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On a Distribution Property of the Residual Order of $a \pmod{p}$ with a Quadratic Residue Condition

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Abstract. Let a be a positive integer with $a \geq 2$ and $Q_a(k,l)$ be the set of odd prime numbers p such that the residual order of a in $\mathbf{Z}/p\mathbf{Z}^\times$ is congruent to l mod k. The natural density of the set $Q_a(q,0)$ (q is a prime) is already known. In this paper, we consider the set $S_{a,b}(k,l)$, which consists of the primes p that belong to $Q_a(k,l)$ and satisfy $\left(\frac{b}{p}\right) = 1$, where $\left(\frac{b}{p}\right)$ is the Legendre symbol and b is a fixed integer. Heuristically, the natural density of $S_{a,b}(k,l)$ is expected to be half of that of $Q_a(k,l)$, but it is not true for some choices of a and b. In this paper, we determine the natural density of $S_{a,b}(k,l)$ for $(k,l) = (2,j), (q,0), (4,l)$, where $j = 0, 1$, q is an odd prime and $l = 0, 2$.

1. Introduction

Let $\mathbf{P}$ be the set of all odd prime numbers and $S \subset \mathbf{P}$. The natural density ΔS of the set S is defined by

$$\Delta S = \lim_{x \to \infty} \frac{\#\{s \in S\,;\ s \leq x\}}{\#\{p \in \mathbf{P}\,;\ p \leq x\}}\,,$$

if it exists.

We take an integer $a \geq 2$. For a prime p with $(a,p) = 1$, we define $D_a(p)$, the residual order of $a \pmod{p}$ by

$$D_a(p) = \#\langle a \pmod{p} \rangle\,,$$

i.e. the order of the subgroup generated by a in the group $\mathbf{Z}/p\mathbf{Z}^\times$. We also introduce the quantity

$$I_a(p) = |\mathbf{Z}/p\mathbf{Z}^\times : \langle a \pmod{p} \rangle|\,,$$

i.e. the residual index of $a \pmod{p}$. We have

$$D_a(p) I_a(p) = p - 1. \tag{1.1}$$

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In this paper, we consider the prime set

$$S_{a,b}(k,l)=\left\{p\in\mathbf{P}\,;\ p\nmid a,b,\ D_a(p)\equiv l\ (\mathrm{mod}\,k),\ \left(\frac{b}{p}\right)=1\right\},$$

where $a,b,k,l\in\mathbf{Z}$, $a\geq 2$, $b\neq 0$ and $(\frac{b}{p})$ is the Legendre symbol. For simplicity, we assume that a and b are square free. We introduce another prime set

$$Q_a(k,l)=\left\{p\in\mathbf{P}\,;\ p\nmid a,\ D_a(p)\equiv l\ (\mathrm{mod}\,k)\right\}.$$

It is known that

$$\Delta Q_a(q,0)=\frac{q}{q^2-1}$$

if $(a,q)\neq(2,2)$ ($\Delta Q_2(2,0)=17/24$, see [2], [3] and [8]). It is also well known that

$$\Delta\left\{p\in\mathbf{P}\,;\ p\nmid b,\ \left(\frac{b}{p}\right)=1\right\}=\frac{1}{2}. \tag{1.2}$$

So, heuristically, we expect that

$$\Delta S_{a,b}(k,l)=\frac{1}{2}\Delta Q_a(k,l)\,. \tag{1.3}$$

In many cases it is true, but this equality does not hold for some choices of a and b.

The aim of this paper is to determine $\Delta S_{a,b}(k,l)$ in the case $(k,l)=(2,j),(q,0),(4,l)$ ($j=0,1$, q is an odd prime, $l=0,2$) and observe the effect of the algebraic interaction between a and b on the density $\Delta S_{a,b}(k,l)$. Let

$$S_{a,b}(x;k,l)=\{p\in S_{a,b}(k,l)\,;\ p\leq x\}\,.$$

The main results are the following:

THEOREM 1. *We assume a, b are square free positive integers with a, $b\geq 2$. Then we have*

$$\#S_{a,b}(x;2,0)=\Delta S_{a,b}(2,0)\,\mathrm{li}\,x+O\left(\frac{x}{\log x\log\log x}\right)\quad(x\to\infty)\,,$$

where $\mathrm{li}\,x=\int_2^x(\log t)^{-1}dt$ *and the density* $\Delta S_{a,b}(2,0)$ *is given by the following:*

$$\Delta S_{2,2}(2,0)=\frac{5}{24}\,;\quad \Delta S_{a,a}(2,0)=\frac{1}{6}\,,\quad \text{if } a\neq 2\,;$$

$$\Delta S_{a,b}(2,0)=\frac{1}{3}\,,\quad \text{if } a,b\neq 2\,,\quad a\neq b,\ a\neq 2b \text{ and } b\neq 2a\,;$$

$$\Delta S_{a,b}(2,0)=\frac{17}{48},$$

if one of the following three conditions holds:

(i) $a, b \neq 2,\ a = 2b,\ or\ b = 2a,$
(ii) $a \neq 2,\ b = 2,$
(iii) $a = 2,\ b \neq 2.$

It is remarkable that the conditions (i) through (iii) turn out to be symmetric with respect to a and b, despite that the initial ones $D_a(p) \equiv 0 \pmod q$ and $\left(\frac{b}{p}\right) = 1$ are not.

By $S_{a,b}(2,1) = \{p \in \mathbf{P}\ ;\ p \nmid b,\ \left(\frac{b}{p}\right) = 1\} - S_{a,b}(2,0)$ and (1.2), we easily obtain the natural densities of all the sets

$$S^{\pm}_{a,b}(2, j) = \left\{p \in \mathbf{P}\ ;\ p \nmid a, b,\ D_a(p) \equiv j \pmod 2,\ \left(\frac{b}{p}\right) = \pm 1\right\} \quad (j = 0, 1)\,.$$

COROLLARY 2. *Let a, b be as above. Then we have*

$$\# S^{\pm}_{a,b}(x; 2, j) = \Delta S^{\pm}_{a,b}(2, j) \operatorname{li} x + O\left(\frac{x}{\log x \log\log x}\right) \quad (x \to \infty)\,,$$

where the density $\Delta S^{\pm}_{a,b}(2, j)$ is given by the following table. The condition $(*)$ means $a, b \neq 2$, $a \neq b$, $a \neq 2b$ and $b \neq 2a$. The condition $(**)$ means one of (i) *and* (ii) *in Theorem* 1 :

