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Cobordism of Algebraic Knots Defined by Brieskorn Polynomials

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Abstract. In this paper we study the cobordism of algebraic knots associated with weighted homogeneous polynomials, and in particular Brieskorn polynomials. Under some assumptions we prove that the associated algebraic knots are cobordant if and only if the Brieskorn polynomials have the same exponents.

1. Introduction

A *Brieskorn polynomial* is a polynomial of the form

$$P(z) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$

with $z = (z_1, z_2, \ldots, z_{n+1})$, $n \geq 1$, where the integers $a_j \geq 2$, $j = 1, 2, \ldots, n+1$, are called the *exponents*. The complex hypersurface in $\mathbf{C}^{n+1}$ defined by $P = 0$ has an isolated singularity at the origin, which is called a *Brieskorn singularity*.

In this paper, we will study Brieskorn singularities up to cobordism. We prove that two Brieskorn singularities have cobordant algebraic knots if and only if they have the same set of exponents, provided that no exponent is a multiple of another for each of the two Brieskorn polynomials. Consequently, for such Brieskorn polynomials the multiplicity is an invariant of the cobordism class of the associated algebraic knot.

To be more precise, let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a holomorphic function germ with an isolated critical point at the origin. We denote by D_ε^{2n+2} the closed ball of radius $\varepsilon > 0$ centred at 0 in $\mathbf{C}^{n+1}$, and by S_ε^{2n+1} its boundary. According to Milnor [11], the oriented homeomorphism class of the pair $(D_\varepsilon^{2n+2}, f^{-1}(0) \cap D_\varepsilon^{2n+2})$ does not depend on the choice of a sufficiently small $\varepsilon > 0$, and by definition it is the *topological type* of f. (For other equivalent definitions, we refer the reader to [7, 16, 18].) The oriented diffeomorphism class

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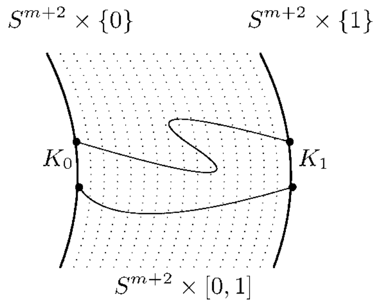

FIGURE 1. A cobordism between K_0 and K_1

of the pair $(S_\varepsilon^{2n+1}, K_f)$, with $K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1}$, is the *algebraic knot* associated with f, where K_f is a closed oriented $(2n-1)$-dimensional manifold. According to Milnor's cone structure theorem [11], the algebraic knot K_f determines the topological type of f. In fact, it is known that the converse also holds.

DEFINITION 1.1. An *m-dimensional knot* (*m-knot*, for short) is a closed oriented m-dimensional submanifold of the oriented $(m+2)$-dimensional sphere S^{m+2}. Two m-knots K_0 and K_1 in S^{m+2} are said to be *cobordant* if there exists a properly embedded oriented $(m+1)$-dimensional submanifold X of $S^{m+2} \times [0, 1]$ such that

(1) X is diffeomorphic to $K_0 \times [0, 1]$, and

(2) $\partial X = (K_0 \times \{0\}) \cup (-K_1 \times \{1\})$,

where $-K_1 \times \{1\}$ denotes the manifold $K_1 \times \{1\}$ with the reversed orientation. A manifold X as above is called a *cobordism* between K_0 and K_1 (see Fig. 1).

In [1], for $n \geq 3$, necessary and sufficient conditions for two algebraic $(2n-1)$-knots to be cobordant have been obtained in terms of Seifert forms (for the definition of the Seifert form, see §2). However, the computation of the Seifert form of a given algebraic knot is very difficult, and an explicit calculation is known only for a very limited class of algebraic knots. (In fact, even for algebraic knots associated with weighted homogeneous polynomials, Seifert forms have not been determined yet, as far as the authors know.) Furthermore, even if we know the Seifert forms explicitly, it is still difficult to see if given two such forms satisfy the algebraic conditions given in [1] or not. So, it is worthwhile to study the conditions for two algebraic knots associated with weighted homogeneous polynomials to be cobordant. We note that cobordism does not necessarily imply isotopy for algebraic knots in general. For details, see the survey article [2].

It is known that cobordant algebraic knots have Witt equivalent Seifert forms (for details, see §2). In this paper, we give a necessary and sufficient condition for two algebraic knots associated with weighted homogeneous polynomials to have Witt equivalent Seifert forms over the real numbers in terms of their weights. Using this result, we give some conditions for two algebraic knots associated with Brieskorn polynomials to be cobordant in terms of the exponents. Under some assumptions, we show that two such knots are cobordant if and only if the Brieskorn polynomials have the same set of exponents.

The paper is organized as follows. In §2, we state our results. We give a necessary and sufficient condition for two nondegenerate weighted homogeneous polynomials to have Witt equivalent Seifert forms over the real numbers, in terms of their weights. Then, we give more explicit results for Brieskorn polynomials. In §3, we prove the results stated in §2. In §4, we give more precise results in the case of two and three variables.

