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The Lefschetz Elements of Coinvariant Algebras of Binary Polyhedral Groups

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Abstract. In this paper, we give a characterization of the Lefschetz elements in a coinvariant algebra of an arbitrary complex reflection group which contains a binary polyhedral group as a subgroup of index 2. In addition, we examine the relation between the set of the non-strong Lefschetz elements and polyhedra.

1. Introduction

The notion of the Lefschetz properties comes from the hard Lefschetz theorem in complex geometry. The hard Lefschetz theorem holds for the flag varieties in particular. The cohomology ring of the flag variety is isomorphic to the coinvariant algebra of the corresponding Weyl group W, which is the regular representation of W.

In general the coinvariant algebras of finite groups are not always isomorphic to the cohomology rings of any manifolds. Indeed the coinvariant algebras of non-crystallographic finite Coxeter groups are not isomorphic to the cohomology rings of any varieties, but have the Lefschetz properties ([6], [9], [10], [13]). See [2], [3], [7], [13], [14], for the study of Lefschetz properties for wider classes of graded Artinian algebras.

In this paper, we give characterizations of the strong Lefschetz elements for an arbitrary complex reflection group G which contains a finite subgroup of $SL(2, \mathbf{C})$ as a subgroup of index 2. Furthermore we also characterize the weak Lefschetz elements for the complex reflection groups, a dihedral group and $G(2n, n, 2)$, which contain a cyclic group and a binary dihedral group respectively. The set of the strong Lefschetz elements is the complement of the union of the zero locus of G-semi-invariant homogeneous polynomials g_i. The set of polynomials g_i contains G-invariants whose zeros correspond to the vertices, the centers of the edges and of the faces of the corresponding regular polyhedrons.

Let $p : S^2 \longrightarrow \mathbf{C} \cup \{\infty\} = \mathbf{P}^1$ be the stereographic projection. If we put $t(g_i) = \{(a : b) \in \mathbf{P}^1 \mid g_i(a, b) = 0\}$, then the inverse image $p^{-1}(t(g_i))$ in S^2 consists of the points corresponding to the vertices, the centers of the edges, the centers of the faces of the regular polyhedrons, and generic points of S^2.

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This paper is organized as follows. In section 2, we collect some basic facts on binary polyhedral groups and complex reflection groups containing binary polyhedral groups. In section 3, we introduce the definition of the coinvariant algebras and the Lefschetz properties. In the remaining sections, we determine the strong Lefschetz elements for an arbitrary complex reflection group containing a finite subgroup of $SL(2, \mathbf{C})$.

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2. Binary polyhedral groups and complex reflection groups

2.1. Binary polyhedral groups. In this section, we explain the basic results on binary polyhedral groups, which show that every finite subgroup of $SL(2, \mathbf{C})$ is isomorphic to a cyclic group or a binary polyhedral group.

It is known that the regular polyhedra are classified into 5 types : regular tetrahedron, cube, regular octahedron, regular dodecahedron, regular icosahedron. When the barycenter of a regular polyhedron is put on the origin of three-dimensional Euclidean space $\mathbf{R}^3$, the vertices of a regular polyhedron are on the sphere S^2.

For a regular polyhedron Δ, let Γ_Δ be the subgroup of rotation group $SO(3)$ consisting of the rotations that transform Δ into itself.

$$\Gamma_\Delta = \{g \in SO(3) \mid g(\Delta) = \Delta\}$$

Besides the groups of the regular polyhedra, there are two more infinite series of finite subgroups of $SO(3)$: cyclic groups and dihedral groups.

Any convex solid has a dual namely the convex hull of the barycenters of its faces. By this convex duality, the octahedron and cube, resp. the icosahedron and dodecahedron, give rise to the same group.

Now S^2 is considered as the Riemannian sphere, i.e. as the complex number plane $\mathbf{C}$ of the complex variable z compactified by one point ∞. We identify the sphere S^2 with $\mathbf{C} \cup \{\infty\}$ by the stereographic projection.

After identifying $\mathbf{C} \cup \{\infty\}$ with the complex projective line $\mathbf{P}^1(\mathbf{C})$, any rotation $g \in SO(3)$ acting on S^2 corresponds to a fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$ of $\mathbf{C} \cup \{\infty\}$. By this correspondence, $SO(3)$ is considered as the subgroup of complex projective transformation group

$$PGL(2, \mathbf{C}) = PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\langle \pm 1 \rangle$$

which is the automorphism group of $\mathbf{P}^1(\mathbf{C})$.

The inverse image of $SO(3)$ under the projection

$$\pi : SL(2, \mathbf{C}) \longrightarrow PSL(2, \mathbf{C})$$

is the special unitary group :

$$SU(2) = \pi^{-1}(SO(3)) .$$

DEFINITION 1. A subgroup of $SL(2, \mathbf{C})$ which is conjugate to the inverse image $\pi^{-1}(\Gamma)$ of the finite subgroup Γ of $SO(3)$ is called a binary subgroup corresponding to Γ. We denote this by G.

When Γ is a polyhedral group, G is called a binary polyhedral group.

Since the kernel of π is $\{I, -I\}$ (I is the unit matrix), we have

$$| G |= 2 | \Gamma | .$$

THEOREM 1 (See e.g. [12]). *There are the following conjugacy classes of finite subgroups of* $SL(2, \mathbf{C})$.

1. *cyclic groups* (*order* n, $n \in \mathbf{N}$)
2. *binary dihedral groups* (*order* $4n$, $n \in \mathbf{N}$)
3. *binary tetrahedral groups* (*order* 24)
4. *binary octahedral groups* (*order* 48)
5. *binary icosahedral groups* (*order* 120)

In this paper, the polyhedral group is the general term for dihedral groups and the groups of the regular polyhedra.

EXAMPLE 1. (1) cyclic group C_n (order n)

$$C_n = \left\{ \begin{pmatrix} \theta^h & 0 \\ 0 & \theta^{-h} \end{pmatrix} \middle| (0 \leq h < n) \right\}, \quad \theta = e^{\frac{2\pi i}{n}}, \quad (n \geq 2)$$

(2) binary dihedral group $\widetilde{D}_n$ (order $4n$)

$$\widetilde{D}_n = \left\langle \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \quad \theta = e^{\frac{\pi i}{n}}, \quad (n \geq 3)$$

2.2. Complex reflection groups including binary polyhedral groups. A reflection on complex vector space is a congruent transformation of finite period that leaves invariant every point of a hyperplane, and it is characterized by the property that all but one of the characteristic roots of the matrix of transformation are equal to unity.

Let G a finite subgroup of $SL(2, \mathbf{C})$. There exists a complex reflection group $\tilde{G}$ of $GL(2, \mathbf{C})$ containing G with $[\tilde{G} : G] = 2$ ([11]).

We list all such pairs G and $\tilde{G}$.

We use the notation in [11].

G	order	$\tilde{G}$	order
cyclic group	n	dihedral group	$2n$
binary dihedral group	$4n$	$G(2n, n, 2)$	$8n$
binary tetrahedral group	24	G_{12}	48
binary octahedral group	48	G_{13}	96
binary icosahedral group	120	G_{22}	240

EXAMPLE 2. (1) dihedral group D_n

$$C_n \subset D_n = \left\langle \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \theta = e^{\frac{2\pi i}{n}}, \quad (n \geq 3)$$

(2) $G(2n, n, 2)$

$$\widetilde{D_n} \subset G(2n, n, 2) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} \right\rangle, \quad \theta = e^{\frac{\pi i}{n}}, \quad (0 \leq k \leq 2n-1)$$

3. Coinvariant algebra

In this section we define the coinvariant algebra for a finite group and introduce the notion of the strong Lefschetz property and the weak Lefschetz property for graded Artinian algebras over the field $\mathbf{C}$.

3.1. Invariant algebra. Let V denote a complex vector space of dimension n and V^* its dual vector space. If $Sym(V^*)$ denotes the symmetric algebra over $V^* = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$, the polynomial algebra on V is

$$S(V) = Sym(V^*) = \bigoplus_{j=0}^{\infty} S_j(V),$$

where $S_j(V)$ is the space of homogeneous polynomials of degree j.

If we choose a basis $\{e_1, e_2, \ldots, e_n\}$ for V and let $\{x_1, x_2, \ldots, x_n\}$ denote the dual basis of V^*, i.e. $x_i(e_j) = \delta_{ij}$, then $S(V)$ can be identified with the polynomial algebra $\mathbf{C}[x_1, x_2, \ldots, x_n]$:

$$S(V) = \mathbf{C}[x_1, x_2, \ldots, x_n] = Sym(V^*)$$

Let F be a finite group acting on a complex vector space V. We define the action of F on $S(V)$ by

$$g \cdot \phi(z) = \phi(g^{-1}(z)), \quad g \in F, \quad \phi \in S(V), \quad z \in V.$$

DEFINITION 2. For all $g \in F$, we call $\phi \in S(V)$ an F-invariant if

$$g \cdot \phi = \phi \,.$$

The set of F-invariants

$$S(V)^F := \{\phi \in S(V) \mid g \cdot \phi = \phi\}$$

forms a subalgebra of $S(V)$ and is called a invariant algebra.

THEOREM 2 (See e.g. [12], [chap. 4]). *The following properties of the finite group F are equivalent.*

1. *F is a finite reflection group.*
2. *$S(V)$ is a free graded module over $S(V)^F$ with a finite basis.*
3. *$S(V)^F$ is generated by n algebraically independent homogeneous elements.*

3.2. Coinvariant algebra and the Lefschetz properties

DEFINITION 3. If F acts on V, let I_F denote the homogeneous ideal in $S(V)$ generated by $S(V)^F_+ = \bigoplus_{i>0} S_i(V)^F$.

Then we call the graded quotient algebra

$$S(V)_F = S(V)/I_F = \bigoplus_{i=0}^{\infty} S_i(V)_F$$

the coinvariant algebra.

For a graded vector space $R = \bigoplus_{i\geq 0} R_i$, over $\mathbf{C}$, let

$$P_R(t) = \sum_{i\geq 0} \dim_{\mathbf{C}}(R_i) t^i$$

be the Poincaré series of R.

By Theorem 2, we have

$$\begin{aligned} P_{S(V)_F}(t) &= \frac{P_{S(V)}(t)}{P_{S(V)^F}(t)} \,, \\ P_{S(V)}(t) &= (1-t)^{-n}, \\ P_{S(V)^F}(t) &= \prod_{i=1}^{n} (1-t^{d_i})^{-1} \,, \end{aligned} \tag{3}$$

where $n = \dim(V)$ and $d_1, d_2, \ldots, d_n$ are the degrees of algebraically independent generators of $S(V)^F$.

