K_2 = 1 + \frac{\|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)}}{\nu-\mathcal{B}_2}$.

THEOREM 2 (weak Harnack inequality). *Let $B_{4R}(x_0) \subset \Omega$. Suppose u is a nonnegative weak supersolution of* (DE) *in $B_{2R}(x_0)$. Then there are positive numbers $p_0 > 0$ and C depending only on n, ν, L, $\|\mathbf{b}_1\|_{L^{n,q}(\Omega)}$, q, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\|V\|_{BMO(\Omega)}$ such that*

$$\left(\frac{1}{R^n} \int_{B_R(x_0)} u^{p_0} \, dx \right)^{\frac{1}{p_0}} \le C \operatorname*{ess\,inf}_{B_{\frac{R}{2}}(x_0)} u.$$

More precisely, p_0 and C can be expressed as $p_0 = \frac{C(n,\nu,L,q)}{K_3}$ and $C = \{C(n, L, q) K_1^{n+1} K_2^{qn}\}^{C(n,\nu,L,q)K_3}$ where $K_3 = 1 + \|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)} + \mathcal{B}_3 + \|V\|_{BMO(\Omega)}$.

REMARK 2. Note that $L^n(\Omega) = L^{n,n}(\Omega)$. Even for the case $\mathbf{b} = \mathbf{b}^{(1)} \in L^n(\Omega)$, Theorem 1 is new and gives a refined subsolution estimate which contains a precise dependence on the quantity $\|\mathbf{b}\|_{L^n(\Omega)}$. Although Stampacchia already proved a subsolution estimate for the case $\mathbf{b} \in L^n(\Omega)$ in [23], the precise dependence of the quantity $\|\mathbf{b}\|_{L^n(\Omega)}$ was not given. Actually, as it was pointed out in [14, p.200], Stampacchia's constant depends on the constant K such that $\|\mathbf{b}\|_{L^n(B_R(x_0)\cap\{|\mathbf{b}|>K\})} \le C(n)\nu$, what $C(n)$ is constant depending only n. Therefore the constant K depends on $B_R(x_0) \subset \Omega$, not on the quantity $\|\mathbf{b}\|_{L^n(\Omega)}$.

REMARK 3. The smallness condition on $\mathcal{B}_2$ is sharp. Let $\mathbf{b}(x) = \nu\gamma \frac{x}{|x|^2}$ with $\gamma \in \mathbb{R}$. When $-\infty < \gamma < \frac{n-2}{2}$, $\mathbf{b}$ satisfies the condition (B) and hence a weak subsolution (supersolution) $W^{1,2}(B_1)$ to $-\nu\Delta u + \mathbf{b}\cdot\nabla u = 0$ in B_1 satisfies the subsolution estimate (the weak Harnack inequality). Actually, $\mathbf{b}$ satisfies the condition (B) as $\mathbf{b}^{(2)} = \mathbf{b}$, $\mathbf{b}^{(1)} = \mathbf{b}^{(3)} = \mathbf{b}^{(4)} = 0$ for the case $|\gamma| < \frac{n-2}{2}$ by using Hardy's inequality:

$$\int_{\mathbb{R}^n} \frac{\phi^2}{|x|^2} \, dx \le \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |\nabla\phi|^2 \, dx, \quad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

and $\mathbf{b}^{(3)} = \mathbf{b}$, $\mathbf{b}^{(1)} = \mathbf{b}^{(2)} = \mathbf{b}^{(4)} = 0$ for the case $\gamma \le 0$, since $\operatorname{div}\mathbf{b} = \nu\gamma\frac{(n-2)}{|x|^2} \le 0$. On the other hand, when $\gamma > \frac{n-2}{2}$, it is easy to see that

$$u(x) = \begin{cases} c|x|^{2-n+\gamma} & \gamma \ne n-2, \\ c\log|x| & \gamma = n-2 \end{cases}$$

belongs to $W^{1,2}(B_1)$ and is a weak solution to $-\nu\triangle u + \mathbf{b}\cdot\nabla u = 0$ in B_1. Since u is not a bounded function, the smallness condition on $\mathcal{B}_2$ is sharp.

REMARK 4. In [5], for a certain $\mathbf{b} \in L^p(\Omega)$ with $p < n$ and div$\mathbf{b} = 0$ the existence of a bounded weak solution u to $-\triangle u + \mathbf{b}\cdot\nabla u = 0$, which is not continuous has been pointed out.

REMARK 5. If $\mathbf{b}^{(1)} \in L^q(\Omega)$ with $q > n$ and $\mathbf{b}^{(2)} = \mathbf{b}^{(3)} = \mathbf{b}^{(4)} = 0$, a sharp form weak of the Harnack inequality is known ([8]). i.e., positive number p_0 in Theorem 2 can be replaced to any $0 < p < \frac{n}{n-2}$. But, as $\mathbf{b}^{(2)}, \mathbf{b}^{(3)} \neq 0$, we cannot expect such a sharp form of the weak Harnack inequality in general. Actually, for $\mathbf{b} = \gamma \frac{x}{|x|^2}$ with $\gamma < 0$ sufficiently small, $u_k = \min\{|x|^{2-n+\gamma}, k\}$ $(k > 1)$ is a nonnegative weak supersolution of the equation $-\triangle u + \mathbf{b}\cdot\nabla u = 0$ in $\Omega = B_{2R}$. Then, in spite of $\operatorname{ess\,inf}_{B_1} u_k = 1$, $\lim_{k\to\infty} \|u_k\|_{L^p(B_1)} = \infty$ for $p = \frac{n}{n-2-\gamma}$. Therefore, the weak Harnack inequality with $p_0 = p$ does not hold.

In this paper, we treat the conditions $\mathbf{b}^{(i)}$ $(i = 1, 2, 3, 4)$ in a unified way. The classes $\mathbf{b}^{(2)}$, $\mathbf{b}^{(3)}$ and $\mathbf{b}^{(4)}$ also have been considered in previous works ([15, 21, 6, 22]) for parabolic equations in $\Omega = \mathbb{R}^n$. Restricting to the elliptic problem, their results yields essentially the same subsolution estimate for weak subsolutions under the assumption $\mathbf{b} = \mathbf{b}^{(2)}, \mathbf{b}^{(3)}$ or $\mathbf{b}^{(4)}$ without the precise dependence of the constant on the quantities $\mathcal{B}_2$, $\mathcal{B}_3$ and $\|V\|_{BMO(\Omega)}$. The method of the proof is slightly different in the following sense. In [15], [21] and [22], since they were mainly concerned with weak solutions, first they established a solution for the approximated equations with smooth vector field $\mathbf{b}$ and then took the limit to obtain the estimate for weak solutions. In [22], they used the higher integrability of the gradient of u to show the parabolic Harnack inequality for suitable weak solutions. Furthermore, in [15], [21] for $\mathbf{b} = \mathbf{b}^{(2)}, \mathbf{b}^{(3)}$ or $\mathbf{b}^{(4)}$, they also proved Hölder continuity of weak solutions by using the estimates for fundamental solutions to parabolic equations. In [6], they proved Hölder continuity by using Caffarelli-Vasseur approach based on the oscillation lemma ([3]). The strategy of this paper is to establish a refined subsolution estimate and a weak Harnack inequality for weak subsolutions and weak supersolutions without using the approximating procedure on the vector field $\mathbf{b}$. Instead of such approximating procedure, we will take care of substituting processes of various test functions in details. We also remark that in [18] and [13] they showed a subsolution estimate for Lipshitz continuous weak solutions under slightly weaker conditions than the one on $\mathbf{b}^{(3)}$.

Combining Theorem 1 with Theorem 2, we obtain following Harnack's inequality immediately.

COROLLARY 1 (Harnack's inequality). *Let $B_{4R}(x_0) \subset \Omega$. Suppose u is a nonnegative weak solution of* (DE) *in $B_{2R}(x_0)$. Then there is a constant C depending only on $n, \nu, L, \|\mathbf{b}_1\|_{L^{n,q}(\Omega)}, q, \mathcal{B}_2, \mathcal{B}_3$ and $\|V\|_{BMO(\Omega)}$ such that*

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}(x_0)} u \leq C \operatorname*{ess\,inf}_{B_{\frac{R}{2}}(x_0)} u.$$

Once we get Corollary 1, we can show the following consequences by using a standard argument (see e.g., [8, 9, 18]). We omit the detail of the proofs.

COROLLARY 2 (Hölder estimate). *Let $B_{4R}(x_0) \subset \Omega$. Suppose u is a weak solution of* (DE) *in $B_{2R}(x_0)$. Then there are positive numbers $\beta \in (0,1)$ and C depending only on n, ν, L, $\|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)}$, q, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\|V\|_{BMO(\Omega)}$ such that*

$$\operatorname*{osc}_{B_\rho(x_0)} u \le C\left(\frac{\rho}{R}\right)^\beta \operatorname*{osc}_{B_R(x_0)} u, \quad 0 < \forall \rho < R.$$

COROLLARY 3 (Liouville). *Let condition* (A) *be satisfied in $\mathbb{R}^n$. Suppose that* (B) *be satisfied in any domain $\Omega \Subset \mathbb{R}^n$ for some fixed $q < \infty$. We define*

$$S(\Omega) := \frac{1 + \|\mathbf{b}^{(1)}\|_{L^{p,q}(\Omega)} + \mathcal{B}_3(\Omega) + \|V\|_{BMO(\Omega)}}{\nu - \mathcal{B}_2(\Omega)}. \tag{6}$$

Also suppose that

$$\liminf_{R\to\infty} \sup_{|x|=R} S(B_{\delta R}(x)) < \infty \tag{7}$$

holds for some $0 < \delta < 1$. If u is a weak solution of (DE) *in $\mathbb{R}^n$ and bounded from below* (*or above*), *then u is a constant.*

REMARK 6. We note several examples of $\mathbf{b}$ satisfying (7). If $\mathbf{b} \in L^{n,q}(\mathbb{R}^n)$, $\mathcal{B}_3(\mathbb{R}^n) < \infty$, $V \in BMO(\mathbb{R}^n)^{n\times n}$ and $\mathcal{B}_2(\mathbb{R}^n) < \nu$, then (7) satisfied for any $0 < \delta < 1$. If $|\mathbf{b}| \le \frac{C}{1+|x|}$ for some $C > 0$, it is easy to see that

$$\liminf_{R\to\infty} \sup_{|x|=R} \|\mathbf{b}\|_{L^{n,q}(B_{\delta R}(x))} < \infty$$

holds for any $0 < \delta < 1$.

