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The Continuity of Distribution-valued Additive Functionals for H_1^β

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Abstract. In [2] and [3], we discuss the existence and the (a,t)-joint continuity of the distribution-valued additive functional $A_T(a:t,\omega)=\int_0^t T(X_s-a)$ for $T\in H_p^\beta$ except for the case of the (a,t)-joint continuity with $p=1$. In this paper, we discuss the (a,t)-joint continuity of the distribution-valued additive functional $A_T(a:t,\omega)$ for $T\in H_1^\beta$.

1. Introduction and preliminaries

In this paper, we discuss the (a,t)-joint continuity of the distribution-valued additive functional $A_T(a:t,\omega)=\int_0^t T(X_s-a)$ for $T\in H_1^\beta$ which is the case we did not finish doing in [2] and [3]. The main results are Theorem 9 and 11 whose proof are produced by Lemma 8.

Throughout this paper, we shall use the same notations as those in [2] and [3]. But we notice some notations and remember the results of [2] and [3].

We denote the Fourier transform of $\phi(a)$ by $\hat{\phi}(\lambda)$:

$$\hat{\phi}(\lambda)=\int\phi(x)e^{i\lambda\cdot x}dx$$

and the Fourier inverse transform of $\psi(\lambda)$ by $\mathcal{F}^{-1}(\psi)(a)$:

$$\mathcal{F}^{-1}(\psi)(a)=\frac{1}{(2\pi)^d}\int\psi(\lambda)e^{-i\lambda\cdot a}d\lambda\,,$$

where $x\cdot y$ ($x\in\mathbf{R}^d$,$y\in\mathbf{R}^d$) denotes the inner product.

Let $T\in\mathcal{S}'$. We denote the Fourier transform of T by $\hat{T}$.

DEFINITION 1. We say that T is an element of H_p^β ($1\le p\le\infty$, $-\infty<\beta<\infty$) if and only if T is an element of $\mathcal{S}'$ and the Fourier transform of T has a version as a function $\hat{T}(\lambda)$ on $\mathbf{R}^d$ such that

$$\hat{T}(\lambda)(1+|\lambda|^2)^{\frac{\beta}{2}}\in L^p.$$

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Then we set

$$\|T\|_{H_p^\beta} = \|\hat{T}(\lambda)(1+|\lambda|^2)^{\frac{\beta}{2}}\|_{L^p}\,.$$

We note $\mathcal{F}^{-1}(T)(\lambda) = (2\pi)^{-d}\hat{T}(-\lambda)$ for $T \in H_p^\beta$.

Let $\{X_s\}$ be the standard Brownian motion on $\mathbf{R}^d$ or one-dimensional real valued stable process with index α $(0 < \alpha < 2)$ or d-dimensional real valued symmetric stable process with index α $(0 < \alpha < 2)$.

We define τ_x and θ_t as follows:

$$\tau_x : X_t(\tau_x\omega) = X_t(\omega) + x$$

and

$$\theta_t : X_s(\theta_t\omega) = X_{t+s}(\omega)\,.$$

We remember preliminary results in [2].

LEMMA 2. *Let $T \in \mathcal{D}'$, $\phi \in \mathcal{D}$ and set $T * \phi(x) = \langle T_y,\ \phi(x-y)\rangle_y$. Then*

$$\langle A_T(t,\omega), \phi\rangle = \int_0^t T * \phi(X_s(\omega))ds$$

is well-defined and we have

$$A_T(t,\omega) \in \mathcal{D}'\,.$$

LEMMA 3.

$$\begin{aligned}\langle A_T(t,\tau_x\omega), \phi\rangle =&\langle A_T(t,\omega), \phi(\cdot + x)\rangle\\ \langle A_T(s+t,\omega), \phi\rangle =&\langle A_T(s,\omega), \phi\rangle + \langle A_T(t,\theta_s\omega), \phi\rangle\,.\end{aligned}$$

LEMMA 4. *Let T be an element of H_p^β. Then $A_T(t,\omega)$ is also an element of H_p^β.*

Now let ρ_ε be the mollifier. We denote

$$A_T^\varepsilon(t,\omega) = \langle A_T(t,\omega), \rho_\varepsilon\rangle$$

and

$$A_T^\varepsilon(a:t,\omega) = A_T^\varepsilon(t,\tau_{-a}\omega)\,.$$

We note that

$$\langle A_T^\varepsilon(t,\omega), \phi\rangle = \langle A_T(t,\omega), \rho_\varepsilon * \phi\rangle\,.$$

Here we emphasize $A_T^\varepsilon(a:t,\omega)$ is a usual function of a. We can take ρ_ε such that $\rho_\varepsilon \to \delta_0$ as $\varepsilon \to 0$ and $\hat{\rho}_\varepsilon$ uniformly converges to one in wider sense tending ε to zero and $\|\hat{\rho}_\varepsilon\|_\infty \le 1$.