$\underline{a = 2}$

	$\Delta S^{+}_{a,b}(2,0)$	$\Delta S^{-}_{a,b}(2,0)$	$\Delta S^{+}_{a,b}(2,1)$	$\Delta S^{-}_{a,b}(2,1)$
$a = b = 2$	$\frac{5}{24}$	$\frac{1}{2}$	$\frac{7}{24}$	0
$a = 2, b \neq 2$	$\frac{17}{48}$	$\frac{17}{48}$	$\frac{7}{48}$	$\frac{7}{48}$

$\underline{a \neq 2}$

	$\Delta S^{+}_{a,b}(2,0)$	$\Delta S^{-}_{a,b}(2,0)$	$\Delta S^{+}_{a,b}(2,1)$	$\Delta S^{-}_{a,b}(2,1)$
$a = b \neq 2$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	0
$(*)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
$(**)$	$\frac{17}{48}$	$\frac{5}{16}\left(=\frac{15}{48}\right)$	$\frac{7}{48}$	$\frac{3}{16}\left(=\frac{9}{48}\right)$

REMARK. (i) We can verify $S^{-}_{a,a}(2,1) = \emptyset$ in an elementary manner: $D_a(p) \equiv 1 \pmod 2$ and (1.1) imply $2 | I_a(p)$, which is equivalent to $\left(\frac{a}{p}\right) = 1$.

(ii) We have $\Delta S^{+}_{a,b}(2, j) = \Delta S^{-}_{a,b}(2, j) = \Delta Q_a(2, j)/2$ when $a = 2$ and $b \neq 2$, or $(*)$ holds.

THEOREM 3. *Let a, b be as above, and q be an odd prime number. Then we have*

$$\#S_{a,b}(x; q, 0) = \Delta S_{a,b}(q, 0)\operatorname{li} x + O\left(\frac{x}{\log x \log\log x}\right) \quad (x \to \infty),$$

where the density $\Delta S_{a,b}(q, 0)$ is given by the following:

$$\Delta S_{a,b}(q, 0) = \frac{q}{q^2 - 1}, \quad \textit{if } b = q, \quad q \equiv 1 \pmod 4;$$

$$\Delta S_{a,b}(q, 0) = \frac{q}{2(q^2 - 1)}, \quad \textit{otherwise}.$$

We know from these theorems that

$$\Delta S_{a,a}(2, 0) = \frac{1}{6} = \frac{1}{4}\Delta Q_a(2, 0),$$

if $a \neq 2$, and

$$\Delta S_{a,a}(q, 0) = \frac{q}{2(q^2 - 1)} = \frac{1}{2}\Delta Q_a(q, 0)$$

if $q \geq 3$ and $a \not\equiv 1 \pmod 4$. It is remarkable that in the latter case, even though $b = a$, the probabilistic argument in (1.3) is true, but in the former case, $\Delta S_{a,a}(2, 0)$ is actually much less than the value expected from (1.3).

The case $q = 4$ can be dealt with in a similar manner and we obtain the following:

THEOREM 4. *Let a, b be as above. Then we have*

$$\#S_{a,b}(x; 4, 0) = \Delta S_{a,b}(4, 0)\operatorname{li} x + O\left(\frac{x}{\log x \log\log x}\right) \quad (x \to \infty),$$

where the density $\Delta S_{a,b}(4, 0)$ is given by the following:

$$\Delta S_{2,2}(4, 0) = \frac{1}{6}; \quad \Delta S_{a,a}(4, 0) = \frac{1}{12}, \quad \textit{if } a \neq 2;$$

$$\Delta S_{a,b}(4, 0) = \frac{1}{6}, \quad \textit{if } a, b \neq 2, \quad a \neq b,\ a \neq 2b \textit{ and } b \neq 2a;$$

$$\Delta S_{a,b}(4, 0) = \frac{5}{24},$$

if one of the following three conditions holds:

(i) $a, b \neq 2,\ a = 2b,\ \textit{or } b = 2a,$

(ii) $a \neq 2,\ b = 2,$

(iii) $a = 2,\ b \neq 2.$

Since $\#S_{a,b}(x; 4, 2) = \#S_{a,b}(x; 2, 0) - \#S_{a,b}(x; 4, 0)$, we easily obtain the following:

COROLLARY 5. *Let a, b be as above. Then we have*

$$\#S_{a,b}(x;4,2)=\Delta S_{a,b}(4,2)\operatorname{li}x+O\left(\frac{x}{\log x\log\log x}\right)\quad(x\to\infty),$$

where the density $\Delta S_{a,b}(4,2)$ is given by the following:

$$\Delta S_{2,2}(4,2)=\frac{5}{24}-\frac{1}{6}=\frac{1}{24};\quad \Delta S_{a,a}(4,2)=\frac{1}{6}-\frac{1}{12}=\frac{1}{12},\quad \textit{if } a\neq 2;$$

$$\Delta S_{a,b}(4,2)=\frac{1}{3}-\frac{1}{6}=\frac{1}{6},\quad \textit{if } a,b\neq 2,\quad a\neq b,\quad a\neq 2b \textit{ and } b\neq 2a;$$

$$\Delta S_{a,b}(4,2)=\frac{17}{48}-\frac{5}{24}=\frac{7}{48},$$

if one of the following three conditions holds:

(i) $a,b\neq 2,\ a=2b,\ b=2a,$

(ii) $a\neq 2,\ b=2,$

(iii) $a=2,\ b\neq 2.$

We obtain from Theorem 4 and Corollary 5 the following tables which show how the sets $Q_a(4,0)$ and $Q_a(4,2)$ are divided by adding the conditions $\left(\frac{b}{p}\right)=1$ or $\left(\frac{b}{p}\right)=-1$. Each value is the density of the primes in $Q_a(4,l)$ satisfying $\left(\frac{b}{p}\right)=\pm 1$. The condition $(*)$ means $a,b\neq 2, a\neq b, a\neq 2b$ and $b\neq 2a$. The condition $(**)$ means one of (i) and (ii) in Theorem 4. We can see from these tables that the "equi-distribution property" holds only in the case of $(*)$.

<u>$a=2$</u>

	$\Delta Q_2(4,0)=5/12$		$\Delta Q_2(4,2)=7/24$	
	$\left(\frac{b}{p}\right)=1$	$\left(\frac{b}{p}\right)=-1$	$\left(\frac{b}{p}\right)=1$	$\left(\frac{b}{p}\right)=-1$
$a=b=2$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{24}$	$\frac{1}{4}$
$a=2, b\neq 2$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{5}{24}$

$\underline{a \neq 2}$

	$\Delta Q_a(4,0) = 1/3$		$\Delta Q_a(4,2) = 1/3$	
	$\left(\frac{b}{p}\right) = 1$	$\left(\frac{b}{p}\right) = -1$	$\left(\frac{b}{p}\right) = 1$	$\left(\frac{b}{p}\right) = -1$
$a = b \neq 2$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{4}$
$(*)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$(**)$	$\frac{5}{24}$	$\frac{1}{8}\left(= \frac{3}{24}\right)$	$\frac{7}{48}$	$\frac{3}{16}\left(= \frac{9}{48}\right)$

This paper is organized as follows: in Section 2, we introduce some preliminary results about algebraic number theory and the prime ideal theorem. In Sections 3, 4 and 5, we prove the main results (Theorems 1, 3 and 4). In Section 6, some results of numerical experiments are shown which support our main theorems.