REMARK 7. Corollaries 1, 2 and 3 are generalization of Theorems 2.5′ and Theorem 2.6′ in [18]. In [18], a generalization of their result to Lorentz spaces was suggested without proof.

In addition, as an application of Theorem 1, we prove the following corollary.

COROLLARY 4 (Higher integrability). *Suppose u is a weak solution of* (DE) *in Ω. Then ∇u belongs to $L^{p_1}_{\mathrm{loc}}(\Omega)$ for some $p_1 > 2$.*

The paper is organized as follows: First, by using the properties of the form $\int \mathbf{b}\cdot\nabla u v\,dx$ (Lemma 7, 8), we prove the Caccioppoli type inequality when $\mathbf{b}^{(1)}$ is small enough (Lemma 1). Also, we get the subsolution estimate when $\mathbf{b}^{(1)}$ is sufficiently small (Lemma 9) using this. Next, using the weak maximum principle and Lemma 2, we prove the subsolution estimate without smallness of $\mathbf{b}^{(1)}$ (Theorem 1). Finally, we show that the BMO estimate of $\log u$ for a positive supersolution u (Lemma 2), using this and the subsolution estimate, we obtain the weak Harnack inequality (Theorem 2). In addition, we show Corollary 4 by applying the subsolution estimate.

We will use the following notation. $B_R(x_0) := \{ x \in \mathbb{R}^n;\ |x - x_0| < R \}$ and $B_R = B_R(0)$. For $B = B_R(x_0)$, we define $2B := B_{2R}(x_0)$. For $x \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$, we define $\mathrm{dist}(x, S) := \inf\{ |x - y|;\ y \in S \}$. For open sets $\Omega', \Omega \subset \mathbb{R}^n$, we denote $\Omega' \Subset \Omega$ if $\bar{\Omega}'$ is compact and $\bar{\Omega}' \subset \Omega$. If $A \subset \mathbb{R}^n$, $|A|$ is the Lebesgue measure of A. $f_+ = \max\{f, 0\}$. $f_A = |A|^{-1} \int_A f\,dx$. $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$ is a standard mollifier.

2. Preliminaries

2.1. Function spaces and imbedding theorem. The Sobolev space $W^{1,2}(\Omega)$ consists of all weakly differentiable functions such that

$$\|u\|_{W^{1,2}(\Omega)}^2 := \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 < \infty\,.$$

The space $W_0^{1,2}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$. We say u belongs to $W_{\mathrm{loc}}^{1,2}(\Omega)$ if $\|u\|_{W^{1,2}(\Omega')} < \infty$ for every $\Omega' \Subset \Omega$. Recall the following properties of $W^{1,2}(\Omega)$. See e.g. [11, p.18, 20] for the proof.

LEMMA 1. *Suppose that* $\{u_j\}_{j=1}^\infty \subset W^{1,2}(\Omega)$, $u \in W^{1,2}(\Omega)$ *and* $u_j \to u$ *in* $W^{1,2}(\Omega)$. *Then* $(u_j)_+ \to u_+$ *in* $W^{1,2}(\Omega)$. *In addition, suppose that* $u_j, u \geq k > 0$ *in* Ω *for some positive constant* k, $f \in C^1(0,\infty)$ *and* f' *is bounded in* $[k,\infty)$. *Then* $\nabla(f \circ u_j) \to \nabla(f \circ u) = f'(u)\nabla u$ *in* $L^2(\Omega)$.

For $0 < p < \infty$ and $0 < q \leq \infty$, we consider the quantity

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left(p \int_0^\infty s^q |\{ x \in \Omega;\ |f(x)| > s \}|^{\frac{q}{p}} \frac{ds}{s}\right)^{\frac{1}{q}} & q < \infty\,, \\ \sup_{s>0} s|\{ x \in \Omega;\ |f(x)| > s \}|^{\frac{1}{p}} & q = \infty\,. \end{cases}$$

The Lorentz space $L^{p,q}(\Omega)$ consists of all measurable functions f satisfying $\|f\|_{L^{p,q}(\Omega)} < \infty$. Note that $L^{p,p}(\Omega) = L^p(\Omega)$ and $L^{p,q}(\Omega) \subsetneq L^{p,r}(\Omega) \subsetneq L^{p,\infty}(\Omega)$ for any $q < r < \infty$ ([10, p.49]). We will use the following lemma to show Theorem 1.

LEMMA 2. *Let* $f \in L^{p,q}(\Omega)$ *with* $p \leq q < \infty$. *For any* $\varepsilon > 0$ *we define* $M = [\varepsilon^{-q}\|f\|_{L^{p,q}(\Omega)}^q] + 1$, *where* $[t]$ *is the integer part of* t. *If* $A_1, \ldots, A_M$ *are disjoint subsets of* Ω, *then* $\|f\|_{L^{p,q}(A_m)} < \varepsilon$ *for some* $m \in \{1, \ldots, M\}$.

PROOF. We note that $[t] + 1 > t$ for any $t \geq 0$. Since $p \leq q$, using the inequality

$\sum_m a_m^\alpha \le (\sum_m a_m)^\alpha$ $(a_m \ge 0,\ \alpha \ge 1)$ as $\alpha = \frac{q}{p}$, we have

$$\begin{aligned}\|f\|^q_{L^{p,q}(\Omega)} &\ge p\int_0^\infty s^q\left(\sum_{m=1}^M |\{x\in A_m;\ |f(x)|>s\}|\right)^{\frac{q}{p}}\frac{ds}{s}\\ &\ge \sum_{m=1}^M p\int_0^\infty s^q|\{x\in A_m;\ |f(x)|>s\}|^{\frac{q}{p}}\frac{ds}{s} = \sum_{m=1}^M \|f\|^q_{L^{p,q}(A_m)}.\end{aligned}$$

If $\varepsilon \le \|f\|_{L^{p,q}(A_m)}$ for all $m = 1, \ldots, M$, then we have $M^{\frac{1}{q}} \le \varepsilon^{-1}\|f\|_{L^{p,q}(\Omega)}$. This inequality contradicts with the definition of M. □

Next lemma is the Sobolev imbedding theorem in Lorentz spaces.

LEMMA 3. *Let $\Omega \subset \mathbb{R}^n$ $(n > 2)$ and $2 \le q \le \infty$. Then there exists a constant $S(n, q)$ depending only n and q such that*

$$\|f\|_{L^{2^*,q}(\Omega)} \le S(n,q)\|\nabla f\|_{L^2(\Omega)}, \quad \forall f \in W_0^{1,2}(\Omega), \tag{8}$$

where $2^* := \frac{2n}{n-2}$.

See e.g. [1] for the proof. Recently, the best constant $S(n, q)$ of (8) was studied in [2]. When $q = 2^*$, (8) is the well-known Sobolev inequality. We denote $C_S(n) := S(n, 2^*)$. The assumption on $\mathbf{b}^{(1)}$, the duality of Lorentz spaces ([10, p.52]) and (8) yield

$$\int_\Omega |\mathbf{b}^{(1)}|^2\phi^2\,dx \le \left(C_B(n,q)\|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)}\right)^2\int_\Omega |\nabla\phi|^2\,dx, \quad \forall\phi \in C_c^\infty(\Omega), \tag{9}$$

where $C_B(n, q) := S(n, \frac{2q}{q-2})$. In the following, we will use these notations.

For a domain $\Omega \subset \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\Omega)$, we define

$$\|f\|_{BMO(\Omega)} := \sup_{2B\subset\Omega}\frac{1}{|B|}\int_B |f(x) - f_B|\,dx,$$

where the supremum is taken over all balls $2B \subset \Omega$. $BMO(\Omega)$ consists of all locally integrable functions f satisfying $\|f\|_{BMO(\Omega)} < \infty$. From the well-known John-Nirenberg inequality $|\{x \in B;\ |f(x) - f_B| > s\}| \le C_1\exp(\frac{-C_2 s}{\|f\|_{BMO(\Omega)}})|B|$ for any $2B \subset \Omega$ (See e.g., [11, p.365]), every $f \in BMO(\Omega)$ has the exponential integrability:

$$\forall 2B \subset \Omega, \quad \int_B \exp\left(\frac{C(n)|f(x) - f_B|}{\|f\|_{BMO(\Omega)}}\right)dx \le C(n)|B|. \tag{10}$$

Especially,

$$\forall 2B \subset \Omega, \quad \left(\frac{1}{|B|}\int_B |f(x) - f_B|^p\,dx\right)^{\frac{1}{p}} \le C(n,p)\|f\|_{BMO(\Omega)}, \tag{11}$$

for any $1 \le p < \infty$. These inequalities are also called as the John-Nirenberg inequality.