We studied the convergence of distribution-valued additive functional $A_T^\varepsilon(a : t, \omega)$ for $T \in H_1^\beta$ in [2] and [3], tending ε to zero. We remember their results.

First, in the case of d-dimensional Brownian motion, we had

THEOREM 5. *For $T \in H_1^\beta$*

$$\lim_{\varepsilon \to 0} A_T^\varepsilon(a : t, \omega) = A_T(a : t, \omega) \quad \text{in } L^2(dP_x)\,,$$

where we take $\beta \geq -1$.

Second, in the case of one-dimensional stable process with index α whose characteristic function is (7), we had

THEOREM 6. *For $T \in H_1^\beta$*

$$\lim_{\varepsilon \to 0} A_T^\varepsilon(a : t, \omega) = A_T(a : t, \omega) \quad \text{in } L^2(dP_x)\,,$$

where we take $\beta \geq -\alpha/2$ but if $\alpha < 1$ and $\gamma_0 \neq 0$ we take $\beta \geq -1/2$. We notice that γ_0 is a real number in (8).

Last, in the case of d-dimensional symmetric stable process with index α, we had

THEOREM 7. *For $T \in H_1^\beta$*

$$\lim_{\varepsilon \to 0} A_T^\varepsilon(a : t, \omega) = A_T(a : t, \omega) \quad \text{in } L^2(dP_x)\,,$$

where we take $\beta \geq -\alpha/2$.

The (a, t)-joint continuity theorem in the case of each is following.

Theorem 9 corresponds to the d-dimensional Brownian motion.

Theorem 11 corresponds to the one-dimensional stable process with index α.

Corollary 12 corresponds to the d-dimensional symmetric stable process with index α.

They will be proved in Section 2.

The following lemma is modified version of the lemma in [2]. This lemma plays important role of proof of the continuity theorems.

LEMMA 8. *Let $p + q \geq 0$ and $p \geq q$. For any $\lambda \in \mathbf{R}^d$,*

$$\sup_{\mu \in \mathbf{R}^d} (1 + |\mu|^2)^{-p}(1 + |\mu + \lambda|^2)^{-q} \asymp (1 + |\lambda|^2)^{-q}. \tag{1}$$

Specially, there exists $C > 0$ such that

$$\sup_{\mu \in \mathbf{R}^d} (1 + |\mu|^2)^{-p}(1 + |\mu + \lambda|^2)^{-q} \leq C(1 + |\lambda|^2)^{-q} \quad \text{for } \lambda \in \mathbf{R}^d. \tag{2}$$

PROOF. The case of $q = 0$ is clear. We will consider the case where q is negative and the other case where q is positive.

First, we consider the case where q is negative. By (1) to the $-1/q$-th power both sides, we have to show that

$$\sup_{\mu\in\mathbf{R}^d}(1+|\mu|^2)^{p/q}(1+|\mu+\lambda|^2)\asymp 1+|\lambda|^2\,, \tag{3}$$

Moreover, for $p+q\geq 0$ we have $-\frac{p}{q}\geq 1$. Then, (3) is rewritten as

$$\sup_{\mu\in\mathbf{R}^d}\frac{(1+|\mu+\lambda|^2)}{(1+|\mu|^2)^m}\asymp 1+|\lambda|^2 \quad \text{for } m\geq 1\,.$$

If we take $\mu=0$,

$$\sup_{\mu\in\mathbf{R}^d}\frac{(1+|\mu+\lambda|^2)}{(1+|\mu|^2)^m}\geq 1+|\lambda|^2\,.$$