For a prime power q^e, $q^e \parallel m$ means that $q^e \mid m$ and $q^{e+1} \nmid m$. We denote Euler's totient by $\varphi(n)$. For $r \in \mathbf{Z}$, let ζ_r be a primitive r-th root of unity.

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2. Preliminaries

In this section, we introduce some preliminary results. First we need the following:

THEOREM 6 (THE PRIME IDEAL THEOREM). *Let K be a finite Galois extension field over $\mathbf{Q}$, $n = [K : \mathbf{Q}]$ and Δ be the discriminant of K. Then under the condition $\exp\left(10n(\log|\Delta|)^2\right) \leq x$, we have*

$$\begin{aligned}\pi_K(x) &= \#\{\mathfrak{p} : \text{a prime ideal in } K\ ;\ N\mathfrak{p} \leq x\} \\ &= \mathrm{li}\, x + O\left(\mathrm{li}\,(x^{\beta_0}) + x \exp\left(-c_1\sqrt{\frac{\log x}{n}}\right)\right),\end{aligned}$$

where $\beta_0 \in \mathbf{R}$,

$$\left(\frac{1}{2} <\right) \beta_0 < \max\left\{1 - \frac{1}{4\log|\Delta|},\ 1 - \frac{1}{c_2|\Delta|^{1/n}}\right\},$$

$c_1,\ c_2 > 0$ *and the constant implied by O-symbol does not depend on n, Δ.*

PROOF. See [5, Theorems 1.3 and 1.4]. ∎

For the calculation of the densities, we need to know the extension degrees of some algebraic number fields.

LEMMA 7. (i) *Let $b \in \mathbf{N}$, $b \geq 2$ and be square free. Then the real quadratic fields which are contained in $\mathbf{Q}(\zeta_{2^j}, \sqrt{b})$ are*

$$\begin{cases} \mathbf{Q}(\sqrt{b}), & \text{if } j = 1, 2, \\ \mathbf{Q}(\sqrt{2}), \mathbf{Q}(\sqrt{b}), \mathbf{Q}(\sqrt{2b}), & \text{if } j \geq 3. \end{cases}$$

(ii) *We have*

$$[\mathbf{Q}(\zeta_{2^j}, \sqrt{b}) : \mathbf{Q}] = \begin{cases} \varphi(2^j), & \text{if } j \geq 3 \quad \text{and} \quad b = 2, \\ 2\varphi(2^j), & \text{otherwise}. \end{cases}$$

PROOF. We give a proof of (i) only. Suppose $\sqrt{c} \in \mathbf{Q}(\zeta_{2^j}, \sqrt{b}) = \mathbf{Q}(\zeta_{2^j})(\sqrt{b})$ ($c \in \mathbf{N}$, $c \geq 2$ and is square free) and is expressed in the form $\sqrt{c} = \alpha + \beta\sqrt{b}$ ($\alpha, \beta \in \mathbf{Q}(\zeta_{2^j})$). If $\alpha \neq 0$, then $\sqrt{c} = (c + \alpha^2 - b\beta^2)/2\alpha \in \mathbf{Q}(\zeta_{2^j})$ and it follows that $j \geq 3$. In this case, the only real quadratic field which is contained in $\mathbf{Q}(\zeta_{2^j})$ is $\mathbf{Q}(\sqrt{2})$, so we have $c = 2$. If $\alpha = 0$, then $\beta = \sqrt{c/b} \in \mathbf{Q}(\zeta_{2^j})$. So, when $j = 1$ or 2, we can conclude $b = c$. When $j \geq 3$, we have $\sqrt{c/b} \in \mathbf{Q}(\sqrt{2})$ and $c = b$, $2b$ or $b/2$. ∎

REMARK. When q is an odd prime and $j \geq 1$, we have

$$[\mathbf{Q}(\zeta_{q^j}, \sqrt{b}) : \mathbf{Q}] = \begin{cases} \varphi(q^j), & \text{if } b = q, \ q \equiv 1 \pmod 4, \\ 2\varphi(q^j), & \text{otherwise}, \end{cases} \tag{2.1}$$

since the quadratic field which is contained in $\mathbf{Q}(\zeta_{q^j})$ is $\mathbf{Q}(\sqrt{q})$ if $q \equiv 1 \pmod 4$ and $\mathbf{Q}(\sqrt{-q})$ if $q \equiv 3 \pmod 4$. We see later that the case $b = q$ and $q \equiv 1 \pmod 4$ can be treated quite easily without using (2.1) (see Section 4).

In Lemmas 8 and 9, we assume $L = \mathbf{Q}(a^{1/q^l})$, $M = \mathbf{Q}(\zeta_{q^j}, \sqrt{b})$ and $K = L \cap M$ (q: prime, $j \geq l \geq 1$).

LEMMA 8. *We have*

$$[LM : \mathbf{Q}] = [\mathbf{Q}(\zeta_{q^j}, \sqrt{b}, a^{1/q^l}) : \mathbf{Q}] = \frac{[L : \mathbf{Q}][M : \mathbf{Q}]}{[K : \mathbf{Q}]}.$$

PROOF. Both L and M are finite extensions over $\mathbf{Q}$ and M is a Galois extension over $\mathbf{Q}$. So LM/L is a Galois extension and we have $\mathrm{Gal}(LM/L) \cong \mathrm{Gal}(M/K)$. Then, $[LM : L] = [M : K]$. Hence,

$$[LM : \mathbf{Q}] = [LM : L][L : \mathbf{Q}] = [M : K][L : \mathbf{Q}] = \frac{[M : \mathbf{Q}]}{[K : \mathbf{Q}]}[L : \mathbf{Q}]. \qquad ∎$$

LEMMA 9. *Let L, M and K be as above.*

(i) *If q is an odd prime, then we have $K = \mathbf{Q}$.*

(ii) *If $q = 2$ and $j = 1, 2$, then we have*

$$K = \begin{cases} \mathbf{Q}(\sqrt{a}), & \text{if } a = b, \\ \mathbf{Q}, & \text{otherwise}. \end{cases}$$