2.2. Some technical facts. We will use the following two technical lemmas in the proof of Theorem 1 and 2. We present these well-known statements for reader's convenience to make it clear the dependence of quantities of $\mathbf{b}$ in our estimates.

LEMMA 4 ([9, p.220] [14, p.66]). *Let $\alpha > 0$ and let $\{x_i\}$ be a sequence of positive numbers, such that*

$$x_{m+1} \leq C b^m x_m^{1+\alpha}, \tag{12}$$

with $C > 0$ and $b > 1$. If $x_0 \leq C^{\frac{-1}{\alpha}} b^{\frac{-1}{\alpha^2}}$, then $\lim_{m\to\infty} x_m = 0$.

LEMMA 5 ([9, p.191] [12, p.76]). *Let $Z(t)$ be a bounded non-negative function in the interval $[\rho, R]$. Assume that for $\rho \leq t < s \leq R$ we have*

$$Z(t) \leq \theta Z(s) + \frac{A}{(s-t)^\alpha}$$

with $A \geq 0, \alpha > 0$ and $0 \leq \theta < 1$. Then there exists a constant $C(\alpha, \theta)$ such that

$$Z(\rho) \leq \frac{C(\alpha, \theta) A}{(R-\rho)^\alpha}.$$

3. Proof of Main theorems

3.1. Basic estimates for $\int \mathbf{b} \cdot \nabla u v \, dx$. Since we do not assume the form boundedness of $|\mathbf{b}^{(4)}|^2$ as in $\mathbf{b}^{(2)}$, $\mathbf{b}^{(3)}$, we must take care of the expressions $\int_\Omega \mathbf{b}^{(4)} \cdot \nabla u v \, dx$ for $u \in W^{1,2}(B_R)$ and $v \in W_0^{1,2}(B_R)$. The following inequality can be found in Maz′ya and Verbitsky ([16]), but we give the proof for completeness.

LEMMA 6. *If $\mathbf{b} = \mathbf{b}^{(4)}$ in the condition* (B) *with $\Omega = \mathbb{R}^n$. Then there exists a constant $C = C(n)$ such that*

$$\left| \int_{\mathbb{R}^n} \mathbf{b}^{(4)} \cdot \nabla u v \, dx \right| \leq C \|V\|_{BMO(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad \forall u, v \in C_c^\infty(\mathbb{R}^n) \tag{13}$$

holds.

PROOF. Since $\mathrm{div}\mathbf{b}^{(4)} = 0$ and $V_{ij} = -V_{ji}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{b}^{(4)} \cdot \nabla u v \, dx &= -\int_{\mathbb{R}^n} \mathbf{b}^{(4)} \cdot u \nabla v \, dx = \frac{1}{2} \int_{\mathbb{R}^n} \mathbf{b}^{(4)} \cdot (\nabla u v - u \nabla v) \, dx \\
&= -\frac{1}{2} \sum_{i,j=1}^n \int_{\mathbb{R}^n} V_{ij} \partial_j \left(\partial_i u v - u \partial_i v\right) dx \\
&= -\frac{1}{2} \int_{\mathbb{R}^n} \sum_{\substack{i,j=1 \\ i \neq j}}^n \underbrace{\left(\partial_i u \partial_j v - \partial_j u \partial_i v\right)}_{=:W_{ij}} dx.
\end{aligned}$$

For $i \neq j$, we take $\vec{f}_{(i,j)} := (0, \dots, \overbrace{-\partial_j u}^{i}, \dots, \overbrace{\partial_i u}^{j}, \dots, 0)^T$. Then $W_{ij} = \vec{f}_{(i,j)} \cdot \nabla v$. Since $\operatorname{div} \vec{f}_{(i,j)} = 0$ and $\|\vec{f}_{(i,j)}\|_{L^2} \leq \|\nabla u\|_{L^2}$, from the div-curl lemma ([4]),

$$\|W_{ij}\|_{\mathcal{H}^1} \leq C \|\vec{f}_{(i,j)}\|_{L^2} \|\nabla v\|_{L^2} \leq C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}.$$

Here, $\mathcal{H}^1$ is the Hardy space. Therefore, by the $\mathcal{H}^1 - BMO$ duality we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \mathbf{b}^{(4)} \cdot \nabla u v \, dx \right| &= \frac{1}{2} \left| \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} V_{ij} W_{ij} \, dx \right| \\ &\leq C \|V\|_{BMO} \|W\|_{\mathcal{H}^1} \leq C \|V\|_{BMO} \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}, \end{aligned}$$

where $\|W\|_{\mathcal{H}^1} = \sum_{i,j} \|W_{ij}\|_{\mathcal{H}^1}$. □

Next lemma is an easy consequence of (13).

LEMMA 7. *Let $B_R \Subset \Omega$. Assume the condition* (B) *on* $\mathbf{b}$. *Then there is a constant C depending only on n, R, $\|\mathbf{b}_1\|_{L^{n,q}(\Omega)}$, q, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\|V\|_{BMO(\Omega)}$ such that*

$$\left| \int_{B_R} \mathbf{b} \cdot \nabla u v \, dx \right| \leq C \|\nabla u\|_{L^2(B_R)} \|\nabla v\|_{L^2(B_R)},$$

for any $u \in W^{1,2}(B_R)$ and any $v \in W_0^{1,2}(B_R) \cap C_c(B_R)$.

PROOF. By the form boundedness condition (9), (4) and (5) and the Cauchy-Schwarz inequality, there is a constant $C = C(n, \|\mathbf{b}_1\|_{L^{n,q}(\Omega)}, q, \mathcal{B}_2, \mathcal{B}_3)$ such that

$$\left| \int_{B_R} \left(\mathbf{b}^{(1)} + \mathbf{b}^{(2)} + \mathbf{b}^{(3)} \right) \cdot \nabla u v \, dx \right| \leq C \|\nabla u\|_{L^2(B_R)} \|\nabla v\|_{L^2(B_R)}$$

holds for $u \in W^{1,2}(B_R)$ and $v \in W_0^{1,2}(B_R)$. It remains to show the inequality for $\mathbf{b}^{(4)}$. We note that there is a skew-symmetric matrix valued function $\widetilde{V} = (\widetilde{V}_{ij})_{1 \leq i,j \leq n} \in BMO(\mathbb{R}^n)^{n \times n}$ such that $\widetilde{V} \equiv V$ in B_R and $\|\widetilde{V}\|_{BMO(\mathbb{R}^n)} \leq C \|V\|_{BMO(\Omega)}$. We define $\widetilde{\mathbf{b}} = (\widetilde{b}_i) := (\sum_j \partial_j \widetilde{V}_{ij})$ (see e.g. [16]). First, we also assume that $u \in W_0^{1,2}(B_{2R})$. Choose a sequence $\{u_\varepsilon\}_{\varepsilon > 0} \subset C_c^\infty(B_{2R})$ such that $u_\varepsilon \to u$ in $W^{1,2}(B_{2R})$ and take $v_\varepsilon := \eta_\varepsilon * v$. Then by (13) we have

$$\begin{aligned} \left| \int_{B_R} \mathbf{b}^{(4)} \cdot \nabla u v \, dx \right| &\leq \left| \int_{B_R} \widetilde{\mathbf{b}} \cdot \nabla u_\varepsilon v_\varepsilon \, dx \right| \\ &+ \left| \int_{B_R} \mathbf{b}^{(4)} \cdot \nabla (u - u_\varepsilon) v_\varepsilon \, dx \right| + \left| \int_{B_R} \mathbf{b}^{(4)} \cdot \nabla u (v - v_\varepsilon) \, dx \right| \end{aligned}$$

$$
\begin{aligned}
&\leq C\|\widetilde{V}\|_{BMO(\mathbb{R}^n)}\|\nabla u_\varepsilon\|_{L^2(\mathbb{R}^n)}\|\nabla v_\varepsilon\|_{L^2(\mathbb{R}^n)}\\
&+\|\mathbf{b}^{(4)}\|_{L^2(B_R)}\|\nabla(u-u_\varepsilon)\|_{L^2(B_R)}\|v_\varepsilon\|_{L^\infty(B_R)}\\
&+\|\mathbf{b}^{(4)}\|_{L^2(B_R)}\|\nabla u\|_{L^2(B_R)}\|v-v_\varepsilon\|_{L^\infty(B_R)}.
\end{aligned}
$$

The right-hand side converges to $C\|\widetilde{V}\|_{BMO(\mathbb{R}^n)}\|\nabla u\|_{L^2(\mathbb{R}^n)}\|\nabla v\|_{L^2(\mathbb{R}^n)}$. The desired inequality is obtained. Next, we treat a general $u \in W^{1,2}(B_R)$. Let $\widetilde{u} \in W^{1,2}_0(B_{2R})$ be an extension of $u - u_{B_R}$. By Poincaré's inequality, we have

$$
\|\widetilde{u}\|_{W^{1,2}(B_{2R})} \leq C(n,R)\|u-u_{B_R}\|_{W^{1,2}(B_R)} \leq C(n,R)'\|\nabla(u-u_{B_R})\|_{L^2(B_R)} = C'\|\nabla u\|_{L^2(B_R)}.
$$