Throughout the paper we work in the smooth category. All the homology groups are with integer coefficients unless otherwise specified.

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2. Results

Let $f(z)$ be a polynomial in $\mathbf{C}^{n+1}$ with an isolated critical point at the origin. We denote by F_f the *Milnor fiber* associated with f, i.e., F_f is the closure of a fiber of the Milnor fibration $\varphi_f : S_\varepsilon^{2n+1} \setminus K_f \to S^1$ defined by $\varphi_f(z) = f(z)/|f(z)|$. According to Milnor [11], F_f is a compact $2n$-dimensional submanifold of S_ε^{2n+1} which is homotopy equivalent to the bouquet of a finite number of copies of the n-dimensional sphere.

The Seifert form

$$L_f : H_n(F_f) \times H_n(F_f) \to \mathbf{Z}$$

associated with f is defined by

$$L_f(\alpha, \beta) = \mathrm{lk}(a_+, b)\,,$$

where a and b are n-cycles representing α and β in $H_n(F_f)$ respectively, a_+ is the n-cycle in S_ε^{2n+1} obtained by pushing a into the positive normal direction of F_f, and lk denotes the linking number of n-cycles in S_ε^{2n+1}. It is known that the isomorphism class of the Seifert form is a topological invariant of f. Furthermore, two algebraic knots K_f and K_g associated with polynomials f and g in $\mathbf{C}^{n+1}$, respectively, with isolated critical points at the origin are isotopic in S_ε^{2n+1} if and only if their Seifert forms L_f and L_g are isomorphic, provided that $n \geq 3$.

In fact, algebraic knots are *simple fibered* knots as follows. We say that an oriented m-knot K is *fibered* if there exists a smooth fibration $\phi : S^{m+2} \setminus K \to S^1$ and a trivialization $\tau : N_K \to K \times D^2$ of a closed tubular neighborhood N_K of K in S^{m+2} such that $\phi|_{N_K \setminus K}$ coincides with $\pi \circ \tau|_{N_K \setminus K}$, where $\pi : K \times (D^2 \setminus \{0\}) \to S^1$ is the composition of the projection to the second factor and the obvious projection $D^2 \setminus \{0\} \to S^1$. Note that then the closure of each fiber of ϕ in S^{m+2} is a compact $(m+1)$-dimensional oriented manifold whose boundary coincides with K. We shall often call the closure of each fiber simply a *fiber*.

Moreover, for $m = 2n - 1 \geq 1$ we say that a fibered $(2n-1)$-knot K is *simple* if each fiber of ϕ is $(n-1)$-connected and K is $(n-2)$-connected. For details we refer the reader to [2]. Note that two simple fibered $(2n-1)$-knots are isotopic if and only if they have isomorphic Seifert forms, provided $n \geq 3$ (see [4, 6]).

DEFINITION 2.1. Two bilinear forms $L_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, defined on free abelian groups G_i of finite ranks are said to be *Witt equivalent* if there exists a direct summand M of $G_0 \oplus G_1$ such that $(L_0 \oplus (-L_1))(x, y) = 0$ for all $x, y \in M$ and twice the rank of M is equal to the rank of $G_0 \oplus G_1$. In this case, M is called a *metabolizer*.

Furthermore, we say that L_0 and L_1 are *Witt equivalent over the real numbers* if there exists a vector subspace $M_{\mathbf{R}}$ of $(G_0 \otimes \mathbf{R}) \oplus (G_1 \otimes \mathbf{R})$ such that $(L_0^{\mathbf{R}} \oplus (-L_1^{\mathbf{R}}))(x, y) = 0$ for all $x, y \in M_{\mathbf{R}}$ and $2 \dim_{\mathbf{R}} M_{\mathbf{R}} = \dim_{\mathbf{R}}(G_0 \otimes \mathbf{R}) + \dim_{\mathbf{R}}(G_1 \otimes \mathbf{R})$, where $L_i^{\mathbf{R}} : (G_i \otimes \mathbf{R}) \times (G_i \otimes \mathbf{R}) \to \mathbf{R}$ is the real bilinear form associated with L_i, $i = 0, 1$.

The following lemma is well known (for example, see [1]).

LEMMA 2.2. *If two simple fibered $(2n-1)$-knots are cobordant, then their Seifert forms are Witt equivalent. In particular, they are Witt equivalent over the real numbers as well.*

Now, let f be a *weighted homogeneous polynomial* in $\mathbf{C}^{n+1}$, i.e., there exist positive rational numbers $(w_1, w_2, \ldots, w_{n+1})$, called *weights*, such that for each monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have

$$\sum_{j=1}^{n+1} \frac{k_j}{w_j} = 1 .$$

We say that f is *nondegenerate* if it has an isolated critical point at the origin. Saito [20] has shown that if f is nondegenerate, then by an analytic change of coordinate, f can be transformed to a nondegenerate weighted homogeneous polynomial such that all the weights are greater than or equal to 2. Furthermore, under the assumption that the weights are all greater than or equal to 2, the weights are analytic invariants of the polynomial.