If $q = 2$ *and* $j \geq 3$, *then we have*

$$K = \begin{cases} \mathbf{Q}(\sqrt{a}), & \text{if } a = 2,\ b,\ 2b,\ b/2, \\ \mathbf{Q}, & \text{otherwise}. \end{cases}$$

PROOF. (i) First note that all the subfields of M are normal extensions over $\mathbf{Q}$, since M is a composition field of $\mathbf{Q}(\zeta_{q^j})$ and $\mathbf{Q}(\sqrt{b})$, which are abelian extensions over $\mathbf{Q}$, and is contained in some cyclotomic field. So, $K = M \cap L \subset M$ is normal over $\mathbf{Q}$. We also note that the maximal normal subfield over $\mathbf{Q}$ which is contained in $\mathbf{Q}(a^{1/u})$ is

$$\begin{cases} \mathbf{Q}, & \text{if } u \text{ is odd}, \\ \mathbf{Q}(\sqrt{a}), & \text{if } u \text{ is even} \end{cases} \tag{2.2}$$

([7, Lemma 3.1]). Then it is clear from (2.2) that $K = \mathbf{Q}$ if q is odd.

(ii) Applying (2.2) to K and L above, we see that

$$K = \begin{cases} \mathbf{Q}(\sqrt{a}), & \text{if } \mathbf{Q}(\sqrt{a}) \subset M, \\ \mathbf{Q}, & \text{otherwise}. \end{cases}$$

So, we get the desired result invoking Lemma 7 (i). ∎

We put

$$K_{a,b,q;j,l} = K_{j,l} = \mathbf{Q}(\zeta_{q^j}, a^{1/q^l}, \sqrt{b}). \tag{2.3}$$

Gathering these results, we get the following proposition which will be used in the subsequent sections:

PROPOSITION 10. (I) *Let q be an odd prime. Then we have*

$$[K_{j,l} : \mathbf{Q}] = \begin{cases} q^l \varphi(q^j) = (q-1)q^{j+l-1}, & \text{if } b = q, \quad q \equiv 1 \pmod 4, \\ 2q^l \varphi(q^j) = 2(q-1)q^{j+l-1}, & \text{otherwise}. \end{cases}$$

(II) *Let* $q = 2$.

(i) *When* $j = 1, 2$,

$$[K_{j,l} : \mathbf{Q}] = \begin{cases} 2^{j+l-1}, & \text{if } a = b, \\ 2^{j+l}, & \text{otherwise}. \end{cases}$$

(ii) *When* $j \geq 3$,

(ii–a) *if* $a = b = 2$, *then*

$$[K_{j,l} : \mathbf{Q}] = 2^{j+l-2},$$

(ii–b) *if one of*
(1) $a \neq 2, b = 2$,
(2) $a = 2, b \neq 2$,
(3) $a = b \neq 2$,
(4) $a, b \neq 2, a = 2b$ *or* $2a = b$

is satisfied, then

$$[K_{j,l} : \mathbf{Q}] = 2^{j+l-1},$$

(ii–c) *if* $a, b \neq 2, a \neq b, a \neq 2b, 2a \neq b$, *then*

$$[K_{j,l} : \mathbf{Q}] = 2^{j+l} .$$

3. Proof of Theorem 1

In this section, we give a proof of Theorem 1. We transform the condition on $D_a(p)$ into some conditions on $I_a(p)$. We consider a prime p such that $2^j \parallel p-1,\ j \geq 1$. From the equation (1.1), we have

$$D_a(p) \equiv 0 \pmod 2 \Leftrightarrow 2^j \nmid I_a(p)$$

and

$$S_{a,b}(x; 2, 0) = \bigcup_{j \geq 1} \left\{ p \leq x \ ;\ 2^j \parallel p-1,\ 2^j \nmid I_a(p),\ \left(\frac{b}{p}\right) = 1 \right\} .$$

Then we have

$$\begin{aligned} \#S_{a,b}(x; 2, 0) = \# &\left\{ p \leq x \ ;\ \left(\frac{b}{p}\right) = 1 \right\} \\ &- \sum_{j \geq 1} \# \left\{ p \leq x \ ;\ p \equiv 1 \ (\mathrm{mod}\ 2^j),\ 2^j \mid I_a(p),\ \left(\frac{b}{p}\right) = 1 \right\} \\ &+ \sum_{j \geq 1} \# \left\{ p \leq x \ ;\ p \equiv 1 \ (\mathrm{mod}\ 2^{j+1}),\ 2^j \mid I_a(p),\ \left(\frac{b}{p}\right) = 1 \right\} . \end{aligned} \tag{3.1}$$

We estimate the former sum in (3.1) (We can estimate the latter sum in (3.1) in a similar manner). Let

$$M_j(x) = \left\{ p \leq x \ ;\ p \equiv 1 \ (\mathrm{mod}\ 2^j),\ 2^j \mid I_a(p),\ \left(\frac{b}{p}\right) = 1 \right\} .$$

We estimate $\sum_{j \geq 1} M_j(x)$. We divide $(0, x]$ into the following three intervals:

$$(0, x] = I_1 \cup I_2 \cup I_3 ,$$

where

$$I_1 = (0, \log\log x],\ I_2 = (\log\log x, \sqrt{x}\log^2 x],\ I_3 = (\sqrt{x}\log^2 x, x]\,.$$

Then

$$\sum_{j\geq 1} \#M_j(x) = \Bigg(\sum_{2^j\in I_1} + \sum_{2^j\in I_2} + \sum_{2^j\in I_3}\Bigg)\#M_j(x)\,.$$

Here we introduce the set

$$M'_j(x) = \left\{p \leq x\ ;\ p \equiv 1 \pmod{2^j},\ 2^j \mid I_a(p)\right\}\,.$$

Then $\#M_j(x) \leq \#M'_j(x)$.