Using the previous inequality, we get the desired inequality. □

LEMMA 8. *Let $B_{2R} \subset \Omega$. Assume the condition* (B) *on* $\mathbf{b}$. *Suppose $\psi \in C^\infty(B_R)$ and $\zeta \in C_c^\infty(B_R)$ are non-negative. Then, for any $\varepsilon_i > 0$ $(i = 2, 3, 4)$ following inequalities hold.*

$$
\int_{B_R} \left|\mathbf{b}^{(1)} \cdot \nabla\psi\right| \psi\zeta^2\,dx \leq 2\mathcal{B}_1 \int_{B_R} |\nabla\psi|^2\zeta^2\,dx + \frac{\mathcal{B}_1}{3}\int_{B_R} \psi^2|\nabla\zeta|^2\,dx, \tag{14}
$$

$$
\begin{aligned}
\int_{B_R} \left|\mathbf{b}^{(2)} \cdot \nabla\psi\right| \psi\zeta^2\,dx \leq& (1+\varepsilon_2)\mathcal{B}_2 \int_{B_R} |\nabla\psi|^2\zeta^2\,dx\\
&+ \frac{(1+\varepsilon_2)\mathcal{B}_2}{2\{(1+\varepsilon_2)^2-1\}}\int_{B_R}\psi^2|\nabla\zeta|^2\,dx,
\end{aligned} \tag{15}
$$

$$
-\int_{B_R} \mathbf{b}^{(3)} \cdot \nabla\psi\psi\zeta^2\,dx \leq \varepsilon_3 \int_{B_R} |\nabla\psi|^2\zeta^2\,dx + \left(\varepsilon_3 + \frac{\mathcal{B}_3^2}{2\varepsilon_3}\right)\int_{B_R}\psi^2|\nabla\zeta|^2\,dx, \tag{16}
$$

$$
\left|\int_{B_R} \mathbf{b}^{(4)} \cdot \nabla\psi\psi\zeta^2\,dx\right| \leq \varepsilon_4 \int_{B_R} |\nabla\psi|^2\zeta^2\,dx + \frac{1}{\varepsilon_4}\int_{B_R}|V - V_{B_R}|^2\psi^2|\nabla\zeta|^2\,dx, \tag{17}
$$

where $\mathcal{B}_1 = C_B(n,q)\|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)}$.

PROOF. First, we recall Young's inequality:

$$
ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad \forall a, b \geq 0, \quad \forall\varepsilon > 0. \tag{18}
$$

Using (18), we have for any $\varepsilon > 0$

$$
\int_{B_R} \left|\mathbf{b}^{(1)} \cdot \nabla\psi\right| \psi\zeta^2\,dx \leq \frac{\varepsilon}{2}\int_{B_R} |\nabla\psi|^2\zeta^2\,dx + \frac{1}{2\varepsilon}\int_{B_R}|\mathbf{b}^{(1)}|^2(\psi\zeta)^2\,dx.
$$

On the other hand, for any $\varepsilon_1 > 0$, using $(a+b)^2 \leq \frac{1}{s}a^2 + \frac{1}{1-s}b^2$ $(a, b \geq 0, 0 < s < 1)$ with $s = (1+\varepsilon_1)^{-2}$, we have

$$
\int_{B_R} |\nabla(\psi\zeta)|^2\,dx \leq (1+\varepsilon_1)^2 \int_{B_R} |\nabla\psi|^2\zeta^2\,dx + C(\varepsilon_1)\int_{B_R}\psi^2|\nabla\zeta|^2\,dx,
$$

where $C(\varepsilon_1) = \frac{(1+\varepsilon_1)^2}{(1+\varepsilon_1)^2-1}$. We combine the two inequalities as $\varepsilon = (1+\varepsilon_1)\mathcal{B}_1$. Taking $\varepsilon_1 = 1$ and using (9), we obtain (14). In the same manner, we get (15). Next, since $\mathrm{div}\mathbf{b}^{(3)} \leq 0$ in the distribution sense, (18) yields

$$-\int_{B_R} \mathbf{b}^{(3)} \cdot \nabla\psi\psi\zeta^2\,dx = -\frac{1}{2}\int_{B_R} \mathbf{b}^{(3)} \cdot \nabla\left(\psi^2\zeta^2\right)dx + \int_{B_R} \mathbf{b}^{(3)} \cdot \psi^2\nabla\zeta\zeta\,dx$$
$$\leq \frac{\varepsilon}{2}\int_{B_R} |\mathbf{b}^{(3)}|^2(\psi\zeta)^2\,dx + \frac{1}{2\varepsilon}\int_{B_R} \psi^2|\nabla\zeta|^2\,dx,$$

for any $\varepsilon > 0$. Let ε_3 be a positive constant. Taking $\varepsilon = \varepsilon_3\mathcal{B}_3^{-2}$, we get (16). In order to prove (17), we use the notation $\widetilde{V} = V - V_{B_R}$ with $V_{B_R} = |B_R|^{-1}\int_{B_R} V\,dx$. By the assumption on $\mathbf{b}^{(4)}$, we have

$$-\int_{B_R} \mathbf{b}^{(4)} \cdot \nabla\psi\psi\zeta^2\,dx = -\frac{1}{2}\sum_{i=1}^{n}\int_{B_R} b_i^{(4)}\partial_i(\psi^2)\zeta^2\,dx$$
$$= \frac{1}{2}\sum_{i,j=1}^{n}\int_{B_R} V_{ij}\partial_j(\partial_i(\psi^2)\zeta^2)\,dx = \frac{1}{2}\sum_{i,j=1}^{n}\int_{B_R} \widetilde{V}_{ij}\partial_j(\partial_i(\psi^2)\zeta^2)\,dx$$
$$= \frac{1}{2}\sum_{i,j=1}^{n}\int_{B_R} \widetilde{V}_{ij}\partial_i(\psi^2)\partial_j(\zeta^2)\,dx + \frac{1}{2}\sum_{i,j=1}^{n}\int_{B_R} \widetilde{V}_{ij}\partial_j\partial_i(\psi^2)\zeta^2\,dx.$$

Here, we have used $\int_{B_R} \partial_j(\partial_i(\psi^2)\zeta^2)\,dx = 0$ from the divergence theorem. Since $\widetilde{V}_{ij} = -\widetilde{V}_{ij}$, the second term of the right-hand side equals to zero. Therefore

$$-\int_{B_R} \mathbf{b}^{(4)} \cdot \nabla\psi\psi\zeta^2\,dx = 2\int_{B_R} (V - V_{B_R})\nabla\psi \cdot \psi\nabla\zeta\zeta\,dx.$$

Using (18) again, we arrive at the desired inequality. □

3.2. Proof of Theorem 1

PROPOSITION 1 (Caccioppoli type inequality). *Let $B_{2R} \subset \Omega$. Assume the condition* (B) *on* $\mathbf{b}$. *We also assume that*

$$\|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)} \leq \frac{\nu - \mathcal{B}_2}{8C_B(n,q)}. \tag{19}$$

Suppose u is a weak subsolution of (DE) *in B_R. Then for any non-negative $\zeta \in C_c^\infty(B_R)$, we have*

$$\int_{B_R} |\nabla u_+|^2\zeta^2\,dx \leq C\left(\frac{L^2}{(\nu-\mathcal{B}_2)^2} + \frac{\mathcal{B}_3^2}{(\nu-\mathcal{B}_2)^2}\right)\int_{B_R} u_+^2|\nabla\zeta|^2\,dx$$
$$+ \frac{C}{(\nu-\mathcal{B}_2)^2}\int_{B_R} |V - V_{B_R}|^2u_+^2|\nabla\zeta|^2\,dx.$$

PROOF. Choose a sequence $\{u_t\}_{t>0} \subset C^\infty(B_R)$ such that $u_t \to u$ in $W^{1,2}(B_R)$ as $t \to 0$. Also, for each $t > 0$, we choose a sequence $\{\psi_{t,s}\}_{s>0} \subset C^\infty(B_R)$ such that $\psi_{t,s} \to (u_t)_+$ in $W^{1,2}(B_R)$ as $s \to 0$ and $\psi_{t,s} \geq 0$ in B_R. By Lemma 1, $\lim_{t\to 0}(\lim_{s\to 0} \psi_{t,s}) = u_+$ in $W^{1,2}(B_R)$. Taking $\phi = \psi_{t,s}\zeta^2$ in (3), we have

$$\begin{aligned}
&\int_{B_R} (A\nabla u) \cdot \nabla\psi_{t,s}\zeta^2 \, dx \leq -2\int_{B_R} (A\nabla u) \cdot \psi_{t,s}\nabla\zeta\,\zeta \, dx - \int_{B_R} \mathbf{b} \cdot \nabla u\psi_{t,s}\zeta^2 \, dx \\
&\quad = -2\int_{B_R} (A\nabla u) \cdot \psi_{t,s}\nabla\zeta\,\zeta \, dx - \int_{B_R} (\mathbf{b}^{(1)} + \mathbf{b}^{(2)} + \mathbf{b}^{(3)} + \mathbf{b}^{(4)}) \cdot \nabla\psi_{t,s}\psi_{t,s}\zeta^2 \, dx \\
&\qquad - \int_{B_R} \mathbf{b} \cdot \nabla(u - \psi_{t,s})\psi_{t,s}\zeta^2 \, dx.
\end{aligned}$$

Using Lemma 8, for any $\varepsilon_i > 0$ $(i = 2, 3, 4)$ we have

$$\begin{aligned}
&\int_{B_R} (A\nabla u) \cdot \nabla\psi_{t,s}\zeta^2 \, dx \leq -2\int_{B_R} (A\nabla u) \cdot \psi_{t,s}\nabla\zeta\,\zeta \, dx \\
&\quad + (2\mathcal{B}_1 + (1+\varepsilon_2)\mathcal{B}_2 + \varepsilon_3 + \varepsilon_4)\int_{B_R} |\nabla\psi_{t,s}|^2\zeta^2 \, dx \\
&\quad + \left(\frac{\mathcal{B}_1}{3} + \frac{(1+\varepsilon_2)\mathcal{B}_2}{2\{(1+\varepsilon_2)^2 - 1\}} + \varepsilon_3 + \frac{\mathcal{B}_3^2}{2\varepsilon_3}\right)\int_{B_R} \psi_{t,s}^2|\nabla\zeta|^2 \, dx \\
&\quad + \frac{1}{\varepsilon_4}\int_{B_R} |V - V_{B_R}|^2\psi_{t,s}^2|\nabla\zeta|^2 \, dx - \int_{B_R} \mathbf{b} \cdot \nabla(u - \psi_{t,s})\psi_{t,s}\zeta^2 \, dx.
\end{aligned} \tag{20}$$