We list the degrees of algebraically independent generators of the complex reflection group $\tilde{G}$ of $GL(2, \mathbf{C})$ containing G ([11]).

TABLE 1

$\tilde{G}$	degree
dihedral group	2,n
$G(2n, n, 2)$	4,2n
G_{12}	6,8
G_{13}	8,12
G_{22}	12,20

DEFINITION 4. Let $A = \bigoplus_{i=0}^{D} A_i$, $A_D \neq 0$, be a commutative Artinian graded algebra.

(1) We say that A has the strong Lefschetz property if there exists an element $l \in A_1$ such that the map given by multiplying l^{D-2i}

$$\sigma_i : A_i \longrightarrow A_{D-i}$$

is bijective for $i = 0, \dots, [\frac{D}{2}]$.

We call $l \in A_1$ with this property a strong Lefschetz element.

(2) We say that A has the weak Lefschetz property if there exists an element $l \in A_1$ such that the map given by multiplying l

$$\tau_i : A_i \longrightarrow A_{i+1}$$

is of full rank for $i = 0, \dots, D-1$.

We call $l \in A_1$ with this property a weak Lefschetz element.

Let G be a finite subgroup of $SL(2, \mathbf{C})$ acting on a 2-dimensional complex vector space V. Let $\tilde{G}$ be a complex reflection group containing G as a subgroup of index 2. Then for the coinvariant algebras

$$S(V)_{\tilde{G}} = \bigoplus_{i=0}^{\bar{d}} S_i(V)_{\tilde{G}}\,, \quad S(V)_G = \bigoplus_{i=0}^{d} S_i(V)_G\,,$$

we have (see e.g. [1])

PROPOSITION 1.
1. $\bar{d} = d + 1$
2. $I_{\tilde{G}} = I_G \bigoplus \mathbf{C}f$, $S_{\bar{d}}(V)_{\tilde{G}} = \mathbf{C}f$.
3. *The algebra homomorphism $\varphi : S(V)_{\tilde{G}} \to S(V)_G$ induces the isomorphisms $S_i(V)_{\tilde{G}} \to S_i(V)_G$ for $i \leq d$ as vector spaces.*

This proposition shows that the studies of the Lefschetz elements of $S(V)_{\tilde{G}}$ and $S(V)_G$ are much the same. Therefore we investigate the Lefschetz properties for $\tilde{G}$. Since the coinvariant algebra $S(V)_{\tilde{G}}$ is the quotient of $\mathbf{C}[x, y]$, $S(V)_{\tilde{G}}$ has the strong Lefschetz property ([2]). In the following sections, we give characterizations of the strong Lefschetz elements for $\tilde{G}$.

For later use, we introduce the following notations.

1. Let $\rho : S(V) \longrightarrow S(V)_G$ the homomorphism from $S(V)$ to the coinvariant algebra $S(V)_G := S(V)/I_G$ given by $x = \rho(X)$, $y = \rho(Y)$ $(X, Y \in S(V))$.
2. Let

$$p : S^2 \longrightarrow \mathbf{C} \cup \{\infty\} = \mathbf{P}^1$$

be the stereographic projection. For a homogeneous polynomial $g_i \in \mathbf{C}[x, y]$, put

$$t(g_i) = \{(a : b) \in \mathbf{P}^1 \mid g_i(a, b) = 0\}$$

and

$$B(g_i) := p^{-1}(t(g_i)) .$$

4. The Lefschetz elements for dihedral group

We recall that the dihedral group of order $2n$ contains a cyclic group of order n as a subgroup of index 2. In this section, we give a characterization of the strong (weak) Lefschetz element of the coinvariant algebras of the dihedral groups.

Let G be a dihedral group :

$$G = \left\langle \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle , \quad \theta = e^{\frac{2\pi i}{n}} , \quad n \geq 3 .$$

The invariant algebra of G is

$$S(V)^G = \mathbf{C}[X^n + Y^n, XY] .$$

By (3) in 3.2 and Table 1, we have

$$P_{S(V)_G}(t) = \frac{(1-t^2)(1-t^n)}{(1-t)^2} = 1 + 2t + 2t^2 + \cdots + 2t^{n-1} + t^n .$$

We choose a basis of $S(V)_G$

$$\{1, x, y, x^2, y^2, \ldots, x^{n-1}, y^{n-1}, x^n\} . \tag{4.0}$$

THEOREM 3. (1) *The element $l = ax + by \in S_1(V)_G$ is the strong Lefschetz element if and only if $\prod_{i=1}^{3} g_i(a, b) \neq 0$, where g_i, $1 \leq i \leq 3$, are the polynomials in the variables a, b given by*

$$g_1 = a , \quad g_2 = b , \quad g_3 = a^n - b^n .$$

(*See the end of* §3 *for the notation.*)

(2) *The element* $l = ax + by \in S_1(V)_G$ *is the weak Lefschetz element if and only if* $a \neq 0$ *and* $b \neq 0$.

(3) *We have the inverse images of* $t(g_i)$ *under the stereographic projection as follows* (*see the end of* §3 *for the notation*) :

$$B(g_1) = \{(0, 0, -1)\},$$

$$B(g_2) = \{(0, 0, 1)\},$$

$$B(g_3) = \left\{ \left(\cos \frac{2\pi k}{n}, \sin \frac{2\pi k}{n}, 0 \right) \middle| \, 0 \leq k \leq n-1 \right\}.$$

We take an n*-sided regular polygon on* XY*-plane. Now if we put a vertex of the regular polygon on* $(1, 0, 0)$, *the set* $B(g_i)$ *are given as follows*:

(i) $B(g_1) \cup B(g_2)$ *is the set of points corresponding to the centers of the faces of the regular polygon.*

(ii) $B(g_3)$ *is the set of* n *vertices of the regular polygon.*

PROOF. (1) Since $I_G = (X^n + Y^n, XY)$, we have

$$xy = 0, \quad x^n + y^n = 0. \tag{4.1}$$

By (4.1), the matrix representation of σ_0 and σ_i $(1 \leq i \leq [\frac{n}{2}])$ with respect to the basis (4.0) are

$$(a^n - b^n), \quad \begin{pmatrix} a^{n-2i} & 0 \\ 0 & b^{n-2i} \end{pmatrix}$$

respectively. Thus we get (1).

(2) By (4.1), the matrix representation of τ_0, τ_i $(1 \leq i \leq n-2)$, τ_{n-1} with respect to the basis (4.0) are

$$\begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad (a \ \ -b)$$

respectively. Thus we get (2).

(3) For a homogeneous polynomial g_i, the points $(a : b) \in t(g_i)$ are as follows:

$$t(g_1) = \{(0 : 1)\},$$

$$t(g_2) = \{(1 : 0)\},$$

$$t(g_3) = \{(\theta^k : 1) \mid k = 0, \ldots, n-1\}, \ \theta = e^{\frac{2\pi\sqrt{-1}}{n}}.$$

We identify the sphere S^2 with $\mathbf{C} \cup \{\infty\}$ by the stereographic projection p :

$$\xi = \frac{2\alpha}{1+\alpha^2+\beta^2}\,, \quad \eta = \frac{2\beta}{1+\alpha^2+\beta^2}\,, \quad \zeta = \frac{-1+\alpha^2+\beta^2}{1+\alpha^2+\beta^2}\,, \qquad (*)$$

where

$$\frac{a}{b} = \alpha + \sqrt{-1}\beta\,.$$

Therefore for g_i, we get the sets $B(g_i)$ $(1 \leq i \leq 3)$ which is written in the theorem above.

□

5. The Lefschetz elements for $G(2n, n, 2)$

We recall that $G(2n, n, 2)$ of order $8n$ contains a binary dihedral group of order $4n$ as a subgroup of index 2. In this section we give a characterization of the strong (weak) Lefschetz element of the coinvariant algebras of $G(2n, n, 2)$.

Let $G = G(2n, n, 2)$:

$$G = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} \right\rangle, \quad (\theta = e^{\frac{\pi i}{n}})\,.$$

The invariant algebra of G is

$$S(V)^G = \mathbf{C}[X^{2n} + Y^{2n}, X^2Y^2]\,.$$

By (3) in 3.2 and Table 1, we have

$$\begin{aligned} P_{S(V)_G}(t) &= \frac{(1-t^4)(1-t^{2n})}{(1-t)^2} \\ &= 1 + 2t + 3t^2 + 4t^3 + \cdots + 4t^{2n-1} + 3t^{2n} + 2t^{2n+1} + t^{2n+2}\,. \end{aligned}$$

We choose a basis of $S(V)_G$

$$\begin{aligned} \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots, x^{2n-1}, x^{2n-2}y, xy^{2n-2}, y^{2n-1}, \\ x^{2n}, x^{2n-1}y, xy^{2n-1}, x^{2n}y, xy^{2n}, x^{2n+1}y\}\,. \end{aligned} \qquad (5.0)$$

THEOREM 4. (1) *The element $l = ax + by \in S_1(V)_G$ is the strong Lefschetz element if and only if $\prod_{i=1}^4 g_i \neq 0$, where g_i are the polynomials in a, b given by*

$$g_1(a,b) = a\,,\ g_2(a,b) = b\,,\ g_3(a,b) = a^{2n} - b^{2n}\,,\ g_4(a,b) = 4n^2a^{2n}b^{2n} + (a^{2n} - b^{2n})^2\,.$$

(*See the end of* §3 *for the notation*)

(2) *The element $l = ax + by \in S_1(V)_G$ is the weak Lefschetz element if and only if $a \neq 0$ and $b \neq 0$.*

(3) *We have the inverse images of $t(g_i)$ under the stereographic projection as follows* (*see the end of* §3 *for the notation*) :

$$B(g_1) = \{(0,0,-1)\},$$
$$B(g_2) = \{(0,0,1)\},$$
$$B(g_3) = \left\{\left(\cos\frac{\pi k}{n}, \sin\frac{\pi k}{n}, 0\right)\right\}, \ (k = 0, \dots, 2n-1).$$

We take an n-sided regular polygon on XY-plane. If we put a vertex of the regular polygon on $(1,0,0)$, *the sets $B(g_i)$ are given as follows* :

(i) *$B(g_1) \cup B(g_2)$ is the set of points corresponding to the centers of the faces of the regular polygon.*

(ii) *$B(g_3)$ is the set of n vertices of the regular polygon and n points corresponding to the centers of the edges of the regular polygon.*

PROOF. (1) Since $I = (X^{2n} + Y^{2n}, X^2Y^2)$, we have

$$x^2y^2 = 0, \quad x^{2n} + y^{2n} = 0. \tag{5.1}$$

By (5.1), the matrix representation of $\sigma_0, \sigma_1, \sigma_2, \sigma_i$ $(3 \le i \le n+1)$ with respect to the basis (5.0) are

$$\left((2n+2)ab(a^{2n}-b^{2n})\right), \begin{pmatrix} 2na^{2n-1}b & a^{2n}-b^{2n} \\ -a^{2n}+b^{2n} & 2nab^{2n-1} \end{pmatrix},$$

$$\begin{pmatrix} a^{2n-2} & 0 & -b^{2n-2} \\ (2n-2)a^{2n-3}b & a^{2n-2} & 0 \\ 0 & b^{2n-2} & (2n-2)ab^{2n-3} \end{pmatrix},$$

$$\begin{pmatrix} a^{2n+2-2i} & 0 & 0 & 0 \\ (2n+2-2i)a^{2n+1-2i}b & a^{2n+2-2i} & 0 & 0 \\ 0 & 0 & b^{2n+2-2i} & (2n+2-2i)ab^{2n+1-2i} \\ 0 & 0 & 0 & b^{2n+2-2i} \end{pmatrix}$$

respectively. Thus we get (1).