Let f be a nondegenerate weighted homogeneous polynomial in $\mathbf{C}^{n+1}$ with weights $(w_1, w_2, \ldots, w_{n+1})$ such that $w_j \geq 2$ for all j. Set

$$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1} .$$

Note that $P_f(t)$ is a polynomial in $t^{1/m}$ over $\mathbf{Z}$ for some positive integer m. It is known that two nondegenerate weighted homogeneous polynomials f and g in $\mathbf{C}^{n+1}$ have the same weights if and only if $P_f(t) = P_g(t)$ (see [23]).

Our first result is the following.

THEOREM 2.3. *Let f and g be nondegenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms are Witt equivalent over the real numbers if and only if* $P_f(t) \equiv P_g(t) \mod t+1$.

REMARK 2.4. The above theorem should be compared with the result, obtained in [19], which states that the Seifert forms associated with nondegenerate weighted homogeneous polynomials f and g are isomorphic over the real numbers if and only if $P_f(t) \equiv P_g(t) \mod t^2-1$.

Let us now consider the case of Brieskorn polynomials. Note that a Brieskorn polynomial is always a nondegenerate weighted homogeneous polynomial and its weights coincide with its exponents.

PROPOSITION 2.5. *Let*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn polynomials. Then, their Seifert forms are Witt equivalent over the real numbers if and only if

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j} \tag{2.1}$$

holds for all odd integer ℓ.

To each polynomial $Q(t) = \prod_{j=1}^{k}(t-\alpha_j)$, with $\alpha_1, \alpha_2, \ldots, \alpha_k$ in $\mathbf{C}^*$, the multiplicative group of nonzero complex numbers, set

$$\operatorname{divisor} Q(t) = \langle \alpha_1 \rangle + \langle \alpha_2 \rangle + \cdots + \langle \alpha_k \rangle\,,$$

which is regarded as an element of the integral group ring $\mathbf{ZC}^*$ and is called the *divisor* of Q. For a positive integer a, set $\Lambda_a = \operatorname{divisor}(t^a - 1)$. For the notation and some properties of Λ_a, we refer the reader to [12].

Let f be a nondegenerate weighted homogeneous polynomial in $\mathbf{C}^{n+1}$ with weights $(w_1, w_2, \ldots, w_{n+1})$ such that $w_j \geq 2$ for all j. Let $\Delta_f(t)$ be the characteristic polynomial of the monodromy of f (see [11]). Then, by Milnor–Orlik [12], we have

$$\operatorname{divisor} \Delta_f(t) = \prod_{j=1}^{n+1} \left(\frac{1}{v_j} \Lambda_{u_j} - 1 \right), \tag{2.2}$$

where $w_j = u_j/v_j$, and u_j and v_j are relatively prime positive integers, $j = 1, 2, \ldots, n+1$. In the case of a Brieskorn polynomial, by virtue of the Brieskorn–Pham theorem (for example, see [11]), we have

$$\operatorname{divisor} \Delta_f(t) = \prod_{j=1}^{n+1} (\Lambda_{a_j} - 1),$$

which can also be deduced from the Milnor–Orlik theorem mentioned above.

PROPOSITION 2.6. (1) *Let f and g be nondegenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$ with weights*

$$(u_1/v_1, u_2/v_2, \ldots, u_{n+1}/v_{n+1}) \quad \textit{and} \quad (u'_1/v'_1, u'_2/v'_2, \ldots, u'_{n+1}/v'_{n+1})$$

respectively, where u_j and v_j (resp. u'_j and v'_j) are relatively prime positive integers, $j = 1, 2, \ldots, n+1$. If their Seifert forms are Witt equivalent over the real numbers, then we have

$$\prod_{j=1}^{n+1} \left(\frac{1}{v_j} \Lambda_{u_j} - 1 \right) \equiv \prod_{j=1}^{n+1} \left(\frac{1}{v'_j} \Lambda_{u'_j} - 1 \right) \pmod 2.$$

(2) *Let f and g be Brieskorn polynomials as in Proposition 2.5. If their Seifert forms are Witt equivalent over the real numbers, then we have*

$$\prod_{j=1}^{n+1} (\Lambda_{a_j} - 1) \equiv \prod_{j=1}^{n+1} (\Lambda_{b_j} - 1) \pmod 2.$$

The following theorem partially answers [2, Problem 11.10] in the positive.

THEOREM 2.7. *Suppose that for each of the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

no exponent is a multiple of another one. Then, the knots K_f and K_g are cobordant if and only if $a_j = b_j$, $j = 1, 2, \ldots, n+1$, up to order.

Concerning [2, Problem 11.9], we have the following. Recall that the multiplicity of a Brieskorn polynomial coincides with the smallest exponent.

PROPOSITION 2.8. *Suppose that for each of the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

the exponents are pairwise distinct. If K_f and K_g are cobordant, then the multiplicities of f and g coincide.

3. Proofs

In this section, we prove the results stated in §2.