First we consider the sum on I_3. It can be estimated in a similar way to [4]. Under $2^j \parallel p-1$,

$$2^j \mid I_a(p) \Leftrightarrow v^{2^j} \equiv a \pmod{p} \text{ is solvable}\,.$$

Then, $a^{2(p-1)/2^j} \equiv 1 \pmod{p}$. Since $(p-1)/2^j < \sqrt{x}/\log^2 x$, p must divide the positive product

$$\prod_{m<\sqrt{x}/\log^2 x} \left(a^{2m}-1\right)\,,$$

so we have

$$2^{\sum_{2^j\in I_3}\#M'_j(x)} \leq \prod_{m<\sqrt{x}/\log^2 x} a^{2m\log x/\log 2}\,.$$

Therefore,

$$\sum_{2^j\in I_3} \#M_j(x) \leq \sum_{2^j\in I_3} \#M'_j(x) \ll \sum_{m<\sqrt{x}/\log^2 x} m\cdot\log x = O\left(\frac{x}{\log^3 x}\right)\,. \tag{3.2}$$

Next we consider the sum on I_2. By the Siegel-Walfisz theorem, for some $\varepsilon_1 > 0$, we have

$$\begin{aligned}\sum_{2^j\in I_2} \#M_j(x) &\leq \sum_{2^j\in I_2} \#\left\{p\leq x\ ;\ p-1\equiv 0 \pmod{2^j}\right\}\\ &= \sum_{2^j\in I_2} \frac{1}{\varphi(2^j)}\left\{\operatorname{li} x + O(xe^{-\varepsilon_1\sqrt{\log x}})\right\} = O\left(\frac{x}{\log x\log\log x}\right)\,.\end{aligned} \tag{3.3}$$

Finally we consider the sum on I_1. Note that

$$p\equiv 1 \pmod{2^j},\ 2^j \mid I_a(p),\ \left(\frac{b}{p}\right) = 1$$

$$\Leftrightarrow p \text{ splits completely in } \mathbf{Q}(\zeta_{2^j}, a^{1/2^j}) \text{ and } \mathbf{Q}(\sqrt{b})$$
$$\Leftrightarrow p \text{ splits completely in } K_{j,j} = \mathbf{Q}(\zeta_{2^j}, a^{1/2^j}, \sqrt{b})\,.$$

When p splits completely in $K_{j,j}$, the number of distinct prime ideals of degree 1 over p is $n_{j,j} = [K_{j,j} : \mathbf{Q}]$. We put

$$\pi^{(1)}_{K_{j,j}}(x) = \#\left\{\mathfrak{p} : \text{a prime ideal in } K_{j,j}\ ;\ N\mathfrak{p} \le x,\ \mathfrak{p} : \text{degree } 1\right\}\,.$$

Then for $\alpha = \log\log\log x/\log 2$, we have

$$\sum_{2^j \in I_1} \#M_j(x) = \sum_{j \le \alpha} \frac{\pi^{(1)}_{K_{j,j}}(x)}{n_{j,j}}\,.$$

Therefore we have to evaluate $\pi^{(1)}_{K_{j,j}}(x)$. This evaluation needs Theorem 6:

$$\pi_{K_{j,j}}(x) = \mathrm{li}\,x + O\left(\mathrm{li}\,(x^{\beta_0}) + x\exp\left(-c_1\sqrt{\frac{\log x}{n_{j,j}}}\right)\right)\,.$$

To estimate β_0, we need the following estimate of the discriminant Δ of $K_{j,j}$:

$$|\Delta| \le (n_{j,j}^2 ab)^{n_{j,j}}\,. \tag{3.4}$$

The formula (3.4) is proved by the chain rule of differents $\mathfrak{d}_{K_{j,j}/\mathbf{Q}} = \mathfrak{d}_{K_{j,j}/F_j}\mathfrak{d}_{F_j/\mathbf{Q}}$ $(F_j = \mathbf{Q}(\zeta_{2^j}, a^{1/2^j}))$. Taking the norm $N = N_{K_{j,j}/\mathbf{Q}}$ of the both sides, we have $|\Delta| = N(\mathfrak{d}_{K_{j,j}/F_j})|D_{F_j}|^2 \le (2b)^{n_{j,j}}([F_j : \mathbf{Q}]^2 a)^{n_{j,j}} \le (n_{j,j}^2 ab)^{n_{j,j}}$, where D_{F_j} is the discriminant of F_j. We have

$$\log|\Delta| \le n_{j,j}\log(n_{j,j}^2 ab) \le d_1 n_{j,j}^2\,,$$
$$c_2|\Delta|^{1/n_{j,j}} \le c_2(n_{j,j}^2 ab) \le d_2 n_{j,j}^2\,.$$

The constants d_1 and d_2 depend only on a and b. The number d_3 below is the same.

$$\begin{aligned}\beta_0 &< \max\left\{1 - \frac{1}{4\log|\Delta|},\ 1 - \frac{1}{c_2|\Delta|^{1/n_{j,j}}}\right\}\\ &\le \max\left\{1 - \frac{1}{d_1 n_{j,j}^2},\ 1 - \frac{1}{d_2 n_{j,j}^2}\right\} \le 1 - \frac{1}{d_3 n_{j,j}^2}\end{aligned}$$

by $\max\left\{4\log|\Delta|,\ c_2|\Delta|^{1/n_{j,j}}\right\} \le \max\left\{d_1 n_{j,j}^2,\ d_2 n_{j,j}^2\right\} \le d_3 n_{j,j}^2$. Using this, we have

$$\mathrm{li}\,(x^{\beta_0}) \ll \frac{x^{\beta_0}}{\log x^{\beta_0}} \le x\exp\left(-\frac{\sqrt{\log x}}{d_3 n_{j,j}^2}\right)\,.$$

Thus we have

$$\pi_{K_{j,j}}(x) = \mathrm{li}\, x + O\biggl(x \exp\biggl(-c\frac{\sqrt{\log x}}{n_{j,j}^2}\biggr)\biggr),$$

where $c > 0$ does not depend on j. Also, since the contribution of prime ideals of degree more than one is $O(n_{j,j}\sqrt{x})$, we have

$$\pi_{K_{j,j}}^{(1)}(x) = \mathrm{li}\, x + O\biggl(n_{j,j} x \exp\biggl(-c\frac{\sqrt{\log x}}{n_{j,j}^2}\biggr)\biggr).$$

We can estimate the sum on I_1 as follows:

$$\begin{aligned}\sum_{2^j \in I_1} \#M_j(x) &= \sum_{2^j \le \log\log x} \biggl\{\frac{1}{n_{j,j}}\mathrm{li}\, x + O\biggl(x \exp\biggl(-c\frac{\sqrt{\log x}}{n_{j,j}^2}\biggr)\biggr)\biggr\} \\ &= \sum_{j \ge 1} \frac{1}{n_{j,j}}\mathrm{li}\, x - \sum_{2^j > \log\log x} \frac{1}{n_{j,j}}\mathrm{li}\, x + O\biggl(\sum_{2^j \le \log\log x} x \exp\biggl(-c\frac{\sqrt{\log x}}{n_{j,j}^2}\biggr)\biggr).\end{aligned}$$

We have

$$\sum_{2^j > \log\log x} \frac{1}{n_{j,j}}\mathrm{li}\, x \ll \mathrm{li}\, x \sum_{2^j > \log\log x} \frac{1}{2^{2j}} \ll \frac{x}{\log x \Bigl(\log\log x\Bigr)^2}.$$