(2) By (5.1), the matrix representation of $\tau_0, \tau_1, \tau_2, \tau_i\,(3 \le i \le 2n-2), \tau_{2n-1}, \tau_{2n}, \tau_{2n+1}$ with respect to the basis (5.0) are

$$\begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ b & a \\ 0 & b \end{pmatrix}, \quad \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & b & a \\ 0 & 0 & b \end{pmatrix}, \quad \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & b & a \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \begin{pmatrix} a & 0 & 0 & -b \\ b & a & 0 & 0 \\ 0 & 0 & b & a \end{pmatrix}$$

$$\begin{pmatrix} b & a & 0 \\ -a & 0 & b \end{pmatrix}, \quad \begin{pmatrix} a & -b \end{pmatrix}$$

respectively. Thus we get (2).

(3) For a homogeneous polynomial g_i, the points $(a : b) \in t(g_i)$ are as follows:

$$t(g_1) = \{(0:1)\},$$
$$t(g_2) = \{(1:0)\},$$
$$t(g_3) = \{(\theta^k : 1) \mid k = 0, \dots, 2n-1\},\ \theta = e^{\frac{\pi\sqrt{-1}}{n}}.$$

We identify the sphere S^2 with $\mathbf{C} \cup \{\infty\}$ by the stereographic projection p (see the relations $(*)$ of the end of §4).

Therefore for g_i, we get the sets $B(g_i)$ $(1 \le i \le 3)$ which is written in the theorem above.

□

6. The Lefschetz elements for G_{12}

We recall that the group G_{12} of order 48 contains a binary tetrahedral group of order 24 as a subgroup of index 2. In this section we give a characterization of the strong Lefschetz element of the coinvariant algebras of G_{12}.

Let $G = G_{12}$:

$$G = \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon^3 & \varepsilon^7 \end{pmatrix} \right\rangle, \quad \left(\varepsilon = e^{\frac{2\pi i}{8}}\right).$$

The invariant algebra of G is

$$S(V)^G = \mathbf{C}\big[XY(X^4 - Y^4), X^8 + 14X^4Y^4 + Y^8\big].$$

By (3) in 3.2 and Table 1, we have

$$\begin{aligned} P_{S(V)_G}(t) &= \frac{(1-t^6)(1-t^8)}{(1-t)^2} \\ &= 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 6t^6 + 6t^7 + 5t^8 + 4t^9 + 3t^{10} + 2t^{11} + t^{12}. \end{aligned}$$

We choose a basis of $S(V)_G$

$$\begin{aligned} \big\{&1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, \\ &x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, x^6, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, \\ &x^7, x^5y^2, x^4y^3, x^3y^4, x^2y^5, y^7, x^8, x^5y^3, x^3y^5, x^2y^6, y^8, \\ &x^9, x^6y^3, x^3y^6, y^9, x^{10}, x^3y^7, y^{10}, x^{11}, y^{11}, y^{12}\big\} \end{aligned} \tag{6, 0}$$

THEOREM 5. (1) *The element* $l = ax + by \in S_1(V)_G$ *is the strong Lefschetz element if and only if* $\prod_{i=1}^{12} g_i \neq 0$, *where* g_i *are the polynomials in* a, b *given by*

$$g_1 = a\,, \quad g_2 = b\,, \quad g_3 = a - b\,, \quad g_4 = a + b\,, \quad g_5 = a^2 + b^2\,,$$

$$g_6 = a^2 + 2ab - b^2\,, \quad g_7 = a^2 - 2ab - b^2\,, \ g_8 = a^4 + b^4\,, \ g_9 = a^4 + 6a^2b^2 + b^4\,,$$

$$g_{10} = a^4 + 2a^3b + 2a^2b^2 - 2ab^3 + b^4\,, \quad g_{11} = a^4 - 2a^3b + 2a^2b^2 + 2ab^3 + b^4\,,$$

$$g_{12} = 5a^{24} + 642a^{20}b^4 + 1227a^{16}b^8 + 16732a^{12}b^{12} + 1227a^8b^{16} + 642a^4b^{20} + 5b^{24}\,.$$

(*See the end of* §3 *for the notation.*)

(2) *We have the inverse images of* $t(g_i)$ $(1 \le i \le 11)$ *under the stereographic projection as follows* (*see the end of* §3 *for the notation*) :

$$B(g_1) = \{(0, 0, -1)\}\,,$$

$$B(g_2) = \{(0, 0, 1)\}\,,$$

$$B(g_3) = \{(1, 0, 0)\}\,,$$

$$B(g_4) = \{(-1, 0, 0)\}\,,$$

$$B(g_5) = \{(0, 1, 0), (0, -1, 0)\}\,,$$

$$B(g_6) = \left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\},$$

$$B(g_7) = \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)\right\},$$

$$B(g_8) = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right\},$$

$$B(g_9) = \left\{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\},$$

$$B(g_{10}) = \left\{\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)\right\},$$

$$B(g_{11}) = \left\{\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)\right\}.$$

There are, *in a cube*, *two tetrahedra whose edges are the diagonals of the faces of the cube. The dual of the cube is the octahedron. Now if we take an octahedron with vertices* $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ *in* R^3, *the sets* $B(g_i)$ *are given as follows* :

(i) *The sets of* $B(g_1), B(g_2), B(g_3), B(g_4), B(g_5)$ *are the six points of* S^2 *corresponding to the centers of the edges of the tetrahedron* (*or the six vertices of the octahedron*).

(ii) *The sets of* $B(g_6), B(g_7), B(g_8), B(g_9)$ *are the twelve points of* S^2 *corresponding to the centers of the edges of the octahedron.*

(iii) *The sets of* $B(g_{10}), B(g_{11})$ *are the four points of* S^2 *corresponding to the vertices of the tetrahedron and the four points of* S^2 *corresponding to the centers of* *the faces of the tetrahedron* (*or the eight points of* S^2 *corresponding to the centers of the faces of the octahedron*).

PROOF. (1) Since $I_G = (XY(X^4 - Y^4), X^8 + 14X^4Y^4 + Y^8)$, we have

$$xy(x^4 - y^4) = 0\,, \tag{6.1}$$

$$x^8 + 14x^4y^4 + y^8 = 0\,. \tag{6.2}$$

For later use, we prepare the following relations. By (6.1), we have

$$x^9y = (xy^5)x^4 = x^5y^5 = (xy^5)y^4 = xy^9. \tag{6.3}$$

Moreover we have

$$\begin{aligned} x^5y^5 &= -\frac{1}{14}xy(x^8 + y^8) \text{ (by (6.2))} \\ &= -\frac{1}{7}x^5y^5. \text{ (by (6.3))} \end{aligned}$$

Thus we have $x^5y^5 = 0$. Therefore

$$x^5y^5 = x^9y = xy^9 = 0\,. \tag{6.4}$$

By (6.1), (6.4), we have

$$x^iy^j = x^{i-4}y^{j+4}, \quad i \geq j \geq 1 \tag{6.5}$$

and

$$x^iy^j = 0, \quad i \geq 9,\ j \geq 1 \text{ or } i \geq 1,\ j \geq 9\,. \tag{6.6}$$

By (6.2), (6.5), we have

$$x^8y^j = x^4y^{j+4} = -\frac{1}{15}y^{8+j}, \quad 1 \leq j \leq 4\,, \tag{6.7}$$

$$x^iy^8 = x^{i+4}y^4 = -\frac{1}{15}x^{8+i}, \quad 1 \leq i \leq 4\,. \tag{6.8}$$

(i) By (6.5), ..., (6.8), we have

$$\begin{aligned} x^{12} = y^{12}, x^{11}y &= x^{10}y^2 = x^9y^3 = x^7y^5 = x^6y^6 = x^5y^7 = x^3y^9 \\ &= x^2y^{10} = xy^{11} = 0, x^8y^4 = x^4y^8 = -\frac{1}{15}x^{12} = -\frac{1}{15}y^{12}. \end{aligned} \tag{6.9}$$

For $i = 0$, by (6.9), the matrix representation of σ_0 with respect to the basis (6, 0) is regular if and only if $a^{12} - 33a^8b^4 - 33a^4b^8 + b^{12} \neq 0$, i.e. $(a^2 + 2ab - b^2)(a^2 - 2ab - b^2)(a^4 - b^4)(a^4 + 6a^2b^2 + b^4) \neq 0$.

(ii) By (6.5), ..., (6.8), we have

$$x^{10}y = x^9y^2 = x^6y^5 = x^5y^6 = x^2y^9 = xy^{10} = 0\,,$$
$$x^8y^3 = x^4y^7 = -\frac{1}{15}y^{11},\, x^7y^4 = x^3y^8 = -\frac{1}{15}y^{12}\,. \tag{6.10}$$

For $i = 1$, by (6.10), the matrix representation of σ_1 with respect to the basis (6, 0) is

$$\begin{pmatrix} a^{10} - 14a^6b^4 - 3a^2b^8 & -8a^3b^3(a^4+b^4) \\ -8a^3b^3(a^4+b^4) & -3a^8b^2 - 14a^4b^6 + b^{10} \end{pmatrix}.$$

Therefore σ_1 is bijective if and only if $-3a^2b^2(a-b)^2(a+b)^2(a^2+b^2)^2(a^4+2a^3b+2a^2b^2-2ab^3+b^4)(a^4-2a^3b+2a^2b^2+2ab^3+b^4) \neq 0$.