PROOF OF THEOREM 2.3. Let $h : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a polynomial with an isolated critical point at the origin. It is known that the Seifert form associated with the polynomial

$$\tilde{h}(z_1, z_2, \dots, z_{n+2}) = h(z_1, z_2, \dots, z_{n+1}) + z_{n+2}^2$$

is naturally isomorphic to $(-1)^{n+1}L_h$ (for example, see [21] or [19, Lemma 2.1]). Furthermore, we have $P_{\tilde{h}}(t) = t^{1/2}P_h(t)$. Hence, by considering $f(z) + z_{n+2}^2$ and $g(z) + z_{n+2}^2$ if necessary, we may assume that n is even.

Recall that

$$H^n(F_h; \mathbf{C}) = \oplus_\lambda H^n(F_h; \mathbf{C})_\lambda\,,$$

where F_h is the Milnor fiber for h, λ runs over all the roots of the characteristic polynomial $\Delta_h(t)$, and $H^n(F_h; \mathbf{C})_\lambda$ is the eigenspace of the monodromy $H^n(F_h; \mathbf{C}) \to H^n(F_h; \mathbf{C})$ corresponding to the eigenvalue λ ($h = f$ or g). It is easy to see that the intersection form $S_h = L_h + {}^tL_h$ of F_h on $H^n(F_h; \mathbf{C})$ decomposes as the orthogonal direct sum of $(S_h)|_{H^n(F_h;\mathbf{C})_\lambda}$. Let $\mu(h)^+_\lambda$ (resp. $\mu(h)^-_\lambda$) denote the number of positive (resp. negative) eigenvalues of $(S_h)|_{H^n(F_h;\mathbf{C})_\lambda}$. The integer

$$\sigma_\lambda(h) = \mu(h)^+_\lambda - \mu(h)^-_\lambda$$

is called the *equivariant signature* of h with respect to λ (for details, see [14, 22]). According to Steenbrink [24], putting $P_h(t) = \sum c_\alpha t^\alpha$, we have

$$\sigma_\lambda(h) = \sum_{\substack{\lambda=\exp(-2\pi\sqrt{-1}\alpha)\\ \lfloor\alpha\rfloor:\ \text{even}}} c_\alpha - \sum_{\substack{\lambda=\exp(-2\pi\sqrt{-1}\alpha)\\ \lfloor\alpha\rfloor:\ \text{odd}}} c_\alpha$$

for $\lambda \neq 1$, where $\lfloor\alpha\rfloor$ is the largest integer not exceeding α.

Now, suppose that the Seifert forms L_f and L_g are Witt equivalent over the real numbers. Then, the equivariant signatures $\sigma_\lambda(f)$ and $\sigma_\lambda(g)$ coincide for all λ (for example, see [3]. See also [9, 10] for the spherical knot case). Note that by [19, Lemma 2.3], the equivariant signature for $\lambda = 1$ is always equal to zero.

Set $P_f(t) = P_f^0(t) + P_f^1(t)$, where $P_f^0(t)$ (resp. $P_f^1(t)$) is the sum of those terms $c_\alpha t^\alpha$ with $\lfloor\alpha\rfloor \equiv 0 \pmod 2$ (resp. $\lfloor\alpha\rfloor \equiv 1 \pmod 2$). We define $P_g^0(t)$ and $P_g^1(t)$ similarly. Since the equivariant signatures of f and g coincide, we have

$$tP_f^0(t) - P_f^1(t) \equiv tP_g^0(t) - P_g^1(t) \mod t^2 - 1$$

and

$$tP_f^1(t) - P_f^0(t) \equiv tP_g^1(t) - P_g^0(t) \mod t^2 - 1$$

(for details, see [13, 19]). Adding up these two congruences, we have

$$(t-1)P_f(t) \equiv (t-1)P_g(t) \mod t^2-1, \tag{3.1}$$

which implies that

$$P_f(t) \equiv P_g(t) \mod t+1. \tag{3.2}$$

Conversely, suppose that (3.2) holds. Then, we have (3.1), which implies that the Seifert forms L_f and L_g have the same equivariant signatures. Then, we see that they are Witt equivalent over the real numbers by virtue of [19, §4]. This completes the proof. □

PROOF OF PROPOSITION 2.5. Note that $P_f(t)$ and $P_g(t)$ are polynomials in $s = t^{1/m}$ for some m. Let us put $Q_f(s) = P_f(t)$ and $Q_g(s) = P_g(t)$. Then, it is easy to see that (3.2) holds if and only if $Q_f(\xi) = Q_g(\xi)$ for all ξ with $\xi^m = -1$. Note that ξ is of the form $\exp(\pi\sqrt{-1}\ell/m)$ with ℓ odd and that

$$\frac{-1-\exp(\pi\sqrt{-1}\ell/a_j)}{\exp(\pi\sqrt{-1}\ell/a_j)-1} = \sqrt{-1}\cot\frac{\pi\ell}{2a_j}.$$

Then, we immediately get Proposition 2.5. □

By considering those odd integers ℓ which give zero in (2.1), we get the following.