When $2^j \le \log\log x$, $n_{j,j}^2 \le (\log\log x)^4$ by $n_{j,j} = m \cdot 2^{2j}$ $(m = 1, 1/2, 1/4)$, so

$$\sum_{2^j \le \log\log x} x \exp\biggl(-c\frac{\sqrt{\log x}}{n_{j,j}^2}\biggr) \le \sum_{2^j \le \log\log x} x \exp\biggl(-c\frac{\sqrt{\log x}}{(\log\log x)^4}\biggr).$$

When x is sufficiently large, for a positive integer N, $\exp(-c\sqrt{\log x}/(\log\log x)^4)$ $\log\log\log x \le 1/(\log x)^N$. Letting $N = 2$, we have

$$\sum_{2^j \le \log\log x} x \exp\biggl(-c\frac{\sqrt{\log x}}{n_{j,j}^2}\biggr) \ll \frac{x}{\log^2 x}.$$

Hence

$$\sum_{2^j \in I_1} \#M_j(x) = \sum_{j \ge 1} \frac{1}{[K_{j,j} : \mathbf{Q}]}\mathrm{li}\, x + O\biggl(\frac{x}{\log x \log\log x}\biggr). \tag{3.5}$$

Gathering (3.2), (3.3) and (3.5), we obtain

$$\sum_{j \ge 1} \#\biggl\{p \le x\ ;\ p \equiv 1 \pmod{2^j},\ 2^j \mid I_a(p),\ \biggl(\frac{b}{p}\biggr) = 1\biggr\}$$

$$= \sum_{j\geq 1} \frac{1}{[K_{j,j}:\mathbf{Q}]}\operatorname{li} x + O\left(\frac{x}{\log x \log\log x}\right).$$

The first term in (3.1) can be estimated by direct application of Theorem 6 with $K = \mathbf{Q}(\sqrt{b})$:

$$\#\left\{p \leq x\,;\ \left(\frac{b}{p}\right) = 1\right\} = \frac{1}{2}\operatorname{li} x + O\left(x \exp\left(-c_2\frac{\sqrt{\log x}}{4}\right)\right) \quad (c_2 > 0)\,.$$

Consequently, we have

$$\#S_{a,b}(x;2,0) = \left\{\frac{1}{2} - \sum_{j\geq 1}\frac{1}{[K_{j,j}:\mathbf{Q}]} + \sum_{j\geq 1}\frac{1}{[K_{j+1,j}:\mathbf{Q}]}\right\}\operatorname{li} x + O\left(\frac{x}{\log x \log\log x}\right).$$

Finally, we calculate the coefficient of $\operatorname{li} x$, that is $\Delta S_{a,b}(2,0)$. By Proposition 10, we have

$$\Delta S_{2,2}(2,0) = \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{8} + \sum_{j\geq 3}\frac{1}{2^{2j-2}}\right) + \left(\frac{1}{4} + \sum_{j\geq 2}\frac{1}{2^{2j-1}}\right) = \frac{5}{24}\,,$$

$$\Delta S_{a,a}(2,0) = \frac{1}{2} - \sum_{j\geq 1}\frac{1}{2^{2j-1}} + \sum_{j\geq 1}\frac{1}{2^{2j}} = \frac{1}{6} \quad (a \neq 2)\,,$$

$$\Delta S_{a,b}(2,0) = \frac{1}{2} - \sum_{j\geq 1}\frac{1}{2^{2j}} + \sum_{j\geq 1}\frac{1}{2^{2j+1}} = \frac{1}{3} \quad (a,b \neq 2,\ a \neq b,\ a \neq 2b,\ b \neq 2a)\,,$$

$$\Delta S_{a,b}(2,0) = \frac{1}{2} - \left(\frac{1}{4} + \frac{1}{16} + \sum_{j\geq 3}\frac{1}{2^{2j-1}}\right) + \left(\frac{1}{8} + \sum_{j\geq 2}\frac{1}{2^{2j}}\right) = \frac{17}{48}$$

(one of the conditions (i)–(iii) in Theorem 1 holds), which give Theorem 1. ∎

4. Proof of Theorem 3

In this section, we outline the proof of Theorem 3. In the case $b = q$ and $q \equiv 1 \pmod 4$, we get $S_{a,b}(x;q,0) = Q_a(x;q,0)$ in an elementary manner. Indeed, since $D_a(p) \equiv 0 \pmod q$ and $D_a(p)I_a(p) = p-1$, we have $p \equiv 1 \pmod q$. So, $q \equiv 1 \pmod 4$ and the quadratic reciprocity law give

$$\left(\frac{b}{p}\right) = \left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = \left(\frac{1}{q}\right) = 1,$$

i.e. $D_a(p) \equiv 0 \pmod q$ always implies $(\frac{b}{p}) = 1$.

Now we proceed to the case where $b = q$ and $q \equiv 1 \pmod 4$ do not hold. By the equation

$$p - 1 = I_a(p)D_a(p),$$

we have $q \mid p-1$. So we assume $q^j \parallel p-1$, $j \geq 1$. Then

$$D_a(p) \equiv 0 \pmod{q} \Leftrightarrow q^j \nmid I_a(p).$$

We can decompose $S_{a,b}(x; q, 0)$ in the same way as in Theorem 1 and get

$$\begin{aligned}\#S_{a,b}(x; q, 0) = \#&\left\{p \leq x\ ;\ p \equiv 1 \pmod{q},\ \left(\frac{b}{p}\right) = 1\right\}\\ &-\sum_{j\geq 1}\#\left\{p \leq x\ ;\ p \equiv 1 \pmod{q^j},\ q^j \mid I_a(p),\ \left(\frac{b}{p}\right) = 1\right\}\\ &+\sum_{j\geq 1}\#\left\{p \leq x\ ;\ p \equiv 1 \pmod{q^{j+1}},\ q^j \mid I_a(p),\ \left(\frac{b}{p}\right) = 1\right\}.\end{aligned}$$