(iii) By (6.5), ..., (6.8), we have

$$x^5y^5 = x^9y = xy^9 = 0\,, \quad x^7y^3 = x^3y^7\,,$$
$$x^8y^2 = x^4y^6 = -\frac{1}{15}y^{10}\,, \quad x^6y^4 = x^2y^8 = -\frac{1}{15}x^{10}\,. \tag{6.11}$$

For $i = 2$, by (6.11), the matrix representation of σ_2 with respect to the basis (6, 0) is

$$\begin{pmatrix} \frac{1}{15}(15a^8 - 70a^4b^4 - b^8) & -\frac{8}{15}ab(7a^4b^2+b^6) & -\frac{28}{15}a^2b^2(a^4+b^4) \\ 8ab(7a^4b^2+b^6) & 28a^2b^2(a^4+b^4) & 8ab(7a^2b^4+a^6) \\ -\frac{28}{15}a^2b^2(a^4+b^4)y^{10} & -\frac{8}{15}ab(7a^2b^4+a^6) & -\frac{1}{15}(a^8+70a^4b^4-15b^8) \end{pmatrix}.$$

Therefore σ_2 is bijective if and only if $\frac{12}{5}a^2b^2(a-b)^2(a+b)^2(a^2+b^2)^2(a^2+2ab-b^2)(a^2-2ab-b^2)(a^4+b^4)(a^4+6a^2b^2+b^4) \neq 0$.

(iv) By (6.5), (6.7), (6.8), we have

$$x^7y^2 = x^3y^6\,, \quad x^6y^3 = x^2y^7,$$
$$x^8y = x^4t^5 = -\frac{1}{15}y^9\,, \quad x^5y^4 = xy^8 = -\frac{1}{15}x^9\,. \tag{6.12}$$

For $i = 3$, by (6.12), the matrix representation of σ_3 with respect to the basis (6, 0) is

$$\begin{pmatrix} a^2(a^4-b^4) & -\frac{4}{3}a^3b^3 & -\frac{1}{15}b^2(15a^4+b^4) & -\frac{2}{5}ab(a^4+b^4) \\ 20a^3b^3 & b^2(15a^4+b^4) & 6ab(a^4+b^4) & a^2(a^4+15b^4) \\ b^2(15a^4+b^4) & 6ab(a^4+b^4) & a^2(a^4+15b^4) & 20a^3b^3 \\ -\frac{2}{5}ab(a^4+b^4) & -\frac{1}{15}a^2(a^4+15b^4) & -\frac{4}{3}a^3b^3 & b^2(-a^4+b^4) \end{pmatrix}.$$

Therefore σ_3 is bijective if and only if $\frac{1}{71}(5a^{24} + 642a^{20}b^4 + 1227a^{16}b^8 + 16732a^{12}b^{12} + 1227a^8b^{16} + 642a^4b^{20} + 5b^{24}) \neq 0$.

(v) By (6.5), we have

$$x^7y = x^3y^5\,, \quad x^6y^2 = x^2y^6\,, \quad x^5y^3 = xy^7\,. \tag{6.13}$$

For $i = 4$, by (6.13), the matrix representation of σ_4 with respect to the basis (6, 0) is

$$\begin{pmatrix} a^4 - \frac{1}{14}b^4 & -\frac{2}{7}ab^3 & -\frac{3}{7}a^2b^2 & -\frac{2}{7}a^3b & -\frac{1}{14}a^4 \\ 4ab^3 & 6a^2b^2 & 4a^3b & a^4+b^4 & 4ab^3 \\ 4a^3b & a^4+b^4 & 4ab^3 & 6a^2b^2 & 4a^3b \\ 6a^2b^2 & 4a^3b & a^4+b^4 & 4ab^3 & 6a^2b^2 \\ -\frac{1}{14}b^4 & -\frac{2}{7}ab^3 & -\frac{3}{7}a^2b^2 & -\frac{2}{7}a^3b & -\frac{1}{14}a^4+b^4 \end{pmatrix}$$

Therefore σ_4 is bijective if and only if $-\frac{1}{14}(a^2+2ab-b^2)(a^2-2ab-b^2)(a^4+b^4)(a^4+6a^2b^2+b^4)(a^4+2a^3b+2a^2b^2-2ab^3+b^4)(a^4-2a^3b+2a^2b^2+2ab^3+b^4) \neq 0$.

(vi) By (6.5), we have

$$x^6y = x^2y^5, xy^6 = x^5y^2. \tag{6.14}$$

For $i = 5$, by (6.14), the matrix representation of σ_5 with respect to the basis (6, 0) is

$$\begin{pmatrix} a^2 & 0 & 0 & 0 & 0 & 0 \\ b^2 & 2ab & a^2 & 0 & b^2 & 2ab \\ 0 & b^2 & 2ab & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 2ab & a^2 & 0 \\ 2ab & a^2 & 0 & b^2 & 2ab & a^2 \\ 0 & 0 & 0 & 0 & 0 & b^2 \end{pmatrix}.$$

Therefore σ_5 is bijective if and only if $a^2b^2(a-b)^2(a+b)^2(a^2+b^2)^2 \neq 0$.

(vii) For $i = 6$, σ_6 is bijective obviously.

From (i), ..., (vii), we get (1).

(2) For a homogeneous polynomial g_i, the points $(a : b) \in t(g_i)$ are as follows:

$$t(g_1) = \{(0:1)\},$$
$$t(g_2) = \{(1:0)\},$$
$$t(g_3) = \{(1:1)\},$$
$$t(g_4) = \{(-1:1)\},$$
$$t(g_5) = \{(\pm i:1)\},$$
$$t(g_6) = \left\{(-1 \pm \sqrt{2}:1)\right\},$$
$$t(g_7) = \left\{(1 \pm \sqrt{2}:1)\right\},$$
$$t(g_8) = \left\{\left(\pm(-1)^{\frac{1}{4}}:1\right), \left(\pm(-1)^{\frac{1}{4}}i:1\right)\right\},$$
$$t(g_9) = \left\{(\pm(\sqrt{2}+1)i:1), (\pm(\sqrt{2}-1)i:1)\right\},$$

$$t(g_{10}) = \left\{ \left(\frac{1}{2}(-1+i \pm \sqrt{-6i}) : 1 \right), \left(\frac{1}{2}(-1-i \pm \sqrt{6i}) : 1 \right) \right\},$$
$$t(g_{11}) = \left\{ \left(\frac{1}{2}(1-i \pm \sqrt{-6i}) : 1 \right), \left(\frac{1}{2}(1+i \pm \sqrt{6i}) : 1 \right) \right\}.$$

We identify the sphere S^2 with $\mathbf{C} \cup \{\infty\}$ by the stereographic projection p (See the relations $(*)$ of the end of §4).

Therefore for g_i, we get the sets $B(g_i)\,(1 \leq i \leq 11)$ which is written in the theorem above. □

7. The Lefschetz elements for G_{13}

We recall that the group G_{13} of order 96 contains a binary octahedral group of order 48 as a subgroup of index 2. In this section we give a characterization of the strong Lefschetz element of the coinvariant algebras of G_{13}.

Let $G = G_{13}$:

$$G = \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^3 & \varepsilon^3 \\ \varepsilon^5 & \varepsilon \end{pmatrix} \right\rangle, (\varepsilon = e^{\frac{2\pi i}{8}}).$$

The invariant algebra of G is

$$S(V)^G = \mathbf{C}[X^8 + 14X^4Y^4 + Y^8, X^{10}Y^2 - 2X^6Y^6 + X^2Y^{10}].$$

By (3) in 3.2 and Table 1, we have

$$\begin{aligned} P_{S(V)_G}(t) &= \frac{(1-t^8)(1-t^{12})}{(1-t)^2} \\ &= 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 7t^6 + 8t^7 + \cdots + 8t^{11} + 7t^{12} + 6t^{13} \\ &\quad + 5t^{14} + 4t^{15} + 3t^{16} + 2t^{17} + t^{18}. \end{aligned}$$

We choose a basis of $S(V)_G$

$$\begin{gathered} \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, \\ y^5, x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7, \\ x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5, x^2y^6, xy^7, y^8, x^8y, x^7y^2, x^6y^3, x^5y^4, x^4y^5, x^3y^6, x^2y^7, xy^8, \\ x^8y^2, x^7y^3, x^6y^4, x^5y^5, x^4y^6, x^3y^7, x^2y^8, xy^9, x^9y^2, x^8y^3, x^7y^4, x^6y^5, x^5y^6, x^4y^7, x^3y^8, \\ x^2y^9, x^9y^3, x^8y^4, x^7y^5, x^5y^7, x^4y^8, x^3y^9, x^2y^{10}, x^{10}y^3, x^9y^4, x^8y^5, x^5y^8, x^4y^9, x^3y^{10}, \\ x^{10}y^4, x^9y^5, x^5y^9, x^4y^{10}, x^3y^{11}, x^{11}y^4, x^{10}y^5, x^5y^{10}, x^4y^{11}, x^{11}y^5, x^5y^{11}, x^4y^{12}, \\ x^{12}y^5, x^5y^{12}, x^{13}y^5\} \end{gathered} \tag{7.0}$$