Next, we prove

$$\lim_{t\to 0}\left(\lim_{s\to 0}\int_{B_R} \mathbf{b} \cdot \nabla(u - \psi_{t,s})\psi_{t,s}\zeta^2 \, dx\right) = 0. \tag{21}$$

We note that $(u_t)_+\zeta^2 \in W_0^{1,2}(B_R) \cap C_c(B_R)$ for any $t > 0$. By Lemma 7, we get

$$\lim_{s\to 0}\int_{B_R} \mathbf{b} \cdot \nabla(u - \psi_{t,s})\psi_{t,s}\zeta^2 \, dx = \int_{B_R} \mathbf{b} \cdot \nabla(u - (u_t)_+)(u_t)_+\zeta^2 \, dx.$$

Since $\nabla(u - (u_t)_+) = \nabla(u - u_t)$ in $\{(u_t)_+ > 0\}$, we have

$$\int_{B_R} \mathbf{b} \cdot \nabla(u - (u_t)_+)(u_t)_+\zeta^2 \, dx = \int_{B_R} \mathbf{b} \cdot \nabla(u - u_t)(u_t)_+\zeta^2 \, dx.$$

Using Lemma 7 again, we obtain (21). Take $s \to 0$ and $t \to 0$ in (20). Hölder' inequality, the John-Nirenberg inequality (11) and (21) yield

$$
\begin{aligned}
\int_{B_R} (A\nabla u)\cdot \nabla u_+ \zeta^2\,dx &\le -2\int_{B_R} (A\nabla u)\cdot u_+\nabla\zeta\,\zeta\,dx \\
&+ (2\mathcal{B}_1 + (1+\varepsilon_2)\mathcal{B}_2 + \varepsilon_3 + \varepsilon_4)\int_{B_R} |\nabla u_+|^2\zeta^2\,dx \\
&+ \left(\frac{\mathcal{B}_1}{3} + \frac{(1+\varepsilon_2)\mathcal{B}_2}{2\{(1+\varepsilon_2)^2-1\}} + \varepsilon_3 + \frac{\mathcal{B}_3^2}{2\varepsilon_3}\right)\int_{B_R} u_+^2|\nabla\zeta|^2\,dx \\
&+ \frac{1}{\varepsilon_4}\int_{B_R} |V - V_{B_R}|^2 u_+^2 |\nabla\zeta|^2\,dx.
\end{aligned}
$$

Since $\nabla u = \nabla u_+$ in $\{u_+ > 0\}$, using Young's inequality (18) and the uniform ellipticity, we get

$$
\begin{aligned}
\nu\int_{B_R} |\nabla u_+|^2\zeta^2\,dx &\le (\varepsilon_0 + 2\mathcal{B}_1 + (1+\varepsilon_2)\mathcal{B}_2 + \varepsilon_3 + \varepsilon_4)\int_{B_R} |\nabla u_+|^2\zeta^2\,dx \\
&+ \left(\frac{L^2}{\varepsilon_0} + \frac{\mathcal{B}_1}{3} + \frac{(1+\varepsilon_2)\mathcal{B}_2}{2\{(1+\varepsilon_2)^2-1\}} + \varepsilon_3 + \frac{\mathcal{B}_3^2}{2\varepsilon_3}\right)\int_{B_R} u_+^2|\nabla\zeta|^2\,dx \\
&+ \frac{1}{\varepsilon_4}\int_{B_R} |V - V_{B_R}|^2 u_+^2|\nabla\zeta|^2\,dx
\end{aligned}
$$

for any $\varepsilon_0 > 0$. Taking

$$
\varepsilon_2 = \frac{1}{2}\left(\frac{\nu}{\mathcal{B}_2} - 1\right), \quad \varepsilon_0 = \varepsilon_3 = \varepsilon_4 = \frac{1}{16}(\nu - \mathcal{B}_2).
$$

and using $\nu \le L$, we arrive at the desired inequality. The proof is complete. □

LEMMA 9. *Let $B_{2R}(x_0) \subset \Omega$. Assume the condition* (B) *on* **b**. *Furthermore we assume* (19). *Suppose u is a weak subsolution of* (DE) *in $B_R(x_0)$. Let $1 < \kappa < \chi := \frac{n}{n-2}$. Then there is a constant C depends only on n, L and κ such that*

$$
\operatorname*{ess\,sup}_{B_{\frac{R}{2}}(x_0)} u_+^{2\kappa} \le C(n, L, \kappa) K_1^{\frac{2\kappa\chi}{\chi-\kappa}} \frac{1}{R^n}\int_{B_R(x_0)} u_+^{2\kappa}\,dx,
$$

where K_1 is the quantity appeared in Theorem 1.

PROOF. We use De Giorgi's method (see e.g., [9, 12]). Without loss of generality, we assume that $x_0 = 0$. Moreover, we may suppose that the right-hand side is positive. From

Proposition 1 and Sobolev's inequality (8), for any $0 < r < R$, we have

$$\begin{aligned}\left(\int_{B_r} (u_+\zeta)^{2\chi}\,dx\right)^{\frac{1}{\chi}} &\le C_S(n)^2 \int_{B_r} |\nabla(u_+\zeta)|^2\,dx \\ &\le C_S(n)^2 \left((C_1+1)\int_{B_r} u_+^2|\nabla\zeta|^2\,dx + C_2\int_{B_r} |V - V_{B_r}|^2 u_+^2 |\nabla\zeta|^2\,dx\right),\end{aligned}$$

for any non-negative $\zeta \in C_c^\infty(B_R)$. Set $R_m = \{\frac{1}{2} + \frac{1}{2^{m+1}}\}R$ for $m = 0, 1, 2, \ldots$. Then, $B_R = B_{R_0} \supset B_{R_1} \supset \cdots \supset B_{R_m} \supset B_{R_{m+1}} \supset \cdots \supset B_{\frac{R}{2}}$. Choosing $\zeta_m \in C_c^\infty(B_R)$ such that

$$\zeta_m|_{B_{Rm+1}} \equiv 1, \quad \operatorname{supp}\zeta_m \subset B_{R_m}, \quad |\nabla\zeta_m| \le \frac{C2^m}{R},$$

we substitute $\zeta = \zeta_m$ as $r = R_m$. From the John-Nirenberg inequality (11) and Hölder's inequality, we get

$$\left(\int_{B_{R_{m+1}}} u_+^{2\chi}\,dx\right)^{\frac{1}{\chi}} \le C_* \|\nabla\zeta_m\|_{L^\infty}^2 |B_{R_m}|^{1-\frac{1}{\kappa}} \left(\int_{B_{R_m}} u_+^{2\kappa}\,dx\right)^{\frac{1}{\kappa}}.$$

Here, $C_* = C(n, L, \kappa)K_1^2$. Therefore, using Hölder's inequality, we have

$$\begin{aligned}\frac{1}{|B_R|}\int_{B_{R_{m+1}}} u_+^{2\kappa}\,dx &\le \left(\frac{1}{|B_R|}\int_{B_{R_{m+1}}} u_+^{2\chi}\,dx\right)^{\frac{\kappa}{\chi}} \cdot \left(\frac{|B_{R_{m+1}} \cap [u > 0]|}{|B_R|}\right)^{1-\frac{\kappa}{\chi}} \\ &\le (C_* 2^{2m})^\kappa \frac{1}{|B_R|}\int_{B_{R_m}} u_+^{2\kappa}\,dx \cdot \left(\frac{|B_{R_{m+1}} \cap [u > 0]|}{|B_R|}\right)^{1-\frac{\kappa}{\chi}},\end{aligned}$$

where $[u > 0] = \{x \in \Omega;\ u(x) > 0\}$. Let $k > 0$ be a positive constant to be chosen later. Put $k_m := (1 - \frac{1}{2^m})k$. We note that $k_0 = 0$ and $k_m \to k$ as $m \to \infty$. Replacing u with $u - k_{m+1}$ and multiplying $k^{-2\kappa}$ to both sides, we have

$$\begin{aligned}&\frac{1}{k^{2\kappa}|B_R|}\int_{B_{R_{m+1}}} (u - k_{m+1})_+^{2\kappa}\,dx \\ &\le (C_* 2^{2m})^\kappa \frac{1}{k^{2\kappa}|B_R|}\int_{B_{R_m}} (u - k_{m+1})_+^{2\kappa}\,dx \cdot \left(\frac{|B_{R_{m+1}} \cap [u > k_{m+1}]|}{|B_R|}\right)^{1-\frac{\kappa}{\chi}}.\end{aligned}$$

From Chebyshev's inequality:

$$\frac{|B_{R_{m+1}} \cap [u > k_{m+1}]|}{|B_R|} \le \frac{(k_{m+1} - k_m)^{-2\kappa}}{|B_R|}\int_{B_{R_{m+1}}} (u - k_m)_+^{2\kappa}\,dx,$$

we have

$$\frac{1}{k^{2\kappa}|B_R|}\int_{B_{R_{m+1}}}(u-k_{m+1})_+^{2\kappa}\,dx$$
$$\leq C_*^{\kappa}2^{2\kappa(1+(1-\frac{\kappa}{\chi}))m}\left(\frac{1}{k^{2\kappa}|B_R|}\int_{B_{R_m}}(u-k_m)_+^{2\kappa}\,dx\right)^{1+(1-\frac{\kappa}{\chi})}.$$