We can estimate the remainder terms similarly to Theorem 1 and obtain

$$\begin{aligned}\#S_{a,b}(x; q, 0) = &\left(\frac{1}{[\mathbf{Q}(\zeta_q, \sqrt{b}) : \mathbf{Q}]} - \sum_{j\geq 1}\frac{1}{[K_{j,j} : \mathbf{Q}]} + \sum_{j\geq 1}\frac{1}{[K_{j+1,j} : \mathbf{Q}]}\right)\mathrm{li}\, x\\ &+ O\left(\frac{x}{\log x \log\log x}\right),\end{aligned}$$

where $K_{j,l} = \mathbf{Q}(\zeta_{q^j}, a^{1/q^l}, \sqrt{b})$. We calculate the coefficients of $\mathrm{li}\, x$ using Proposition 10. Then we have

$$\Delta S_{a,b}(q, 0) = \frac{1}{2(q-1)} - \sum_{j\geq 1}\frac{1}{2(q-1)q^{2j-1}} + \sum_{j\geq 1}\frac{1}{2(q-1)q^{2j}} = \frac{q}{2(q^2-1)}.$$

Thus we have proved Theorem 3. ∎

5. Proof of Theorem 4

In this section, we describe a proof of Theorem 4. The proof is similar to those of the previous theorems, so we give an outline only. By $D_a(p) \equiv 0 \pmod{4}$, $p-1 = D_a(p)I_a(p) \equiv 0 \pmod{4}$. So we assume $2^j \parallel p-1$, $j \geq 2$. Then we have

$$D_a(p) \equiv 0 \pmod{4} \Leftrightarrow 2^{j-1} \nmid I_a(p)\,.$$

We can proceed in the same way as in Theorem 1 and get

$$\begin{aligned}\#S_{a,b}(x; 4, 0) = \#&\left\{p \leq x\ ;\ p \equiv 1 \pmod{4},\ \left(\frac{b}{p}\right) = 1\right\}\\ &-\sum_{j\geq 1}\#\left\{p \leq x\ ;\ p \equiv 1 \pmod{2^{j+1}},\ 2^j \mid I_a(p),\ \left(\frac{b}{p}\right) = 1\right\}\end{aligned}$$

$$+\sum_{j\geq 1}\#\left\{p \leq x\ ;\ p \equiv 1 \ (\mathrm{mod}\ 2^{j+2}),\ 2^j \mid I_a(p),\ \left(\frac{b}{p}\right)=1\right\}.$$

Estimating the remainder terms, we get

$$\#S_{a,b}(x;4,0)=\left\{\frac{1}{[\mathbf{Q}(\zeta_4,\sqrt{b}):\mathbf{Q}]}-\frac{1}{[K_{j+1,j}:\mathbf{Q}]}+\frac{1}{[K_{j+2,j}:\mathbf{Q}]}\right\}\mathrm{li}\,x + O\left(\frac{x}{\log x\log\log x}\right),$$

where $K_{j,l}=\mathbf{Q}(\zeta_{2^j}, a^{1/2^l}, \sqrt{b})$. The densities are given by the following:

$$\Delta S_{2,2}(4,0)=\frac{1}{4}-\left(\frac{1}{4}+\sum_{j\geq 2}\frac{1}{2^{2j-1}}\right)+\sum_{j\geq 1}\frac{1}{2^{2j}}=\frac{1}{6},$$

$$\Delta S_{a,a}(4,0)=\frac{1}{4}-\sum_{j\geq 1}\frac{1}{2^{2j}}+\sum_{j\geq 1}\frac{1}{2^{2j+1}}=\frac{1}{12}\quad (a\neq 2),$$

$$\Delta S_{a,b}(4,0)=\frac{1}{4}-\sum_{j\geq 1}\frac{1}{2^{2j+1}}+\sum_{j\geq 1}\frac{1}{2^{2j+2}}=\frac{1}{6}\quad (a,b\neq 2,\ a\neq b,\ a\neq 2b,\ b\neq 2a),$$

$$\Delta S_{a,b}(4,0)=\frac{1}{4}-\left(\frac{1}{8}+\sum_{j\geq 2}\frac{1}{2^{2j}}\right)+\sum_{j\geq 1}\frac{1}{2^{2j+1}}=\frac{5}{24}$$

(one of the conditions (i)–(iii) in Theorem 4 holds). This completes the proof Theorem 4. ∎

6. Numerical examples

In this section, we give some results of numerical experiments on the densities $\Delta S_{a,b}(k,l)$. Each table shows the values $\#S_{a,b}(x;k,l)/\pi(x)$ for $x=10^m$ ($m=3,4,\dots,8$). The theoretical densities which are obtained in the previous sections are also shown.

(I) The case $(k,l)=(2,0)$

This case corresponds to Theorem 1.

(I–i) The case $a=b$

We show the data for $(a,b)=(2,2),(3,3)$ and $(6,6)$. The theoretical densities are

$5/24 \approx 0.208333$ for $(a, b) = (2, 2)$ and $1/6 \approx 0.166667$ for other cases.

x	$(a, b) = (2, 2)$	$(a, b) = (3, 3)$	$(a, b) = (6, 6)$
10^3	0.179641	0.132530	0.138554
10^4	0.206026	0.164629	0.162999
10^5	0.207069	0.164234	0.165693
10^6	0.207320	0.165856	0.166187
10^7	0.208054	0.166599	0.166288
10^8	0.208284	0.166595	0.166656

(I–ii) The case $a, b \neq 2, a \neq b, a \neq 2b, b \neq 2a$

This is the standard case in $(k, l) = (2, 0)$. The theoretical density is $1/3 \approx 0.333333$ for all cases.

x	$(a, b) = (3, 5)$	$(a, b) = (3, 10)$	$(a, b) = (5, 3)$
10^3	0.303030	0.327273	0.321212
10^4	0.331158	0.334421	0.327080
10^5	0.332464	0.331526	0.332881
10^6	0.332735	0.332913	0.333435
10^7	0.333309	0.333181	0.333158
10^8	0.333295	0.333313	0.333348

x	$(a, b) = (6, 5)$	$(a, b) = (6, 10)$	$(a, b) = (10, 3)$
10^3	0.296970	0.315152	0.296970
10^4	0.333605	0.332790	0.331158
10^5	0.332151	0.331526	0.332256
10^6	0.332161	0.332658	0.332798
10^7	0.333098	0.332799	0.333408
10^8	0.333297	0.333346	0.333218

(I–iii) None of the above cases

This case includes (a) $a, b \neq 2$, $a = 2b$ or $b = 2a$, (b) $a \neq 2$, $b = 2$, (c) $a = 2$, $b \neq 2$. The theoretical density is $17/48 \approx 0.354167$.

x	$(a, b) = (3, 6)$	$(a, b) = (6, 3)$	$(a, b) = (7, 14)$	$(a, b) = (14, 7)$
10^3	0.331325	0.325301	0.361446	0.343373
10^4	0.352893	0.356153	0.352893	0.359413
10^5	0.353180	0.352555	0.355474	0.351825
10^6	0.353368	0.353547	0.354298	0.352961
10^7	0.354093	0.353963	0.354004	0.354066
10^8	0.354136	0.354138	0.354040	0.354046

x	$(a,b)=(3,2)$	$(a,b)=(6,2)$	$(a,b)=(2,3)$	$(a,b)=(2,6)$
10^3	0.331325	0.337349	0.313253	0.325301
10^4	0.349633	0.352893	0.352078	0.352078
10^5	0.352868	0.353806	0.352138	0.353702
10^6	0.353496	0.353585	0.353508	0.353419
10^7	0.354042	0.353945	0.353954	0.353988
10^8	0.354116	0.354121	0.354160	0.354163

(II) The case $(k,l)=(q,0)$ (q is an odd prime)

This case corresponds to Theorem 3.