THEOREM 6. (1) *The element* $l = ax + by \in S_1(V)_G$ *is the strong Lefschetz element if and only if* $\prod_{i=1}^{17} g_i \neq 0$, *where* g_i *are the polynomials in* a, b *given by*

$$g_1 = a\,, \quad g_2 = b\,, \quad g_3 = a - b\,, \quad g_4 = a + b\,, \quad g_5 = a^2 + b^2\,,$$

$$g_6 = a^2 + 2ab - b^2\,,\ g_7 = a^2 - 2ab - b^2\,,\ g_8 = a^4 + b^4\,,\ g_9 = a^4 + 6a^2b^2 + b^4\,,$$

$$g_{10} = a^4 + 2a^3b + 2a^2b^2 - 2ab^3 + b^4, \quad g_{11} = a^4 - 2a^3b + 2a^2b^2 + 2ab^3 + b^4\,,$$

$$g_{12} = a^{24} + 366a^{20}b^4 - 705a^{16}b^8 + 4772a^{12}b^{12} - 705a^8b^{16} + 366a^4b^{20} + b^{24}\,,$$

$$g_{13} = 2a^{24} + 3a^{20}b^4 + 1506a^{16}b^8 + 5170a^{12}b^{12} + 1506a^8b^{16} + 3a^4b^{20} + 2b^{24}\,,$$

$$g_{14} = 2a^{24} - 637a^{20}b^4 + 4066a^{16}b^8 + 1330a^{12}b^{12} + 4066a^8b^{16} - 637a^4b^{20} + 2b^{24}\,,$$

$$g_{15} = a^{24} - 470a^{20}b^4 + 2639a^{16}b^8 - 244a^{12}b^{12} + 2639a^8b^{16} - 470a^4b^{20} + b^{24},$$

$$\begin{aligned} g_{16} = 4a^{48} &- 468a^{44}b^4 + 39633a^{40}b^8 - 590924a^{36}b^{12} + 3414840a^{32}b^{16} \\ &+ 16202208a^{28}b^{20} + 28978278a^{24}b^{24} + 16202208a^{20}b^{28} + 3414840a^{16}b^{32} \\ &- 590924a^{12}b^{36} + 39633a^8b^{40} - 468a^4b^{44} + 4b^{48}\,, \end{aligned}$$

$$\begin{aligned} g_{17} = a^{48} &+ 4312a^{44}b^4 + 348186a^{40}b^8 + 392504a^{36}b^{12} + 8766479a^{32}b^{16} \\ &- 37985808a^{28}b^{20} + 73725868a^{24}b^{24} - 37985808a^{20}b^{28} + 8766479a^{16}b^{32} \\ &+ 392504a^{12}b^{36} + 348186a^8b^{40} + 4312a^4b^{44} + b^{48}\,. \end{aligned}$$

(*See the end of* §3 *for the notation.*)

(2) *We have the inverse images of* $t(g_i)$ $(1 \leq i \leq 11)$ *under the stereographic projection as follows* (*see the end of* §3 *for the notation*) :

$$B(g_1) = \{(0, 0, -1)\}\,,$$

$$B(g_2) = \{(0, 0, 1)\}\,,$$

$$B(g_3) = \{(1, 0, 0)\}\,,$$

$$B(g_4) = \{(-1, 0, 0)\}\,,$$

$$B(g_5) = \{(0, 1, 0), (0, -1, 0)\}\,,$$

$$B(g_6) = \left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\},$$

$$B(g_7) = \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)\right\},$$

$$B(g_8) = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right\},$$

$$B(g_9) = \left\{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\},$$

$$B(g_{10}) = \left\{\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)\right\},$$

$$B(g_{11}) = \left\{\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)\right\}.$$

Now if we take the octahedron and the tetrahedron in R^3 *in the same way as* Theorem 5 (2), *the sets* $B(g_i)$ *are given as follows* :

(i) *The sets of* $B(g_1)$, $B(g_2)$, $B(g_3)$, $B(g_4)$, $B(g_5)$ *are the six vertices of the octahedron* (*the six points of* S^2 *corresponding to the centers of the edges of the tetrahedron*).

(ii) *The sets of* $B(g_6)$, $B(g_7)$, $B(g_8)$, $B(g_9)$ *are the twelve points of* S^2 *corresponding to the centers of the edges of the octahedron.*

(iii) *The sets of* $B(g_{10})$, $B(g_{11})$ *are the eight points of* S^2 *corresponding to the centers of the faces of the octahedron* (*the four points of* S^2 *corresponding to the vertices of the tetrahedron and the four points of* S^2 *corresponding to the centers of the faces of the tetrahedron*).

PROOF. (1) Since $I_G = (X^8 + 14X^4Y^4 + Y^8, X^{10}Y^2 - 2X^6Y^6 + X^2Y^{10})$, we have

$$x^8 + 14x^4y^4 + y^8 = 0\,, \tag{7.1}$$

$$x^{10}y^2 - 2x^6y^6 + x^2y^{10} = 0\,. \tag{7.2}$$

For later use, we introduce the following relations. By (7.2), we have

$$x^6y^6 = \frac{1}{2}(x^{10}y^2 + x^2y^{10})\,. \tag{7.3}$$

Since $x^4y^4 = -\frac{1}{14}(x^8 + y^8)$ by (7.1), we have

$$x^6y^6 = x^2y^2\left\{-\frac{1}{14}(x^8 + y^8)\right\} = -\frac{1}{14}(x^{10}y^2 + x^2y^{10})\,. \tag{7.4}$$

Therefore by (7.3), (7.4), we have

$$x^6y^6 = x^{10}y^2 + x^2y^{10} = 0\,. \tag{7.5}$$

By (7.5), we have

$$x^{14}y^2 = x^4(-x^2y^{10}) = 0\,, \quad x^2y^{14} = y^4(-x^{10}y^2) = 0\,. \tag{7.6}$$

Therefore by (7.5), (7.6), we have

$$x^iy^j = 0\,, \quad i, j \geq 6 \quad \text{or} \quad i \geq 14, j \geq 2 \quad \text{or} \quad i \geq 2, j \geq 14 \tag{7.7}$$

and

$$x^iy^j = -x^{i-8}y^{j+8}\,, \quad i \geq 10\,, j \geq 2\,. \tag{7.8}$$

By (7.1), (7.7), we have

$$x^{18} = y^{18} = 0\,. \tag{7.9}$$

Also by (7.1), (7.7), (7.8), we have

$$\begin{aligned} x^i y^j &= -14x^{i-4}y^{j+4}\,, \quad i \geq 14\,, \\ x^i y^j &= -14x^{i+4}y^{j-4}\,, \quad j \geq 14\,. \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} x^i y^j &= -14x^{i-4}y^{j+4} - x^{i-8}y^{j+8}\,, \quad i \geq 8\,, \\ x^i y^j &= -14x^{i+4}y^{j-4} - x^{i+8}y^{j-8}\,, \quad j \geq 8\,. \end{aligned} \tag{7.11}$$

(i) By (7.7), ..., (7.10), we have

$$\begin{gathered} x^{12}y^6 = x^{11}y^7 = x^{10}y^8 = x^9y^9 = x^8y^{10} = x^7y^{11} = x^6y^{12} = 0\,, \\ x^{16}y^2 = x^{15}y^3 = x^{14}y^4 = x^4y^{14} = x^3y^{15} = x^2y^{16} = 0\,, \\ x^{18} = y^{18} = 0\,,\ x^{17}y = -14x^{13}y^5 = 14x^5y^{13} = -xy^{17}\,. \end{gathered} \tag{7.12}$$

For $i = 0$, by (7.12), the matrix representation of σ_0 with respect to the basis (7.0) is regular if and only if $-252ab(a+b)(a-b)(a^2+b^2)(a^2-2ab-b^2)(a^2+2ab-b^2)(a^4+b^4)(a^4+6a^2b^2+b^4) \neq 0$.

(ii) By (7.7), (7.8), (7.10), we have

$$\begin{aligned} & x^{11}y^6 = x^{10}y^7 = x^9y^8 = x^8y^9 = x^7y^{10} = x^6y^{11} = 0\,, \\ & x^{15}y^2 = x^{14}y^3 = x^3y^{14} = x^2y^{15} = 0\,, \\ & \quad x^{17} = -14x^{13}y^4 = 14x^5y^{12} = -xy^{16}\,, \\ & \quad y^{17} = -14x^4y^{13} = 14x^{12}y^5 = -x^{16}y\,. \end{aligned} \tag{7.13}$$

For $i = 1$, by (7.13), the matrix representation of σ_1 with respect to the basis (7.0) is

$$\begin{pmatrix} -112a^3b(2a^{12}-39a^8b^4+5b^{12}) & -14(a^{16}-130a^{12}b^4+130a^4b^{12}-b^{16}) \\ 14(a^{16}-130a^{12}b^4+130a^4b^{12}-b^{16}) & -112ab^3(5a^{12}-39a^4b^8+2b^{12}) \end{pmatrix}.$$

Therefore σ_1 is bijective if and only if $196(a^4+2a^3b+2a^2b^2-2ab^3+b^4)(a^4-2a^3b+2a^2b^2+2ab^3+b^4)(a^{24}+366a^{20}b^4-705a^{16}b^8+4772a^{12}b^{12}-705a^8b^{16}+366a^4b^{20}+b^{24}) \neq 0$.

(iii) By (7.7), (7.8), (7.10), we have

$$\begin{gathered} x^{10}y^6 = x^9y^7 = x^8y^8 = x^7y^9 = x^6y^{10} = 0\,, \quad x^{14}y^2 = x^2y^{14} = 0\,, \\ x^{16} = -14x^{12}y^4 = 14x^4y^{12} = -y^{16}\,, \\ x^{15}y = -14x^{11}y^5 = 14x^3y^{13}, \quad xy^{15} = -14x^5y^{11} = 14x^{13}y^3\,. \end{gathered} \tag{7.14}$$

For $i = 2$, by (7.14), the matrix representation of σ_2 with respect to the basis (7.0) is

$$\begin{pmatrix} -14ab(14a^{12}-143a^8b^4+b^{12}) & -a^2(14a^{12}-1001a^8b^4+91b^{12}) & 364a^3b^3(a^8-b^8) \\ -364a^3b^3(a^8-b^8) & -b^2(91a^{12}-1001a^4b^8+14b^{12}) & -14ab(a^{12}-143a^4y^8+14b^{12}) \\ a^2(14a^{12}-1001a^8b^4+91b^{12}) & -364a^3b^3(a^8-b^8) & -b^2(91a^{12}-1001a^4b^8+14b^{12}) \end{pmatrix}.$$

Therefore σ_2 is bijective if and only if $1372ab(a-b)(a+b)(a^2+b^2)(a^2+2ab-b^2)(a^2-2ab-b^2)(a^4+b^4)(a^4+6a^2b^2+b^4)(2a^{24}+3a^{20}b^4+1506a^{16}b^8+5170a^{12}b^{12}+1506a^8b^{16}+3a^4b^{20}+2b^{24}) \neq 0$.