Put

$$C=C_*^{\kappa},\quad b=2^{2\kappa(1+(1-\frac{\kappa}{\chi}))},\quad \alpha=(1-\frac{\kappa}{\chi}),$$

$$x_m=k^{-2\kappa}\frac{1}{|B_R|}\int_{B_{R_m}}(u-k_m)_+^{2\kappa}\,dx.$$

Then the above inequality is rewritten as (12). Furthermore, let

$$k^{2\kappa}=C^{\frac{1}{\alpha}}b^{\frac{1}{\alpha^2}}\frac{1}{|B_R|}\int_{B_R}u_+^{2\kappa}\,dx.$$

Since

$$x_0=k^{-2\kappa}\frac{1}{|B_R|}\int_{B_R}u_+^{2\kappa}\,dx=C^{\frac{-1}{\alpha}}b^{\frac{-1}{\alpha^2}},$$

$x_m\to 0$ as $m\to\infty$ by Lemma 4. On the other hand, if $|\{x\in B_{\frac{R}{2}};\ u(x)\geq k+\varepsilon\}|>0$ for some $\varepsilon>0$, then

$$x_m\geq k^{-2\kappa}\frac{1}{|B_R|}\int_{B_{\frac{R}{2}}}(u-k)_+^{2\kappa}\,dx\geq k^{-2\kappa}\frac{1}{|B_R|}\varepsilon^{2\kappa}|\{x\in B_{\frac{R}{2}};\ u(x)\geq k+\varepsilon\}||>0$$

for all $m=0,1,\ldots$. This contradicts to $x_m\to 0$ as $m\to\infty$. Therefore, we obtain

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}}u_+\leq k.$$

Since $C^{\frac{1}{\alpha}}b^{\frac{1}{\alpha^2}}=C_*^{\kappa(\frac{\chi}{\chi-\kappa})}\cdot 2^{2\kappa\{1+(1-\frac{\kappa}{\chi})\}(\frac{\chi}{\chi-\kappa})^2}\leq C(n,L,\kappa)K_1^{\frac{2\kappa\chi}{\chi-\kappa}}$, we arrived at

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}}u_+^{2\kappa}\leq C(n,L,\kappa)K_1^{\frac{2\kappa\chi}{\chi-\kappa}}\frac{1}{|B_R|}\int_{B_R}u_+^{2\kappa}\,dx.$$

The proof is complete. □

PROOF OF THEOREM 1. Without loss of generality, we assume that $x_0=0$. We use Safonov's idea [20].

Step 1. We will split $B_R\setminus B_{\sigma R}$ into several disjoint spherical shells A_m with the same thickness. We choose $M=[(\frac{\nu-\mathcal{B}_2}{8C_B(n,q)})^{-q}\|\mathbf{b}_1\|_{L^{n,q}(\Omega)}^q]+1$. For $1\leq m\leq M$, we defined as $R_m := \sigma R + (2m-1)\frac{1-\sigma}{2M}R$. We also take $A_m := \{ x \in \Omega ;\ \mathrm{dist}(x, \partial B_{R_m}) < \frac{1-\sigma}{2M} R \}$. Then, by Lemma 2, there exists some $m_* \in \{1, \dots, M\}$ such that $\|\mathbf{b}_1\|_{L^{n,q}(A_{m*})} \le \frac{\nu - \mathcal{B}_2}{8C_B(n,q)}$. Let $B_* := B_{R_{m*}}$. We apply Lemma 9 with $\kappa = \frac{n^2+n}{n^2+n-2}$. We note that $1 < \kappa < \chi$ and $\frac{2\kappa\chi}{\chi-\kappa} = n+1$. Then, for any $y \in \partial B_*$, we have

$$
\begin{aligned}
\operatorname*{ess\,sup}_{B_{\frac{1-\sigma}{4M}R}(y)} u_+^{2\kappa} &\le C(n, L) K_1^{n+1} \frac{1}{(\frac{1-\sigma}{2M}R)^n} \int_{B_{\frac{1-\sigma}{2M}R}(y)} u_+^{2\kappa}\, dx \\
&\le C(n, L) K_1^{n+1} (2M)^n \frac{1}{(1-\sigma)^n R^n} \int_{B_R} u_+^{2\kappa}\, dx. \qquad (22)
\end{aligned}
$$

In particular, u is bounded from above on $A_* := \{ x \in \Omega ;\ \mathrm{dist}(x, \partial B_*) < \frac{1-\sigma}{4M} R \}$.

Step 2. Let us show the inequality

$$
\operatorname*{ess\,sup}_{A_*} u_+ \ge \bar{k} := \operatorname*{ess\,sup}_{B_*} u. \qquad (23)
$$

We use the method for proving the weak maximum principle (see e.g., [8, p.179]). Suppose $\operatorname{ess\,sup}_{A_*} u_+ < \bar{k}$. We choose $\operatorname{ess\,sup}_{A_*} u_+ \le k < \bar{k}$ and define

$$
\psi_{t,s} = \begin{cases} \eta_s * (\eta_t * (u-k))_+ & \text{if } x \in B_*, \\ 0 & \text{otherwise.} \end{cases}
$$

At this, $\psi_{t,s} \in C_c^\infty(\Omega)$ for sufficiently small s and t. Taking $\phi = \psi_{t,s}$ in (3) we have

$$
\begin{aligned}
\int_{B_R} (A\nabla u)\nabla \psi_{t,s}\, dx &\le - \int_{B_R} \mathbf{b} \cdot \nabla u \psi_{t,s}\, dx \\
&= - \int_{B_R} (\mathbf{b}^{(1)} + \mathbf{b}^{(2)}) \cdot \nabla \psi_{t,s} \psi_{t,s}\, dx - \frac{1}{2} \int_{B_R} (\mathbf{b}^{(3)} + \mathbf{b}^{(4)}) \cdot \nabla \psi_{t,s}^2\, dx \\
&\quad - \int_{B_R} \mathbf{b} \cdot \nabla (u - \psi_{t,s}) \psi_{t,s}\, dx.
\end{aligned}
$$

Since $\mathrm{div}(\mathbf{b}^{(3)} + \mathbf{b}^{(4)}) \le 0$ in the distribution sense, the second term of the right-hand side is less than or equal to 0. In the same manner as the proof of Proposition 1, we obtain

$$
\lim_{t \to 0} \left(\lim_{s \to 0} \int_{B_R} \mathbf{b} \cdot \nabla (u - \psi_{t,s}) \psi_{t,s}\, dx \right) = 0 .
$$

Therefore, taking the limits $s \to 0$ and $t \to 0$, we have

$$
\begin{aligned}
\int_{B_*} (A\nabla (u-k)_+) \cdot \nabla (u-k)_+\, dx &= \int_{B_*} (A\nabla u) \cdot \nabla (u-k)_+\, dx \\
&\le - \int_{B_*} (\mathbf{b}^{(1)} + \mathbf{b}^{(2)}) \cdot \nabla (u-k)_+ (u-k)_+\, dx.
\end{aligned}
$$

We split the domain of integration into two parts $\{|\mathbf{b}^{(1)}| > C_*\}$ and $\{|\mathbf{b}^{(1)}| \le C_*\}$ for some $C_* > 0$ to be chosen later. Condition (9) and (4), Hölder's inequality and Sobolev's inequality yield

$$\begin{aligned}
&\nu\left(\int_{B_*}|\nabla(u-k)_+|^2\,dx\right)^{\frac{1}{2}}\\
&\le \left(C_B(n,q)\|\mathbf{b}^{(1)}\|_{L^{n,q}(\{|\mathbf{b}^{(1)}|>C_*\})}+\mathcal{B}_2\right)\left(\int_{B_*}|\nabla(u-k)_+|^2\,dx\right)^{\frac{1}{2}}\\
&+C_*C_S(n)\left(\int_{B_*}|\nabla(u-k)_+|^2\,dx\right)^{\frac{1}{2}}|\Gamma_k|^{\frac{1}{n}}.
\end{aligned}$$

where $\Gamma_k := \{\nabla(u-k)_+ \ne 0\} = \{\, x \in B_*;\ \nabla(u-k)_+(x) \ne 0\,\}$. By the definition of the Lorentz norm, we have $\lim_{C_*\to\infty}\|\mathbf{b}_1\|_{L^{n,q}(\{|\mathbf{b}_1|>C_*\})} = 0$. Thus, by choosing C_* large enough we get

$$|\Gamma_k| \ge C(n,\nu,\mathbf{b}_1,q,\mathcal{B}_2)^{-n} > 0.$$

Here, the constant C does not depend on the selection of $k < \bar{k}$. Since it's well-known $\nabla u = 0$ a.e. on $\{u = \bar{k}\}$, we may assume that $\{u \le k \text{ or } u = \bar{k}\} \subset \{\nabla(u-k)_+ = 0\}$ for any $k < \bar{k}$. Therefore

$$\begin{aligned}
|B_*| &= \left|\left(\bigcup_{k<\bar{k}}\{u\le k\}\right)\cup\{u=\bar{k}\}\right| \le \left|\bigcup_{k<\bar{k}}\{\nabla(u-k)_+=0\}\right|\\
&= |B_*\setminus \bigcap_{k<\bar{k}}\Gamma_k| \le |B_*| - C^{-n} < |B_*|.
\end{aligned}$$

This is impossible. (23) was obtained.