PROPOSITION 3.1. *Let f and g be the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}.$$

If their Seifert forms are Witt equivalent over the real numbers, then we have

$$\begin{aligned}&\{\ell \in \mathbf{Z} \mid \ell \text{ is odd and is a multiple of some } a_j\}\\ &= \{\ell \in \mathbf{Z} \mid \ell \text{ is odd and is a multiple of some } b_j\}.\end{aligned}$$

In particular, if a_j is odd for some j, then b_k is odd for some k, and the minimal odd exponent for f coincides with that for g.

REMARK 3.2. For nondegenerate weighted homogeneous polynomials, we also have results similar to Propositions 2.5 or 3.1. However, the statement becomes complicated, so we omit them here (compare this with [19, Proposition 2.6]).

Now, Proposition 2.6 is a consequence of the Milnor–Orlik and Brieskorn–Pham theorems on the characteristic polynomials [11, 12] together with the Fox–Milnor type relation. Here, a Fox–Milnor type relation for two polynomials f and g with Witt equivalent Seifert forms means that there exists a polynomial $\gamma(t)$ such that $\Delta_f(t)\,\Delta_g(t) = \pm t^{\deg(\gamma)}\gamma(t)\,\gamma(t^{-1})$ (for details, see [2], for example). Here we give another proof, using Theorem 2.3, as follows.

PROOF OF PROPOSITION 2.6. Since $P_f(t) \equiv P_g(t) \mod t+1$, there exists a polynomial $R(t) \in \mathbf{Z}[t^{1/m}]$ for some m such that

$$P_f(t) - P_g(t) = (t+1)R(t) = (t-1)R(t) + 2R(t).$$

Therefore, for each $\lambda \in S^1$, the multiplicities of λ in the characteristic polynomials $\Delta_f(t)$ and $\Delta_g(t)$ are congruent modulo 2 to each other (for details, see [13, 19], for example). Then, the result follows in view of the Milnor–Orlik formula (2.2) for the characteristic polynomial. □

For the proof of Theorem 2.7, we need the following.

LEMMA 3.3. *For integers $2 \leq a_1 < a_2 < \cdots < a_p$ and $2 \leq b_1 < b_2 < \cdots < b_q$, we have*

$$\sum_{j=1}^{p} \Lambda_{a_j} \equiv \sum_{j=1}^{q} \Lambda_{b_j} \pmod 2 \tag{3.3}$$

if and only if $p = q$ and $a_j = b_j$ for all j.

PROOF. Suppose that $a_p < b_q$. Then the coefficient of $\langle \exp(2\pi\sqrt{-1}/b_q) \rangle$ on the right hand side of (3.3) is equal to 1, while the corresponding coefficient on the left hand side is equal to 0. This is a contradiction. So, we must have $a_p = b_q$. Then we have

$$\sum_{j=1}^{p-1} \Lambda_{a_j} \equiv \sum_{j=1}^{q-1} \Lambda_{b_j} \pmod 2 .$$

Therefore, by induction, we get the desired conclusion. □

PROOF OF THEOREM 2.7. Suppose that the algebraic knots K_f and K_g are cobordant. We may assume $a_1 < a_2 < \cdots < a_{n+1}$ and $b_1 < b_2 < \cdots < b_{n+1}$. By Proposition 2.6 (2), we have

$$\prod_{j=1}^{n+1} (\Lambda_{a_j} - 1) - (-1)^{n+1} \equiv \prod_{j=1}^{n+1} (\Lambda_{b_j} - 1) - (-1)^{n+1} \pmod 2 . \tag{3.4}$$

Recall that for positive integers a and b, we have

$$\Lambda_a \Lambda_b = (a, b) \Lambda_{[a,b]} ,$$

where (a, b) is the greatest common divisor of a and b, and $[a, b]$ denotes the least common multiple of a and b.

By considering the term of the form Λ_d with the smallest d on both sides of (3.4), we see that $a_1 = b_1$ by Lemma 3.3. By subtracting Λ_{a_1} from the both sides of (3.4), we see $a_2 = b_2$, since a_2 (or b_2) is not a multiple of a_1 (resp. b_1). Then, by further subtracting $\Lambda_{a_2} + (a_1, a_2)\Lambda_{[a_1,a_2]}$ from (3.4), we see $a_3 = b_3$, since a_3 (or b_3) is not a multiple of a_1 or a_2 (resp. b_1 or b_2). Repeating this procedure, we see that $a_j = b_j$ for all j.

Conversely, if f and g have the same set of exponents, then K_f and K_g are isotopic and hence cobordant. This completes the proof. □

PROOF OF PROPOSITION 2.8. In the proof of Theorem 2.7, we proved that the smallest exponents of f and g are equal, provided that there is only one smallest exponent for each of f and g. Since we assume that the exponents of f (or g) are pairwise distinct, the same argument works. □

REMARK 3.4. Theorem 2.7 implies that two algebraic knots K_f and K_g associated with certain Brieskorn polynomials are isotopic if and only if they are cobordant. Recall that according to Yoshinaga–Suzuki [25], two algebraic knots associated with Brieskorn polynomials in general are isotopic if and only if they have the same set of exponents. In fact, they showed that the characteristic polynomials coincide if and only if the Brieskorn polynomials have the same set of exponents.