(iv) By (7.7), (7.8), (7.10), we have

$$\begin{aligned} &x^9y^6 = x^8y^7 = x^7y^8 = x^6y^9 = 0\,, \\ &\;x^{15} = -14x^{11}y^4 = 14x^3y^{12}\,, \quad x^{14}y = -14x^{10}y^5 = 14x^2y^{13}\,, \\ &xy^{14} = -14x^5y^{10} = 14x^{13}y^2\,, \quad y^{15} = -14x^4y^{11} = 14x^{12}y^3\,. \end{aligned} \tag{7.15}$$

For $i = 3$, by (7.15), the matrix representation of σ_3 with respect to the basis (7.0) is

$$\begin{pmatrix} -(14a^{12}-495a^8b^4+b^{12}) & 4ab^3(55a^8-3b^8) & 66a^2b^2(a^8-b^8) & 4a^3b(3a^8-55b^8) \\ -24a^7b(7a^4-33b^4) & -(14a^{12}-495a^8b^4+b^{12}) & 4ab^3(55a^8-3b^8) & 66a^2b^2(a^8-b^8) \\ -66a^2b^2(a^8-b^8) & -4a^3b(3a^8-55b^8) & -(a^{12}-495a^4b^8+14b^{12}) & 24ab^7(33a^4-7b^4) \\ -4ab^3(55a^8-3b^8) & -66a^2b^2(a^8-b^8) & -4a^3b(3a^8-55b^8) & -(a^{12}-495a^4b^8+14b^{12}) \end{pmatrix}.$$

Therefore σ_3 is bijective if and only if $49(4a^{48}-468a^{44}b^4+39633a^{40}b^8-590924a^{36}b^{12}+3414840a^{32}b^{16}+16202208a^{28}b^{20}+28978278a^{24}b^{24}+16202208a^{20}b^{28}+3414840a^{16}b^{32}-590924a^{12}b^{36}+39633a^8b^{40}-468a^4b^{44}+4b^{48}) \neq 0$.

(v) By (7.7), (7.8), (7.10), (7.11), we have

$$\begin{aligned} &x^8y^6 = x^7y^7 = x^6y^8 = 0\,, \quad x^{14} = -14x^{10}y^4 = 14x^2y^{12}\,, \\ &x^{13}y = -14x^9y^5 - x^5y^9, \quad x^{11}y^3 = -x^3y^{11}\,, \\ &xy^{13} = -14x^5y^9 - x^9y^5, \quad y^{14} = -14x^4y^{10} = 14x^{12}y^2\,. \end{aligned} \tag{7.16}$$

For $i = 4$, by (7.16), the matrix representation of σ_4 with respect to the basis (7.0) is

$$\begin{pmatrix} -14a^6(a^4-15b^4) & 120a^7b^3 & b^2(45a^8-b^8) & 10ab(a^8-b^8) & a^2(a^8-45b^8) \\ -28a^5b(5a^4-9b^4) & -14a^6(a^4-15b^4) & 120a^7b^3 & b^2(45a^8-b^8) & 10ab(a^8-b^8) \\ -10ab(a^8-b^8) & -a^2(a^8-45b^8) & 120a^3b^7 & 14b^6(15a^4-b^4) & 28ab^5(9a^4-5b^4) \\ -b^2(45a^8-b^8) & -10ab(a^8-b^8) & -a^2(a^8-45b^8) & 120a^3b^7 & 14b^6(15a^4-b^4) \\ -120a^7b^3 & -b^2(45a^8-b^8) & -10ab(a^8-b^8) & -a^2(a^8-45b^8) & 120a^3b^7 \end{pmatrix}.$$

Therefore σ_4 is bijective if and only if $-70ab(a-b)(a+b)(a^2+b^2)(a^2+2ab-b^2)(a^2-2ab-b^2)(a^4+b^4)(a^4+6a^2b^2+b^4)(a^4+2a^3b+2a^2b^2-2ab^3+b^4)(a^4-2a^3b+2a^2b^2+2ab^3+b^4)(2a^{24}-637a^{20}b^4+4066a^{16}b^8+1330a^{12}b^{12}+4066a^8b^{16}-637a^4b^{20}+2b^{24}) \neq 0$.

(vi) By (7.7), (7.8), (7.11), we have

$$\begin{gathered} x^7y^6 = x^6y^7 = 0\,, \quad x^{13} = -14x^9y^4 - x^5y^8\,, \\ x^{12}y = -14x^8y^5 - x^4y^9, \quad x^{11}y^2 = -x^3y^{10}\,, \\ x^2y^{11} = -x^{10}y^3, \quad xy^{12} = -14x^5y^8 - x^9y^4\,, \\ y^{13} = -14x^4y^9 - x^8y^5\,. \end{gathered} \tag{7.17}$$

For $i = 5$, by (7.17), the matrix representation of σ_5 with respect to the basis (7.0) is

$$\begin{pmatrix} 56a^5b^3 & 28a^6b^2 & 8a^7b & a^8-b^8 & -8ab^7 & -28a^2b^6 \\ -14(a^4-5b^4) & 56a^5b^3 & 28a^6b^2 & 8a^7b & a^8-b^8 & -8ab^7 \\ -56a^3b(2a^4-b^4) & -14(a^4-5b^4) & 56a^5b^3 & 28a^6b^2 & 8a^7b & a^8-b^8 \\ -(a^8-b^8) & 8ab^7 & 28a^2b^6 & 56a^3b^5 & 14b^4(5a^4-b^4) & 56ab^3(a^4-2b^4) \\ -8a^7b & -(a^8-b^8) & 8ab^7 & 28a^2b^6 & 56a^3b^5 & 14b^4(5a^4-b^4) \\ -28a^6b^2 & -8a^7b & -(a^8-b^8) & 8ab^7 & 28a^2b^6 & 56a^3b^5 \end{pmatrix}.$$

Therefore σ_5 is bijective if and only if $a^{48}+4312a^{44}b^4+348186a^{40}b^8+392504a^{36}b^{12}+8766479a^{32}b^{16}-37985808a^{28}b^{20}+73725868a^{24}b^{24}-37985808a^{20}b^{28}+8766479a^{16}b^{32}+392504a^{12}b^{36}+348186a^8b^{40}+4312a^4b^{44}+b^{48} \neq 0$.

(vii) By (7.11), we have

$$\begin{gathered} x^{12} = -14x^8y^4 - x^4y^8\,, \quad x^{11}y = -14x^7y^5 - x^3y^9\,, \\ xy^{11} = -14x^5y^7 - x^9y^3\,, \quad y^{12} = -14x^4y^8 - x^8y^4\,. \end{gathered} \tag{7.18}$$

For $i = 6$, by (7.18), the matrix representation of σ_6 with respect to the basis (7.0) is

$$\begin{pmatrix} 20a^3b^3 & 15a^4b^2 & 6a^5b & a^6 & 0 & -b^6 & -6ab^5 \\ -a^2(14a^4-15b^4) & 20a^3b^3 & 15a^4b^2 & 6a^5b & a^6 & 0 & -b^6 \\ -6ab(14a^4-b^4) & -a^2(14a^4-15b^4) & 20a^3b^3 & 15a^4b^2 & 6a^5b & a^6 & 0 \\ 0 & b^6 & 6ab^5 & 15a^2b^4 & 20a^3b^3 & b^2(15a^4-14b^4) & 6ab(a^4-14b^4) \\ -a^6 & 0 & b^6 & 6ab^5 & 15a^2b^4 & 20a^3b^3 & b^2(15a^4-14b^4) \\ -6a^5b & -a^6 & 0 & b^6 & 6ab^5 & 15a^2b^4 & 20a^3b^3 \\ -15a^4b^2 & -6a^5b & -a^6 & 0 & b^6 & 6ab^5 & 15a^2b^4 \end{pmatrix}.$$

Therefore σ_6 is bijective if and only if $-6ab(a-b)(a+b)(a^2+b^2)(a^2+2ab-b^2)(a^2-2ab-b^2)(a^4+b^4)(a^4+6a^2b^2+b^4)(a^{24}-470a^{20}b^4+2639a^{16}b^8-244a^{12}b^{12}+2639a^8b^{16}-470a^4b^{20}+b^{24}) \neq 0$.

(viii) By (7.11), we have

$$\begin{gathered} x^{11} = -14x^7y^4 - x^3y^8\,, \quad x^{10}y = -14x^6y^5 - x^2y^9\,, \\ xy^{10} = -14x^5y^6 - x^9y^2\,, \quad y^{11} = -14x^4y^7 - x^8y^3\,. \end{gathered} \tag{7.19}$$

For $i = 7$, by (7.19), the matrix representation of σ_7 with respect to the basis (7.0) is

$$\begin{pmatrix} 6a^2b^2 & 4a^3b & a^4 & 0 & 0 & 0 & -b^4 & -4ab^3 \\ 4ab^3 & 6a^2b^2 & 4a^3b & a^4 & 0 & 0 & 0 & -b^4 \\ -(14a^4 - b^4) & 4ab^3 & 6a^2b^2 & 4a^3b & a^4 & 0 & 0 & 0 \\ -56a^3b & -(14a^4 - b^4) & 4ab^3 & 6a^2b^2 & 4a^3b & a^4 & 0 & 0 \\ 0 & 0 & b^4 & 4ab^3 & 6a^2b^2 & 4a^3b & a^4 - 14b^4 & -56ab^3 \\ 0 & 0 & 0 & b^4 & 4ab^3 & 6a^2b^2 & 4a^3b & a^4 - 14b^4 \\ -a^4 & 0 & 0 & 0 & b^4 & 4ab^3 & 6a^2b^2 & 4a^3b \\ -4a^3b & -a^4 & 0 & 0 & 0 & b^4 & 4ab^3 & 6a^2b^2 \end{pmatrix}.$$

Therefore σ_7 is bijective if and only if $(a^4 + 2a^3b + 2a^2b^2 - 2ab^3 + b^4)^4(a^4 - 2a^3b + 2a^2b^2 + 2ab^3 + b^4)^4 \neq 0$.

(ix) By (7.11), we have

$$\begin{aligned} &x^{10} = -14x^6y^4 - x^2y^8, \quad x^9y = -14x^5y^5 - xy^9, \\ &y^{10} = -14x^4y^6 - x^8y^2. \end{aligned} \tag{7.20}$$

For $i = 8$, by (7.20), the matrix representation of σ_8 with respect to the basis (7.0) is

$$\begin{pmatrix} 2ab & a^2 & 0 & 0 & 0 & 0 & 0 & -b^2 \\ b^2 & 2ab & a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & b^2 & 2ab & a^2 & 0 & 0 & 0 & 0 \\ -14a^2 & 0 & b^2 & 2ab & a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & b^2 & 2ab & a^2 & 0 & -14b^2 \\ 0 & 0 & 0 & 0 & b^2 & 2ab & a^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & b^2 & 2ab & a^2 \\ -a^2 & 0 & 0 & 0 & 0 & 0 & b^2 & 2ab \end{pmatrix}.$$

Therefore σ_8 is bijective if and only if $(a^4 + 2a^3b + 2a^2b^2 - 2ab^3 + b^4)^2(a^4 - 2a^3b + 2a^2b^2 + 2ab^3 + b^4)^2 \neq 0$.

(x) For $i = 9$, σ_9 is bijective obviously.