Step 3. Let us combine (22) and (23). Since $B_{\sigma R} \subset B_*$, we have

$$\operatorname*{ess\,sup}_{B_{\sigma R}} u_+^{2\kappa} \le \operatorname*{ess\,sup}_{B_*} u_+^{2\kappa} \le \operatorname*{ess\,sup}_{A_*} u_+^{2\kappa} \le \frac{C}{(1-\sigma)^n R^n}\int_{B_R} u_+^{2\kappa}\,dx,$$

where $C = C(n,L,q)K_1^{n+1}K_2^{qn}$. If $p \ge 2\kappa$, Hölder's inequality yields desired estimate. Let $0 < p < 2\kappa$, from the well-known argument (see, [9, p.223]), for any $\sigma R \le t < s \le R$ we have

$$\operatorname*{ess\,sup}_{B_t} u_+^p \le \frac{1}{2}\operatorname*{ess\,sup}_{B_s} u_+^p + C(p)\frac{C}{(s-t)^n}\int_{B_R} u_+^p\,dx.$$

Using Lemma 5, we arrive at the desired estimate. □

3.3. Proof of Theorem 2

PROPOSITION 2. *Let $B_{4R} \subset \Omega$. Assume the condition* (B) *on* $\mathbf{b}$. *Suppose u is a weak supersolution of* (DE) *in B_{2R} and there exists a positive constant $k > 0$ such that $u \ge k$ on*

B_{2R}. Then there are positive number $p_0 > 0$ and constant C depending only on n, ν, L, $\|\mathbf{b}_1\|_{L^{n,q}(\Omega)}$, q, $\mathcal{B}_2$, $\mathcal{B}_3$, $\|V\|_{BMO(\Omega)}$ such that

$$\left(\frac{1}{|B_R|}\int_{B_R} u^{p_0}\,dx\right)\left(\frac{1}{|B_R|}\int_{B_R} u^{-p_0}\,dx\right) \le C(n).$$

More precisely, $p_0 = C(n, \nu, L, q)K_3^{-1}$, where K_3 is the quantity appeared in Theorem 2.

PROOF. We take any ball $B_{2r}(y) \subset B_{2R}$. We choose $\zeta \in C_c^\infty(B_{2R})$ as the following:

$$\zeta|_{B_r(y)} \equiv 1, \quad \operatorname{supp}\zeta \subset B_{2r}(y), \quad |\nabla\zeta| \le \frac{2}{r}.$$

Let $\widetilde{u} \in W^{1,2}(\mathbb{R}^n)$ be a extension of u and take $u_\varepsilon := \eta_\varepsilon * \max\{\widetilde{u}, k\}$. We note that $\{u_\varepsilon\}_{\varepsilon>0} \subset C^\infty(B_{2R})$, $u_\varepsilon \ge k$ in B_{2R} and $u_\varepsilon \to u$ in $W^{1,2}(B_{2R})$. Taking $\phi = u_\varepsilon^{-1}\zeta^2$ in (3), we have

$$\begin{aligned}
&-\int_{B_{2R}} (A\nabla u)\cdot\nabla(u_\varepsilon^{-1})\zeta^2\,dx \\
&\le 2\int_{B_{2R}} (A\nabla u)\cdot u_\varepsilon^{-1}\nabla\zeta\,\zeta\,dx + \int_{B_{2R}} \mathbf{b}\cdot\nabla u u_\varepsilon^{-1}\zeta^2\,dx \\
&= 2\int_{B_{2R}} (A\nabla u)\cdot u_\varepsilon^{-1}\nabla\zeta\,\zeta\,dx \\
&\quad + \int_{B_{2R}} \left(\mathbf{b}^{(1)} + \mathbf{b}^{(2)} + \mathbf{b}^{(3)} + \mathbf{b}^{(4)}\right)\cdot\nabla u_\varepsilon u_\varepsilon^{-1}\zeta^2\,dx \\
&\quad + \int_{B_{2R}} \mathbf{b}\cdot\nabla(u - u_\varepsilon)u_\varepsilon^{-1}\zeta^2\,dx.
\end{aligned} \tag{24}$$

For $i = 1, 2, 3$, using the Cauchy-Schwarz inequality, we have

$$\left|\int_{B_{2R}} \mathbf{b}^{(i)}\cdot\nabla u_\varepsilon u_\varepsilon^{-1}\zeta^2\,dx\right| \le \left(\int_{B_{2R}} |\nabla(\log u_\varepsilon)|^2\zeta^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_{2R}} |\mathbf{b}^{(i)}|^2\zeta^2\,dx\right)^{\frac{1}{2}}.$$

Furthermore, in a similar manner as in the proof of Lemma 8, we have

$$\begin{aligned}
&\int_{B_{2R}} \mathbf{b}^{(4)}\cdot\nabla u_\varepsilon u_\varepsilon^{-1}\zeta^2\,dx = \int_{B_{2R}} \mathbf{b}^{(4)}\cdot\nabla(\log u_\varepsilon)\zeta^2\,dx \\
&= \sum_{i=1}^n \int_{B_{2R}} \mathbf{b}_i^{(4)}\partial_i(\log u_\varepsilon)\zeta^2\,dx = -\sum_{i,j=1}^n \int_{B_{2R}} V_{ij}\partial_j(\partial_i(\log u_\varepsilon)\zeta^2)\,dx \\
&= -2\int_{B_{2R}} (V - V_{B_{2r}(y)})\nabla(\log u_\varepsilon)\cdot\nabla\zeta\,\zeta\,dx.
\end{aligned}$$

It follows that

$$\left|\int_{B_{2R}} \mathbf{b}^{(4)} \cdot \nabla u_\varepsilon u_\varepsilon^{-1} \zeta^2\, dx\right|$$
$$\leq 2\left(\int_{B_{2R}} |\nabla(\log u_\varepsilon)|^2 \zeta^2\, dx\right)^{\frac{1}{2}} \left(\int_{B_{2R}} |V - V_{B_{2r}(y)}|^2 |\nabla\zeta|^2\, dx\right)^{\frac{1}{2}}.$$

After combining these inequalities, we take the limit $\varepsilon \to 0$. Since $u_\varepsilon^{-1} \leq k^{-1}$, the last term of (24) converges to 0 from Hölder's inequality. Therefore Lemma 1 and the uniform ellipticity yield

$$\nu\left(\int_{B_{2r}(y)} |\nabla(\log u)|^2 \zeta^2\, dx\right)^{\frac{1}{2}}$$
$$\leq C\left(L + C_B(n,q)\|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)} + \mathcal{B}_2 + \mathcal{B}_3 + C(n)\|V\|_{BMO(\Omega)}\right) r^{\frac{n}{2}-1}.$$

Let $v := \log u$. Using Poincaré's inequality, Hölder's inequality and $\mathcal{B}_2 < \nu \leq L$, we get

$$\frac{1}{r^n}\int_{B_r(y)} |v - v_{B_r(y)}|\, dx \leq \frac{C(n,q)}{\nu}\left(L + \|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)} + \mathcal{B}_3 + \|V\|_{BMO(\Omega)}\right).$$

Therefore, we obtain $\|v\|_{BMO(B_{2R})} \leq C(n,\nu,L,q)K_3$. From the John-Nirenberg inequality (10), we have

$$\int_{B_R} \exp\left(p_0|v - v_{B_R}|\right)\, dx \leq C_2(n)R^n,$$

where $p_0 = C(n,\nu,L,q)K_3^{-1}$. Therefore, it follows that

$$\int_{B_R} u^{p_0}\, dx \cdot \int_{B_R} u^{-p_0}\, dx = \int_{B_R} \exp(p_0 v)\, dx \cdot \int_{B_R} \exp(-p_0 v)\, dx$$
$$= \int_{B_R} \exp(p_0(v - v_{B_R}))\, dx \cdot \int_{B_R} \exp(-p_0(v - v_{B_R}))\, dx \leq C^2 R^{2n}.$$

The proof is complete. □

PROOF OF THEOREM 2. Without loss of generality, we assume $x_0 = 0$. Let $k > 0$ and take $\bar{u} = u + k$. Choose a sequence $\{\bar{u}_\varepsilon\}_{\varepsilon>0} \subset C^\infty(B_{2R})$ such that $\bar{u}_\varepsilon \to \bar{u}$ in $W^{1,2}(B_{2R})$ as $\varepsilon \to 0$ and $\bar{u}_\varepsilon \geq k$ in B_{2R}. We may assume that $\bar{u}_\varepsilon \to \bar{u}$ a.e. in B_{2R}. For any $\xi \in C_c^\infty(B_{2R})$ with $\xi \geq 0$, we take $\phi = \bar{u}_\varepsilon^{-p}\xi$ $(p > 1)$ in (3). Then

$$-\int_{B_{2R}} ((A\nabla\bar{u}) \cdot \nabla\xi + \mathbf{b} \cdot \nabla\bar{u}\xi)\, \bar{u}_\varepsilon^{-p}\, dx \leq \int_{B_{2R}} (A\nabla\bar{u}) \cdot \nabla(\bar{u}_\varepsilon^{-p})\xi\, dx.$$

Let $\varepsilon \to 0$. Since $\bar{u}_\varepsilon^{-p} \leq k^{-p}$, we can apply Lebesgue's dominated convergence theorem on the left-hand side. On the other hand, by Lemma 1, $\nabla\bar{u}_\varepsilon^{-p} \to \nabla\bar{u}^{-p} = -p\bar{u}^{-p-1}\nabla\bar{u}$ in