REMARK 3.5. For the case where $n = 2$ and the knots are homology spheres, Theorem 2.7 has been obtained in [17] by using the Fox–Milnor type relation.

EXAMPLE 3.6. For all integers $p_1, p_2, \dots, p_{n-3} \geq 2$, $n \geq 3$, the product of the characteristic polynomials of the algebraic knots associated with

$$f(z) = z_1^{p_1} + z_2^{p_2} + \cdots + z_{n-3}^{p_{n-3}} + z_{n-2}^8 + z_{n-1}^8 + z_n^4 + z_{n+1}^4$$

and

$$g(z) = z_1^{p_1} + z_2^{p_2} + \cdots + z_{n-3}^{p_{n-3}} + z_{n-2}^6 + z_{n-1}^6 + z_n^6 + z_{n+1}^6$$

is a square. This means that the characteristic polynomials $\Delta_f(t)$ and $\Delta_g(t)$ of the algebraic knots K_f and K_g, respectively, satisfy the Fox–Milnor type relation, although their exponents are distinct. Thus the assumptions in Theorem 2.7 and Proposition 2.8 are necessary, as long as the proof depends only on the Fox–Milnor type relation.

4. Further results

In this section, we give some more precise results for the case of two or three variables.

PROPOSITION 4.1. *Let $f(z) = z_1^{a_1} + z_2^{a_2}$ and $g(z) = z_1^{b_1} + z_2^{b_2}$ be Brieskorn polynomials of two variables. If the Seifert forms L_f and L_g are Witt equivalent over the real numbers, then $a_j = b_j$, $j = 1, 2$, up to order.*

PROOF. If a_1 or a_2 is odd, then by Proposition 3.1 we may assume that $a_1 = b_1$ is odd. Then by Proposition 2.5, we have

$$\cot \frac{\pi}{2a_2} = \cot \frac{\pi}{2b_2},$$

which implies that $a_2 = b_2$.

Therefore, we may assume that all the exponents for f and g are even. Then by Proposition 2.6 (2), we have

$$(\Lambda_{a_1} - 1)(\Lambda_{a_2} - 1) \equiv (\Lambda_{b_1} - 1)(\Lambda_{b_2} - 1) \pmod 2,$$

which implies that

$$\Lambda_{a_1} + \Lambda_{a_2} \equiv \Lambda_{b_1} + \Lambda_{b_2} \pmod 2.$$

If $a_1 \neq a_2$, then we see that $b_1 \neq b_2$, and $a_j = b_j$, $j = 1, 2$, up to order by Lemma 3.3. If $a_1 = a_2$, then we must have $b_1 = b_2$. In this case, by Proposition 2.5, we have

$$\cot^2 \frac{\pi}{2a_1} = \cot^2 \frac{\pi}{2b_1},$$

which implies that $a_1 = b_1$. This completes the proof. □

PROPOSITION 4.2. *Let $f(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ and $g(z) = z_1^{b_1} + z_2^{b_2} + z_3^{b_3}$ be Brieskorn polynomials of three variables. If the Seifert forms L_f and L_g are Witt equivalent over the real numbers, then $a_j = b_j$, $j = 1, 2, 3$, up to order.*

PROOF. First suppose that a_1, a_2 and a_3 are all even. Then by Proposition 3.1, b_1, b_2 and b_3 are all even. In this case, by Proposition 2.6 (2), we have

$$\Lambda_{a_1} + \Lambda_{a_2} + \Lambda_{a_3} \equiv \Lambda_{b_1} + \Lambda_{b_2} + \Lambda_{b_3} \pmod 2.$$

Thus, we may assume that $a_1 = b_1$ by Lemma 3.3. Then by Proposition 2.5, we have

$$\cot \frac{\pi\ell}{2a_2} \cot \frac{\pi\ell}{2a_3} = \cot \frac{\pi\ell}{2b_2} \cot \frac{\pi\ell}{2b_3}$$

for all odd integers ℓ. Then, by Proposition 4.1, we see that $a_j = b_j$, $j = 1, 2, 3$, up to order.

Now suppose that a_1, a_2 or a_3 is odd. Then, by Proposition 3.1, we may assume that $a_1 = b_1$ is odd and $a_2 \leq a_3$ and $b_2 \leq b_3$.

Then by Proposition 2.5, we have

$$\cot \frac{\ell\pi}{2a_2} \cot \frac{\ell\pi}{2a_3} = \cot \frac{\ell\pi}{2b_2} \cot \frac{\ell\pi}{2b_3} \tag{4.1}$$

for all odd integers ℓ that are not a multiple of $a_1 = b_1$. If $a_2 = b_2$, then putting $\ell = 1$, we get $a_3 = b_3$. So, suppose that $a_2 < b_2$. Then by (4.1) with $\ell = 1$, we have $a_2 < b_2 \leq b_3 < a_3$.