By (i)$\sim$(x) , we get (1).

(2) For $1 \leq i \leq 11$, we get the same polynomials g_i as in Theorem 6.1(1). Thus we have the same $B(g_i)$ $(1 \leq i \leq 11)$ as in Theorem 6.1 (2). $\square$

8. The Lefschetz elements for G_{22}

We recall that the group G_{22} of order 240 contains a binary icosahedral group of order 120 as a subgroup of index 2. In this section we give a characterization of the strong Lefschetz element of the coinvariant algebras of G_{22}. Let $G = G_{22}$:

$$G = \left\langle \frac{1}{\sqrt{5}} i \begin{pmatrix} \eta^4 - \eta & \eta^2 - \eta^3 \\ \eta^2 - \eta^3 & \eta - \eta^4 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \eta^2 - \eta^4 & \eta^4 - 1 \\ 1 - \eta & \eta^3 - \eta \end{pmatrix} \right\rangle, \quad (\eta = e^{\frac{2\pi i}{5}}).$$

The invariant algebra of G is

$$S(V)^G = \mathbf{C}[X^{11}Y + 11X^6Y^6 - XY^{11}, -(X^{20} + Y^{20}) + 228(X^{15}Y^5 - X^5Y^{15}) - 494X^{10}Y^{10}].$$

By (3) in 3.2 and Table 1, we have

$$
\begin{aligned}
P_{S(V)_G}(t) &= \frac{(1-t^{12})(1-t^{20})}{(1-t)^2} \\
&= 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 7t^6 + 8t^7 + 9t^8 + 10t^9 + 11t^{10} + 12t^{11} \\
&\quad + \cdots + 12t^{19} + 11t^{20} + 10t^{21} + 9t^{22} + 8t^{23} + 7t^{24} + 6t^{25} + 5t^{26} + 4t^{27} \\
&\quad + 3t^{28} + 2t^{29} + t^{30}\,.
\end{aligned}
$$

We choose a basis of $S(V)_G$

$$
\begin{aligned}
&\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, \\
&x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6, x^7, x^6y, x^5y^2, x^4y^3, x^3y^4, x^2y^5, xy^6, y^7, x^8, x^7y, x^6y^2, \\
&x^5y^3, x^4y^4, x^3y^5, x^2y^6, xy^7, y^8, x^9, x^8y, x^7y^2, x^6y^3, x^5y^4, x^4y^5, x^3y^6, x^2y^7, xy^8, y^9, x^{10}, \\
&x^9y, x^8y^2, x^7y^3, x^6y^4, x^5y^5, x^4y^6, x^3y^7, x^2y^8, xy^9, y^{10}, x^{11}, x^{10}y, x^9y^2, x^8y^3, x^7y^4, x^6y^5, \\
&x^5y^6, x^4y^7, x^3y^8, x^2y^9, xy^{10}, y^{11}, x^{12}, x^{11}y, x^{10}y^2, x^9y^3, x^8y^4, x^7y^5, x^5y^7, x^4y^8, x^3y^9, \\
&x^2y^{10}, xy^{11}, y^{12}, x^{13}, x^{12}y, x^{11}y^2, x^{10}y^3, x^9y^4, x^8y^5, x^5y^8, x^4y^9, x^3y^{10}, x^2y^{11}, xy^{12}, y^{13}, \\
&x^{14}, x^{13}y, x^{12}y^2, x^{11}y^3, x^{10}y^4, x^9y^5, x^5y^9, x^4y^{10}, x^3y^{11}, x^2y^{12}, xy^{13}, y^{14}, x^{15}, x^{14}y, x^{13}y^2, \\
&x^{12}y^3, x^{11}y^4, x^{10}y^5, x^5y^{10}, x^4y^{11}, x^3y^{12}, x^2y^{13}, xy^{14}, y^{15}, x^{16}, x^{15}y, x^{14}y^2, x^{13}y^3, x^{12}y^4, \\
&x^{11}y^5, x^5y^{11}, x^4y^{12}, x^3y^{13}, x^2y^{14}, xy^{15}, y^{16}, x^{17}, x^{16}y, x^{15}y^2, x^{14}y^3, x^{13}y^4, x^{12}y^5, x^5y^{12}, \\
&x^4y^{13}, x^3y^{14}, x^2y^{15}, xy^{16}, y^{17}, x^{18}, x^{17}y, x^{16}y^2, x^{15}y^3, x^{14}y^4, x^{13}y^5, x^5y^{13}, x^4y^{14}, x^3y^{15}, \\
&x^2y^{16}, xy^{17}, y^{18}, x^{19}, x^{18}y, x^{17}y^2, x^{16}y^3, x^{15}y^4, x^{14}y^5, x^5y^{14}, x^4y^{15}, x^3y^{16}, x^2y^{17}, xy^{18}, \\
&y^{19}, x^{19}y, x^{18}y^2, x^{17}y^3, x^{16}y^4, x^{15}y^5, x^5y^{15}, x^4y^{16}, x^3y^{17}, x^2y^{18}, xy^{19}, y^{20}, x^{20}y, x^{19}y^2, \\
&x^{18}y^3, x^{17}y^4, x^{16}y^5, x^5y^{16}, x^4y^{17}, x^3y^{18}, x^2y^{19}, xy^{20}, x^{20}y^2, x^{19}y^3, x^{18}y^4, x^{17}y^5, x^5y^{17}, \\
&x^4y^{18}, x^3y^{19}, x^2y^{20}, xy^{21}, x^{21}y^2, x^{20}y^3, x^{19}y^4, x^{18}y^5, x^5y^{18}, x^4y^{19}, x^3y^{20}, x^2y^{21}, x^{21}y^3, \\
&x^{20}y^4, x^{19}y^5, x^5y^{19}, x^4y^{20}, x^3y^{21}, x^2y^{22}, x^{22}y^3, x^{21}y^4, x^{20}y^5, x^5y^{20}, x^4y^{21}, x^3y^{22}, x^{22}y^4, \\
&x^{21}y^5, x^5y^{21}, x^4y^{22}, x^3y^{23}, x^{23}y^4, x^{22}y^5, x^5y^{22}, x^4y^{23}, x^{23}y^5, x^5y^{23}, x^4y^{24}, x^{24}y^5, x^5y^{24}, \\
&x^5y^{25}\}
\end{aligned}
\tag{8.0}
$$

THEOREM 7. (1) *The element $l = ax+by \in S_1(V)_G$ is the strong Lefschetz element if and only if $\prod_{i=1}^{21} g_i \neq 0$, where g_i is a polynomial in a, b given by*

$g_1(a, b) = a\,,$

$g_2(a, b) = b\,,$

$g_3(a, b) = a^2 + ab - b^2\,,$

$g_4(a, b) = a^2 + b^2\,,$

$g_5(a, b) = a^4 - 3a^3b + 4a^2b^2 - 2ab^3 + b^4\,,$

$$
\begin{aligned}
&g_6(a,b) = a^4 + 2a^3b + 4a^2b^2 + 3ab^3 + b^4\,,\\
&g_7(a,b) = a^4 - 3a^3b - a^2b^2 + 3ab^3 + b^4\,,\\
&g_8(a,b) = a^4 + 2a^3b - 6a^2b^2 - 2ab^3 + b^4\,,\\
&g_9(a,b) = a^8 + 4a^7b + 7a^6b^2 + 2a^5b^3 + 15a^4b^4 - 2a^3b^5 + 7a^2b^6 - 4ab^7 + b^8\,,\\
&g_{10}(a,b) = a^8 - a^7b + 7a^6b^2 + 7a^5b^3 - 7a^3b^5 + 7a^2b^6 + ab^7 + b^8\,,\\
&g_{11}(a,b) = a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8\,,\\
&g_{12}(a,b) = a^8 + 4a^7b + 17a^6b^2 + 22a^5b^3 + 5a^4b^4 - 22a^3b^5 + 17a^2b^6 - 4ab^7 + b^8\,,\\
&g_{13}(a,b) = a^8 - 6a^7b + 17a^6b^2 - 18a^5b^3 + 25a^4b^4 + 18a^3b^5 + 17a^2b^6 + 6ab^7 + b^8\,,\\
&g_{14}(a,b) = 56a^{60} - 93411a^{55}b^5 + 5785419a^{50}b^{10} - 767941695a^{45}b^{15}\\
&\quad + 3616437090a^{40}b^{20} - 11153212659a^{35}b^{25} - 9286142171a^{30}b^{30} + 111532126594a^{25}b^{35}\\
&\quad + 3616437090a^{20}b^{40} + 767941695a^{15}b^{45} + 5785419a^{10}b^{50} + 93411a^5b^{55} + 56b^{60}\,,\\
&g_{15}(a,b) = 21a^{60} + 124424a^{55}b^5 + 10939454a^{50}b^{10} - 95837120a^{45}b^{15}\\
&\quad + 3443405315a^{40}b^{20} + 6913090956a^{35}b^{25} + 18005759264a^{30}b^{30} - 6913090956a^{25}b^{35}\\
&\quad + 3443405315a^{20}b^{40} + 95837120a^{15}b^{45} + 10939454a^{10}b^{50} - 124424a^5b^{55} + 21b^{60}\,,\\
&g_{16}(a,b) = 781a^{60} - 168636a^{55}b^5 + 143062194a^{50}b^{10} - 9343436820a^{45}b^{15}\\
&\quad + 65281931715a^{40}b^{20} - 76630172184a^{35}b^{25} + 23325796604a^{30}b^{30} + 76630172184a^{25}b^{35}\\
&\quad + 65281931715a^{20}b^{40} + 9343436820a^{15}b^{45} + 143062194a^{10}b^{50} + 168636a^5b^{55} + 781b^{60}\,,\\
&g_{17}(a,b) = 781a^{60} - 2856636a^{55}b^5 - 4777806a^{50}b^{10} - 12582476820a^{45}b^{15}\\
&\quad + 30096011715a^{40}b^{20} - 263674652184a^{35}b^{25} - 338911771396a^{30}b^{30} + 263674652184a^{25}b^{35}\\
&\quad + 30096011715a^{20}b^{40} + 12582476820a^{15}b^{45} - 4777806a^{10}b^{50} + 2856636a^5b^{55} + 781b^{60}\,,\\
&g_{18}(a,b) = 781a^{60} + 9833796a^{55}b^5 + 693195954a^{50}b^{10} + 2709493740a^{45}b^{15}\\
&\quad + 196213766595a^{40}b^{20} + 619389058536a^{35}b^{25} + 1371263535356a^{30}b^{30} - 619389058536a^{25}b^{35}\\
&\quad + 196213766595a^{20}b^{40} - 2709493740a^{15}b^{45} + 693195954a^{10}b^{50} - 9833796a^5b^{55} + 781b^{60}\,,\\
&g_{19}(a,b) = a^{60} - 184a^{55}b^5 + 184934a^{50}b^{10} - 11924960a^{45}b^{15}\\
&\quad + 84005495a^{40}b^{20} - 95896644a^{35}b^{25} + 34168576a^{30}b^{30} + 95896644a^{25}b^{35}\\
&\quad + 84005495a^{20}b^{40} + 11924960a^{15}b^{45} + 184934a^{10}b^{50} + 184a^5b^{55} + b^{60}\,,\\
&g_{20}(a,b) = 781a^{120} + 1056728a^{115}b^5 + 1690618404a^{110}b^{10}\\
&\quad + 335172561432a^{105}b^{15} + 36048914871226a^{100}b^{20} - 3695470971608808a^{95}b^{25}\\
&\quad + 130053725826919236a^{90}b^{30} - 1594868948748657512a^{85}b^{35} + 8098914234602656899a^{80}b^{40}\\
&\quad - 5824418933856860288a^{75}b^{45} + 30737529948917758104a^{70}b^{50} + 11758899072173963712a^{65}b^{55}\\
&\quad + 38124609592305613676a^{60}b^{60} - 11758899072173963712a^{55}b^{65} + 30737529948917758104a^{50}b^{70}\\
&\quad + 5824418933856860288a^{45}b^{75} + 8098914234602656899a^{40}b^{80} + 1594868948748657512a^{35}b^{85}
\end{aligned}
$$