(II–i) The case $b=q$, $q\equiv 1 \pmod 4$

In this case, $\Delta S_{a,b}(q,0)=\Delta Q_a(q,0)$ holds. We give the examples for $b=q=5$ and $b=q=13$. Theoretical densities are $5/24\approx 0.208333$ for $b=q=5$ and $13/168\approx 0.077381$ for $b=q=13$.

The case $b=q=5$

x	$a=2$	$a=3$	$a=5$
10^3	0.204819	0.212121	0.204819
10^4	0.205379	0.211256	0.211084
10^5	0.209906	0.208259	0.208551
10^6	0.208584	0.208128	0.208686
10^7	0.208223	0.208340	0.208275
10^8	0.208351	0.208354	0.208311

The case $b=q=13$

x	$a=2$	$a=3$	$a=5$
10^3	0.078313	0.072727	0.066667
10^4	0.076610	0.076672	0.073409
10^5	0.077372	0.076963	0.077693
10^6	0.077087	0.077636	0.077725
10^7	0.077454	0.077507	0.077413
10^8	0.077374	0.077406	0.077420

(II–ii) The general cases

We give the examples where $b=q$ and $q\equiv 1 \pmod 4$ do not hold. In this case, $\Delta S_{a,b}(q,0)=\Delta Q_a(q,0)/2$. We give some results for $q=3$ and $q=5$. The theoretical densities are $3/16=0.1875$ for $q=3$ and $5/48\approx 0.104167$ for $q=5$.

The case $q=3$

x	$(a,b)=(2,2)$	$(a,b)=(2,3)$	$(a,b)=(2,6)$	$(a,b)=(3,2)$	$(a,b)=(3,3)$
10^3	0.167665	0.168675	0.180723	0.168675	0.186747
10^4	0.183225	0.182559	0.184189	0.190709	0.191524
10^5	0.189553	0.187070	0.187904	0.188843	0.186340
10^6	0.187434	0.187182	0.187513	0.187653	0.186914
10^7	0.187659	0.187614	0.187628	0.187309	0.187179
10^8	0.187520	0.187502	0.187509	0.187495	0.187469

x	$(a,b)=(5,2)$	$(a,b)=(5,3)$	$(a,b)=(5,5)$	$(a,b)=(6,2)$	$(a,b)=(6,3)$
10^3	0.156627	0.157576	0.180723	0.144578	0.156627
10^4	0.185004	0.185971	0.185004	0.182559	0.191524
10^5	0.187278	0.186881	0.186861	0.187070	0.186548
10^6	0.187105	0.186687	0.187589	0.187041	0.186990
10^7	0.187480	0.187339	0.187576	0.187367	0.187579
10^8	0.187456	0.187485	0.187469	0.187444	0.187408

The case $q = 5$

Note that we must exclude $b = 5$.

x	$(a,b)=(2,2)$	$(a,b)=(2,3)$	$(a,b)=(2,6)$	$(a,b)=(3,2)$	$(a,b)=(3,3)$
10^3	0.095808	0.078313	0.078313	0.102410	0.084337
10^4	0.100163	0.103504	0.096170	0.101874	0.105134
10^5	0.104369	0.105839	0.106257	0.104484	0.103754
10^6	0.104399	0.103954	0.104948	0.104859	0.103814
10^7	0.104037	0.104126	0.104152	0.104102	0.104197
10^8	0.104156	0.104156	0.104206	0.104165	0.104173

x	$(a,b)=(5,2)$	$(a,b)=(5,3)$	$(a,b)=(6,2)$	$(a,b)=(6,3)$
10^3	0.084337	0.090909	0.096386	0.078313
10^4	0.103504	0.106036	0.104319	0.108394
10^5	0.104901	0.105016	0.103441	0.103128
10^6	0.104630	0.103306	0.104974	0.103852
10^7	0.104039	0.104101	0.104158	0.104271
10^8	0.104125	0.104099	0.104132	0.104188

(III) The case $(k,l) = (4,0), (4,2)$

This case corresponds to Theorem 4 and Corollary 5. We give four typical examples.

$(a,b) = (2,2)$

x	$l=0$	$l=2$
10^3	0.167665	0.011976
10^4	0.162866	0.043160
10^5	0.167136	0.039933
10^6	0.166134	0.041186
10^7	0.166597	0.041456
10^8	0.166669	0.041614

Theoretical densities:
$\Delta S_{2,2}(4,0) = 1/6 \approx 0.166667$,
$\Delta S_{2,2}(4,2) = 1/24 \approx 0.041667$.

$(a,b) = (3,3)$

x	$l=0$	$l=2$
10^3	0.060241	0.072289
10^4	0.079055	0.085574
10^5	0.083107	0.081126
10^6	0.082641	0.083214
10^7	0.083259	0.083340
10^8	0.083262	0.083333

Theoretical densities:
$\Delta S_{3,3}(4,0) = 1/12 \approx 0.083333$,
$\Delta S_{3,3}(4,2) = 1/12 \approx 0.083333$.

$(a, b) = (5, 7)$

x	$l = 0$	$l = 2$
10^3	0.145455	0.175757
10^4	0.164763	0.163132
10^5	0.165606	0.166441
10^6	0.165769	0.166444
10^7	0.166769	0.166473
10^8	0.166593	0.166685

Theoretical densities:
$\Delta S_{5,7}(4, 0) = 1/6 \approx 0.166667$,
$\Delta S_{5,7}(4, 2) = 1/6 \approx 0.166667$.

$(a, b) = (5, 10)$

x	$l = 0$	$l = 2$
10^3	0.192771	0.156627
10^4	0.204564	0.150774
10^5	0.208342	0.145047
10^6	0.208087	0.145511
10^7	0.208173	0.145527
10^8	0.208333	0.145860

Theoretical densities:
$\Delta S_{5,10}(4, 0) = 5/24 \approx 0.208333$,
$\Delta S_{5,10}(4, 2) = 7/48 \approx 0.145833$.

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