Let us consider the characteristic polynomials $\Delta_f(t)$ and $\Delta_g(t)$. We have

$$\begin{aligned}
\operatorname{divisor} \Delta_f(t) &= (\Lambda_{a_1} - 1)(\Lambda_{a_2} - 1)(\Lambda_{a_3} - 1) \\
&= (a_1, a_2)([a_1, a_2], a_3)\Lambda_{[a_1,a_2,a_3]} - (a_1, a_2)\Lambda_{[a_1,a_2]} - (a_1, a_3)\Lambda_{[a_1,a_3]} \\
&\quad -(a_2, a_3)\Lambda_{[a_2,a_3]} + \Lambda_{a_1} + \Lambda_{a_2} + \Lambda_{a_3} - 1
\end{aligned}$$

and

$$\text{divisor } \Delta_g(t) = (b_1, b_2)([b_1, b_2], b_3)\Lambda_{[b_1,b_2,b_3]} - (b_1, b_2)\Lambda_{[b_1,b_2]} - (b_1, b_3)\Lambda_{[b_1,b_3]} \\ -(b_2, b_3)\Lambda_{[b_2,b_3]} + \Lambda_{b_1} + \Lambda_{b_2} + \Lambda_{b_3} - 1\,.$$

Since $[a_1, a_2, a_3]$, $[a_1, a_3]$, $[a_2, a_3]$, a_3, $[b_1, b_2, b_3]$, $[b_1, b_2]$, $[b_1, b_3]$, $[b_2, b_3]$, b_2 and b_3 are all strictly greater than a_2, by Proposition 2.6 (2) together with $a_1 = b_1$, we must have $[a_1, a_2] = a_2$. Thus a_2 is a multiple of a_1. Then by Proposition 2.6 (2) again, we have

$$\Lambda_{[a_1,a_3]} + \Lambda_{a_3} \equiv ([b_1, b_2], b_3)\Lambda_{[b_1,b_2,b_3]} + \Lambda_{[b_1,b_2]} + \Lambda_{[b_1,b_3]} \\ +(b_2, b_3)\Lambda_{[b_2,b_3]} + \Lambda_{b_2} + \Lambda_{b_3} \pmod 2\,,$$

since $a_1 = b_1$ is odd.

If $b_2 < b_3$, then we must have $[b_1, b_2] = b_2$, i.e., b_2 is a multiple of b_1. Then, we see that $[a_1, a_3] = a_3$ and $[b_1, b_3] = b_3$. Therefore, a_2, a_3, b_2 and b_3 are all multiples of $a_1 = b_1$. Since a_1 is odd and $a_1 \geq 3$, there exists an odd integer ℓ $(= a_2 + 1$ or $a_2 + 2)$ which is not a multiple of a_1 such that $a_2 < \ell < b_2$. Then for this ℓ, the left hand side of (4.1) is negative, while the right hand side is positive. This is a contradiction.

If $b_2 = b_3$, then we have

$$\Lambda_{[a_1,a_3]} + \Lambda_{a_3} \equiv b_2\Lambda_{[b_1,b_2]} + b_2\Lambda_{b_2} \pmod 2\,.$$

Thus, $[a_1, a_3] = a_3$, and a_3 is a multiple of a_1. Then, using an odd integer ℓ $(= a_2 + 1$ or $a_2 + 2)$ which is not a multiple of a_1 such that $a_2 < \ell < a_3$ in (4.1), we again get a contradiction, since $b_2 = b_3$.

Therefore, we must have $a_2 = b_2$ and $a_3 = b_3$. This completes the proof. □

PROPOSITION 4.3. *Let f and g be nondegenerate weighted homogeneous polynomials of two variables with weights (w_1, w_2) and (w_1', w_2'), respectively, with $w_j, w_j' \geq 2$. If their Seifert forms are Witt equivalent over the real numbers, then $w_j = w_j'$, $j = 1, 2$, up to order.*

PROOF. Set $w_j = u_j/v_j$ and $w_j' = u_j'/v_j'$, $j = 1, 2$, where u_j and v_j (resp. u_j' and v_j') are relatively prime positive integers. Let m be a common multiple of u_1, u_2, u_1' and u_2'. Then, by the same argument as in the proof of [19, Lemma 3.1], we see that the polynomial

$$F(\eta) = -\eta^{m/w_1+m/w_2+m/w_1'} - \eta^{m/w_1+m/w_2+m/w_2'} \\ +\eta^{m/w_1+m/w_1'+m/w_2'} + \eta^{m/w_2+m/w_1'+m/w_2'} \\ +\eta^{m/w_1} + \eta^{m/w_2} - \eta^{m/w_1'} - \eta^{m/w_2'}$$

in η is divisible by $\eta^m + 1$. Note that $F(\eta)$ corresponds to $F(z)$ in the notation of [19].