$$
\begin{aligned}
&+ 130053725862919236a^{30}b^{90} + 3695470971608808a^{25}b^{95} + 36048914871226a^{20}b^{100}\\
&- 335172561432a^{15}b^{105} + 1690618404a^{10}b^{100} - 1056728a^{5}b^{115} + 781b^{120}\,,\\
&g_{21}(a,b) = 781a^{120} - 37434408a^{115}b^{5} + 28669521444a^{110}b^{10}\\
&- 4018508591592a^{105}b^{15} + 231220616958906a^{100}b^{20} + 13323015986481432a^{95}b^{25}\\
&+ 4830395349952a^{90}b^{30} + 1093401918957423768a^{85}b^{35} + 8911688602181278851a^{80}b^{40}\\
&- 7486305076756601356\,8a^{75}b^{45} - 21436724021848920936a^{70}b^{50} + 96989209532481420844\,8a^{65}b^{55}\\
&- 5444953249643015261\,64a^{60}b^{60} - 96989209532481420844\,8a^{55}b^{65} - 21436724021848920936a^{50}b^{70}\\
&+ 7486305076756601356\,8a^{45}b^{75} + 8911688602181278851a^{40}b^{80} - 1093401918957423768a^{35}b^{85}\\
&+ 483039534995219524a^{30}b^{90} - 13323015986481432a^{25}b^{95} + 231220616958906a^{20}b^{100}\\
&+ 4018508591592a^{15}b^{105} + 28669521444a^{10}b^{110} + 37434408a^{5}b^{115} + 781b^{120}\,.
\end{aligned}
$$

(*See the end of* §3 *for the notation.*)

(2) *We take an icosahedron in R^3. Put Z-axis on a diameter of S^2 passing through* 2 *vertices of the icosahedron and Y-axis on a diameter of S^2 passing through* 2 *centers of edges of the icosahedron which is perpendicular to Z-axis. Then the sets $B(g_i)$, which are the inverse images of $t(g_i)$ $(1 \le i \le 13)$ under the stereographic projection, are given as follows* (*see the end of* §3 *for the notation*) :

(i) $B(g_1), B(g_2), B(g_3), B(g_5), B(g_6)$ *are the twelve vertices of the icosahedron.*

(ii) $B(g_4), B(g_8), B(g_{11}), B(g_{12}), B(g_{13})$ *are the thirty points of S^2 corresponding to the centers of the edges of the icosahedron.*

(iii) $B(g_7), B(g_9), B(g_{10})$ *are the twenty points of S^2 corresponding to the centers of the faces of the icosahedron.*

PROOF. (1) Since $I_G = (X^{11}Y + 11X^6Y^6 - XY^{11}, -(X^{20}+Y^{20}) + 228(X^{15}Y^5 - X^5Y^{15}) - 494X^{10}Y^{10})$, we have

$$x^{11}y + 11x^6y^6 - xy^{11} = 0\,, \tag{8.1}$$

$$-(x^{20}+y^{20}) + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10} = 0\,. \tag{8.2}$$

For later use, we prepare several relations. By (8.1), we have

$$x^6y^6 = -\frac{1}{11}(x^{11}y - xy^{11})\,. \tag{8.3}$$

By (8.2), (8.3), we have

$$
\begin{aligned}
x^{20} &= -y^{20} + 228(x^{15}y^5 - x^5y^{15}) + 494x^4y^4 \cdot \frac{1}{11}(x^{11}y - xy^{11})\\
&= -y^{20} + \frac{3002}{11}(x^{15}y^5 - x^5y^{15}).
\end{aligned} \tag{8.4}
$$

By (8.3), (8.4), we have

$$\begin{aligned} x^{11}y^{11} &= -\frac{1}{11}xy(x^{15}y^5 - x^5y^{15}) \\ &= -\frac{1}{11}xy \cdot \frac{11}{3002}(x^{20} + y^{20}) \\ &= -\frac{1}{3002}(x^{21}y + xy^{21}) . \end{aligned} \tag{8.5}$$

Moreover since

$$\begin{aligned} x^{11}y^{11} &= -\frac{1}{11}(x^{16}y^6 - x^6y^{16}) \\ &= -\frac{1}{11}\left(-x^{10} \cdot \frac{1}{11}(x^{11}y - xy^{11}) + y^{10} \cdot \frac{1}{11}(x^{11}y - xy^{11})\right) \\ &= \frac{1}{121}(x^{21}y + xy^{21} - 2x^{11}y^{11}) , \end{aligned}$$

we have

$$x^{11}y^{11} = \frac{1}{123}(x^{21}y + xy^{21}) . \tag{8.6}$$

Therefore by (8.5), (8.6), we have

$$x^{11}y^{11} = x^{21}y + xy^{21} = 0 . \tag{8.7}$$

By (8.3), (8.7), we have

$$\begin{aligned} x^{16}y^6 &= -\frac{1}{11}(x^{21}y - x^{11}y^{11}) \\ &= -\frac{1}{11}x^{21}y \\ &= \frac{1}{11}xy^{21} , \\ x^6y^{16} &= -\frac{1}{11}(x^{11}y^{11} - xy^{21}) \\ &= xy^{21}. \end{aligned}$$

Therefore we have

$$x^{16}y^6 = x^6y^{16} = -\frac{1}{11}x^{21}y = \frac{1}{11}xy^{21} . \tag{8.8}$$

Multiplying (8.8) by x^5 and using (8.7), we have

$$x^{26}y = x^{21}y^6 = x^6y^{21} = xy^{26} = 0 . \tag{8.9}$$

Therefore by (8.7), (8.9), we have

$$x^i y^j = x^j y^i = 0\,, \quad i, j \geq 11 \tag{8.10}$$

and

$$x^i y^j = x^j y^i = 0\,, \quad i \geq 26\,, \quad j \geq 1 \quad \text{or} \quad i \geq 21\,, \quad j \geq 6\,. \tag{8.11}$$

Since $x^{21}y = -xy^{21}$, we have

$$x^{i+21}y^{j+1} = -x^{i+1}y^{j+21}\,. \tag{8.12}$$

By (8.8), we have

$$x^i y^j = x^{i+10}y^{j-10} = -\frac{1}{11}x^{i+15}y^{j-15} = \frac{1}{11}x^{i-5}y^{j+5}\,, \quad i \geq 6,\ j \geq 16\,, \tag{8.13}$$

$$x^i y^j = x^{i-10}y^{j+10} = -\frac{1}{11}x^{i+5}y^{j-5} = \frac{1}{11}x^{i-15}y^{j+15}\,, \quad i \geq 16,\ j \geq 6\,. \tag{8.14}$$

It follows from (8.4), (8.10), (8.13), (8.14) that

$$x^{30-i} = 273x^{25-i}y^5, \quad y^{30-i} = -273x^5y^{25-i}\,. \tag{8.15}$$

By (8.3), we have

$$x^{15-i}y^{10-j} = \frac{1}{122}(-11x^{20-i}y^{5-j} + x^{5-i}y^{20-j})\,, \quad 0 \leq i, j \leq 4\,, \tag{8.16}$$

$$x^{10-i}y^{15-j} = \frac{1}{122}(x^{20-i}y^{5-j} + 11x^{5-i}y^{20-j})\,, \quad 0 \leq i, j \leq 4\,. \tag{8.17}$$

It follows from (8.4), (8.16), (8.17) that

$$x^{25-i} = \frac{11}{61}(-142x^{5-i}y^{20} + 1501x^{20-i}y^5)\,, \quad 0 \leq i, j \leq 4\,, \tag{8.18}$$

$$y^{25-i} = -\frac{11}{61}(142x^{20}y^{5-i} + 1501x^5y^{20-i})\,, \quad 0 \leq i, j \leq 4\,. \tag{8.19}$$

By (8.3), we have

$$x^{10-i}y^{10-j} = -\frac{1}{11}(x^{15-i}y^{5-j} - x^{5-i}y^{15-j})\,, \quad 0 \leq i, j \leq 4\,, \tag{8.20}$$

By using (8.10), ..., (8.20), we get the matrix representations of $\rho_i, i = 0, \dots, 15$. But we omit the details for brevity's sake.

(2) For $1 \leq i \leq 21$, we have the following :

$$g_1g_2g_3g_5g_6 = a^{11}b + 11a^6b^6 - ab^{11}\,,$$

$$g_4g_8g_{11}g_{12}g_{13} = -(a^{20} + b^{20}) + 228(a^{15}b^5 - a^5b^{15}) - 494a^{10}b^{10}\,,$$

$$g_7g_9g_{10} = (a^{30} + b^{30}) + 522(a^{25}b^5 - a^5b^{25}) - 10005(a^{20}b^{10} + a^{10}b^{20})\,.$$

These polynomials are generators of the invariant algebra of the binary icosahedral group ([5], chap. 2). The sets of zeros of these invariants correspond to the set of vertices, the centers of the edges and the centers of the faces of the icosahedron respectively. □

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