$L^2(B_{2R})$. Therefore, the uniform ellipticity implies

$$\begin{aligned}
&\frac{1}{p-1}\int_{B_{2R}} (A\nabla(\bar{u}^{-p+1}))\cdot\nabla\xi + \mathbf{b}\cdot\nabla(\bar{u}^{-p+1})\xi\,dx \\
&\quad= -\int_{B_{2R}} ((A\nabla\bar{u})\cdot\nabla\xi + \mathbf{b}\cdot\nabla\bar{u}\xi)\,\bar{u}^{-p}\,dx \\
&\quad\le -p\int_{B_{2R}} (A\nabla\bar{u})\cdot\nabla\bar{u}\bar{u}^{-p-1}\xi\,dx \le -p\nu\int_{B_{2R}} |\nabla\bar{u}|^2\bar{u}^{-p-1}\xi\,dx \le 0.
\end{aligned}$$

Hence $\bar{u}^{-p+1}$ is a weak subsolution of (DE) in B_{2R}. Take $p-1=\frac{p_0}{\kappa}$, where p_0 appeared in Proposition 2 and $\kappa=\frac{n^2+n}{n^2+n-2}$ appeared in proof of Lemma 9. Applying Theorem 1 with $p=\kappa$, we have

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}} \bar{u}^{-1} \le C_*^{\frac{1}{p_0}}\left(\frac{1}{R^n}\int_{B_R}\bar{u}^{-p_0}\,dx\right)^{\frac{1}{p_0}}.$$

where $C_*=C(n,L,q)K_1^{n+1}K_2^{qn}$. This and Proposition 2 yield

$$\begin{aligned}
\operatorname*{ess\,inf}_{B_{\frac{R}{2}}} \bar{u} &\ge \frac{1}{C_*^{\frac{1}{p_0}}}\left(\frac{1}{R^n}\int_{B_R}\bar{u}^{-p_0}\,dx\right)^{\frac{-1}{p_0}} \\
&= \frac{1}{C_*^{\frac{1}{p_0}}}\left(\frac{1}{R^n}\int_{B_R}\bar{u}^{-p_0}\,dx\cdot\frac{1}{R^n}\int_{B_R}\bar{u}^{p_0}\,dx\right)^{\frac{-1}{p_0}}\left(\frac{1}{R^n}\int_{B_R}\bar{u}^{p_0}\,dx\right)^{\frac{1}{p_0}} \\
&\ge \frac{1}{(C_*C(n))^{\frac{1}{p_0}}}\left(\frac{1}{R^n}\int_{B_R}\bar{u}^{p_0}\,dx\right)^{\frac{1}{p_0}}.
\end{aligned}$$

Letting $k\to 0$, we can complete the proof of the theorem. □

3.4. Proof of Corollary 4

LEMMA 10. *Let $B_{2R}(x_0)\subset\Omega$. Assume the condition* (B) *on* $\mathbf{b}$. *Suppose u is a weak subsolution of* (DE) *in $B_R(x_0)$. Then there are constants C_1 and C_2 depending only on n, ν, L, $\|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)}$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\|V\|_{BMO(\Omega)}$ such that*

$$\begin{aligned}
\int_{B_{\frac{R}{2}}(x_0)} |\nabla u_+|^2\,dx &\le \frac{C_1}{R^2}\int_{B_R(x_0)\setminus\overline{B_{\frac{R}{2}}(x_0)}} u_+^2\,dx \\
&\quad+\frac{C_2}{R^2}\int_{B_R(x_0)\setminus\overline{B_{\frac{R}{2}}(x_0)}} |V-V_{B_R}|^2u_+^2\,dx.
\end{aligned}$$

PROOF. Without loss of generality, we assume $x_0=0$. We note that

$$\int_{B_R}|\mathbf{b}^{(1)}\cdot\nabla\psi|\psi\zeta^2\,dx \le \frac{\varepsilon_1}{2}\int_{B_R}|\nabla\psi|^2\zeta^2\,dx + \frac{1}{2\varepsilon_1}\int_{B_R}|\mathbf{b}^{(1)}|^2\psi^2\zeta^2\,dx$$

for any $\psi \in C^\infty(B_R)$, $\zeta \in C_c^\infty(B_R)$ and $\varepsilon_1 > 0$ from Young's inequality (18). Therefore, in a similar manner as in the proof of Proposition 1, we obtain

$$\int_{B_R} |\nabla u_+|^2 \zeta^2 \, dx \le C \left(\frac{L^2}{(\nu - \mathcal{B}_2)^2} + \frac{\mathcal{B}_3^2}{(\nu - \mathcal{B}_2)^2} \right) \int_{B_R} u_+^2 |\nabla \zeta|^2 \, dx$$
$$+ \frac{C}{(\nu - \mathcal{B}_2)^2} \int_{B_R} |V - V_{B_R}|^2 u_+^2 |\nabla \zeta|^2 \, dx + \frac{C}{(\nu - \mathcal{B}_2)^2} \int_{B_R} |\mathbf{b}^{(1)}|^2 u_+^2 \zeta^2 \, dx.$$

The last term can be estimated by

$$\int_{B_R} |\mathbf{b}^{(1)}|^2 u_+^2 \zeta^2 \, dx \le \left(C_B(n,q) \|\mathbf{b}^{(1)}\|_{L^{n,q}(\Omega)} \right)^2 \|\nabla \zeta\|_{L^\infty}^2 |B_R| \operatorname*{ess\,sup}_{\operatorname{supp} \zeta} u_+^2.$$

On the other hand, in a similar manner as in the Step 2 of the proof of Theorem 1, $\operatorname{ess\,sup}_{B_{\frac{3R}{4}}} u_+ \le \operatorname{ess\,sup}_{B_{\frac{7R}{8}} \setminus \overline{B_{\frac{5R}{8}}}} u_+$ holds. Using Theorem 1, we get

$$\operatorname*{ess\,sup}_{B_{\frac{3R}{4}}} u_+^2 \le \operatorname*{ess\,sup}_{B_{\frac{7R}{8}} \setminus \overline{B_{\frac{5R}{8}}}} u_+^2 \le \frac{C}{R^n} \int_{B_R \setminus \overline{B_{\frac{R}{2}}}} u_+^2 \, dx.$$

Thus, taking

$$\zeta|_{B_{\frac{R}{2}}} \equiv 1, \quad \operatorname{supp} \zeta \subset B_{\frac{3R}{4}}, \quad |\nabla \zeta| \le \frac{C}{R},$$

and combining these inequalities, we arrive at the desired inequality. □

Recall the following lemma.

LEMMA 11 ([7, p.114, Theorem 6.33]). *Let $g \in L^q_{\mathrm{loc}}(\Omega)$, $q > 1$, be a nonnegative function. Suppose that for some constant $b > 1$ and $R_0 > 0$*

$$\left(\frac{1}{|B_R|} \int_{B_R(x_0)} g^q \, dx \right)^{\frac{1}{q}} \le \frac{b}{|B_{2R}|} \int_{B_{2R}(x_0)} g \, dx$$

holds for all $x_0 \in \Omega$, $0 < R < \min\{R_0, \frac{\operatorname{dist}(x_0, \partial\Omega)}{2}\}$. Then $g \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > q$ and there is a constant $c = c(n, q, p, b)$ such that

$$\left(\frac{1}{|B_R|} \int_{B_R(x_0)} g^p \, dx \right)^{\frac{1}{p}} \le c \left(\frac{1}{|B_{2R}|} \int_{B_{2R}(x_0)} g^q \, dx \right)^{\frac{1}{q}}.$$

PROOF OF COROLLARY 4. Let $B_{2R}(x_0) \subset \Omega$ and $B_r(y) \subset B_R(x_0)$. We use Lemma

10 to $u - u_{B_r(y)}$ and $u_{B_r(y)} - u$ in $B_r(y)$. Then we have

$$\int_{B_{\frac{r}{2}}(y)} |\nabla u|^2 \, dx \le \frac{C}{r^2} \int_{B_r(y)} |u - u_{B_r(x_0)}|^2 \, dx + \frac{C}{r^2} \int_{B_r(y)} |V - V_{B_r(y)}|^2 |u - u_{B_r(x_0)}|^2 \, dx.$$

Let $p := \frac{2n}{n+1} < 2$. Note that $p^* = \frac{np}{n-p} = \frac{2n^2}{n^2-n} > 2$. By Hölder's inequality and the John-Nirenberg inequality (11) with $p = \frac{2p^*}{p^*-2}$,

$$\frac{1}{|B_r(y)|} \int_{B_r(y)} |V - V_{B_r(y)}|^2 |u - u_{B_r(x_0)}|^2 \, dx \le C(n, p) \|V\|^2_{BMO(\Omega)} \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} |u - u_{B_r(x_0)}|^{p^*} \, dx \right)^{\frac{2}{p^*}}.$$

Thus, we have

$$\left(\frac{1}{|B_{\frac{r}{2}}(y)|} \int_{B_{\frac{r}{2}}(y)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \le \frac{C}{r} \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} |u - u_{B_r(x_0)}|^{p^*} \, dx \right)^{\frac{1}{p^*}}.$$

Using Sobolev-Poincaré's inequality:

$$\left(\frac{1}{|B_r(y)|} \int_{B_r(y)} |u - u_{B_r(x_0)}|^{p^*} \, dx \right)^{\frac{1}{p^*}} \le C(n) r \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$

we obtain

$$\left(\frac{1}{|B_{\frac{r}{2}}(y)|} \int_{B_{\frac{r}{2}}(y)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \le C \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$

Applying Lemma 11 with $g^q = |\nabla u|^2$, we have

$$\left(\frac{1}{|B_{\frac{R}{2}}(x_0)|} \int_{B_{\frac{R}{2}}(x_0)} |\nabla u|^{p_1} \, dx \right)^{\frac{1}{p_1}} \le C \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}},$$

for some $p_1 > 2$. The proof is complete. $\square$

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