Since

$$\cot\frac{\pi}{2w_1}\cot\frac{\pi}{2w_2} = \cot\frac{\pi}{2w_1'}\cot\frac{\pi}{2w_2'}\,,$$

we may assume that $w_1 \geq w'_1 \geq w'_2 \geq w_2$. Furthermore, if $w_1 = w'_1$, then we have $w_2 = w'_2$. Therefore, we may assume

$$w_1 > w'_1 \geq w'_2 > w_2 (\geq 2)\,.$$

Note that then the highest degree of F is equal to $m/w_2 + m/w'_1 + m/w'_2$, while the lowest one is equal to m/w_1. Set $V(\eta) = \eta^{-m/w_1} F(\eta)$, which is a polynomial in η of degree

$$\frac{m}{w_2} + \frac{m}{w'_1} + \frac{m}{w'_2} - \frac{m}{w_1}\,,$$

and which is divisible by $\eta^m + 1$. Note that $V(\eta)$ corresponds to $V(z)$ in the notation of [19].

If we have $\deg V < m$, then by the same argument as in the proof of [19, Lemma 3.1], we have the desired conclusion.

If $\deg V \geq m$, then we have the congruence

$$\begin{aligned} V(\eta) \equiv & -\eta^{m/w_2+m/w'_1} - \eta^{m/w_2+m/w'_2} + \eta^{m/w'_1+m/w'_2} \\ & -\eta^{m/w_2+m/w'_1+m/w'_2-m/w_1-m} + 1 + \eta^{m/w_2-m/w_1} \\ & -\eta^{m/w'_1-m/w_1} - \eta^{m/w'_2-m/w_1} \quad \bmod \eta^m + 1. \end{aligned} \tag{4.2}$$

Note that all the terms appearing on the right hand side of (4.2) have nonnegative degrees strictly less than m.

Let us consider the monomial $-\eta^{m/w_2+m/w'_1+m/w'_2-m/w_1-m}$ on the right hand side of (4.2), with negative sign. In order that $V(\eta)$ be divisible by $\eta^m + 1$, a term with positive sign must cancel with $-\eta^{m/w_2+m/w'_1+m/w'_2-m/w_1-m}$. Therefore, we have the following three cases.

Case 1. $1/w_2 + 1/w'_1 + 1/w'_2 - 1/w_1 - 1 = 1/w'_1 + 1/w'_2$.

This does not occur, since $w_1 > w_2 \geq 2$.

Case 2. $1/w_2 + 1/w'_1 + 1/w'_2 - 1/w_1 - 1 = 0$.

In this case, we have

$$\begin{aligned} V(\eta) \equiv & -\eta^{m/w_2+m/w'_1} - \eta^{m/w_2+m/w'_2} + \eta^{m/w'_1+m/w'_2} \\ & +\eta^{m/w_2-m/w_1} - \eta^{m/w'_1-m/w_1} - \eta^{m/w'_2-m/w_1} \quad \bmod \eta^m + 1 \\ = & \,\eta^{m/w'_1-m/w_1}(-\eta^{m/w_1+m/w_2} - \eta^{m/w_1+m/w_2+m/w'_2-m/w'_1} \\ & +\eta^{m/w_1+m/w'_2} + \eta^{m/w_2-m/w'_1} - 1 - \eta^{m/w'_2-m/w'_1}). \end{aligned}$$

Note that the difference of the highest and the lowest degrees of the last polynomial is equal to $m/w_1 + m/w_2 + m/w'_2 - m/w'_1$, which is strictly positive and is strictly smaller than m, since $1/w_2 + 1/w'_2 = 1/w_1 - 1/w'_1 + 1$. This means that $V(\eta)$ cannot be divisible by $\eta^m + 1$. This is a contradiction.

Case 3. $1/w_2 + 1/w'_1 + 1/w'_2 - 1/w_1 - 1 = 1/w_2 - 1/w_1$.

In this case, we have $1/w'_1 + 1/w'_2 = 1$, which implies that $w'_1 = w'_2 = 2$. This is a contradiction, since $w'_2 > w_2 \geq 2$.

Therefore, we must have $w_1 = w'_1$ and $w_2 = w'_2$. This completes the proof. □

By using exactly the same argument as in [19, Lemma 3.1], we have the following.

PROPOSITION 4.4. *Let f and g be nondegenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$ with weights $(w_1, w_2, \ldots, w_{n+1})$ and $(w'_1, w'_2, \ldots, w'_{n+1})$, respectively, such that $w_j \geq 2$ and $w'_j \geq 2$ for all j. Suppose that the Seifert forms of f and g are Witt equivalent over the real numbers. If*

$$\sum_{j=1}^{n+1} \frac{1}{w_j} + \sum_{j=1}^{n+1} \frac{1}{w'_j} - 2\min\left\{\frac{1}{w_1}, \ldots, \frac{1}{w_{n+1}}, \frac{1}{w'_1}, \ldots, \frac{1}{w'_{n+1}}\right\} < 1\,,$$

then we have $w_j = w'_j$, $j = 1, 2, \ldots, n+1$, up to order.

REMARK 4.5. By Proposition 4.3, we see that if the algebraic knots associated with two weighted homogeneous polynomials of two variables are cobordant, then the polynomials have the same set of weights. In fact, this fact itself is a consequence of already known results as follows.

If two algebraic knots in S^3 are cobordant, then they are in fact isotopic by virtue of the results of Lê [8] and Zariski [27] (for details, see [2, §4]). Then, by Yoshinaga–Suzuki [26] (see also [5, 15]), they have the same set of weights.

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