By $m\geq 1$ we get

$$\begin{aligned}
&\sup_{\mu\in\mathbf{R}^d}\frac{(1+|\mu+\lambda|^2)}{(1+|\mu|^2)^m}\\
\leq{}&\sup_{\mu\in\mathbf{R}^d}\frac{(1+|\mu+\lambda|^2)}{1+|\mu|^2}\\
\leq{}&\sup_{\mu\in\mathbf{R}^d}\frac{(1+2|\mu|^2+2|\lambda|^2)}{1+|\mu|^2}\\
\leq{}&\sup_{\mu\in\mathbf{R}^d}\frac{2(1+|\mu|^2)(1+2|\lambda|^2)}{1+|\mu|^2}\\
\leq{}&2(1+|\lambda|^2)\,.
\end{aligned}$$

Hence we get (1), if $q<0$.

Second, we consider the case where q is positive. In a similar way where q is negative, we have to show that

$$\inf_{\mu\in\mathbf{R}^d}(1+|\mu|^2)^{p/q}(1+|\mu+\lambda|^2)\asymp 1+|\lambda|^2\,.$$

If we take $\mu=0$,

$$\inf_{\mu\in\mathbf{R}^d}(1+|\mu|^2)^{p/q}(1+|\mu+\lambda|^2)\leq 1+|\lambda|^2\,.$$

Next, since $p/q\geq 1$ we get

$$\begin{aligned}
&\inf_{\mu\in\mathbf{R}^d}(1+|\mu|^2)^{p/q}(1+|\mu+\lambda|^2)\\
\geq{}&\inf_{\mu\in\mathbf{R}^d}(1+|\mu|^2)(1+|\mu+\lambda|^2)
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{\mu\in\mathbf{R}^d}\left(1+\left|\mu-\frac{\lambda}{2}\right|^2\right)\left(1+\left|\mu+\frac{\lambda}{2}\right|^2\right)\\
&= \inf_{\mu\in\mathbf{R}^d}\left(1+2|\mu|^2+\frac{|\lambda|^2}{2}+\left|\mu-\frac{\lambda}{2}\right|^2\left|\mu+\frac{\lambda}{2}\right|^2\right)\\
&\geq 1+\frac{|\lambda|^2}{2}\,.
\end{aligned}
$$

Therefore we get Lemma 8. □

2. Continuity theorems

2.1. The case of d-dimensional Brownian motion. Let P_x be the probability measure of the d-dimensional standard Brownian motion $\{X_t\}$ starting from x. We notice that the characteristic function of X_s is

$$E_x[e^{i\lambda X_s}] = \exp\left\{-\frac{|\lambda|^2}{2}s + i\lambda x\right\}.$$

THEOREM 9. *Let $T \in H_1^\beta$ where we take $\beta > -1$. Suppose that $\delta = \min(1, \beta+1)$. Then $A_T(a : t, \omega)$ has (a,t)-jointly continuous modification, which is locally Hölder-continuous with exponent γ, where $0 < \gamma < \delta$.*

PROOF. We will estimate

$$E_x[(A_T^\varepsilon(a : t, \omega) - A_T^\varepsilon(b : s, \omega))^{2n}]$$

and then we apply Kolmogorov–Čentsov theorem([1, P. 55, Problem 2.9]) to get the joint continuity.

Without loss of generality, for fixed $N > 0$ we take t and s such that $N > t > s$ and we suppose that Brownian motion starts from zero and $b = 0$.

We set

$$
\begin{aligned}
&E_0[(A_T^\varepsilon(a : t, \omega) - A_T^\varepsilon(0 : s, \omega))^{2n}]\\
&\quad \leq 2^{2n}|E_0[(A_T^\varepsilon(a : t, \omega) - A_T^\varepsilon(0 : t, \omega))^{2n}]| + 2^{2n}|E_0[(A_T^\varepsilon(0 : t, \omega) - A_T^\varepsilon(0 : s, \omega))^{2n}]|\\
&\quad = 2^{2n}|I_a| + 2^{2n}|I_t|\,, \quad \text{say}\,.
\end{aligned}
$$

First we estimate I_a. Using Parseval's equality we get

$$
\begin{aligned}
I_a = \frac{(2n)!}{(2\pi)^{2nd}} &\int d\lambda_1 \cdots \int d\lambda_{2n} \int_0^t du_1 \int_{u_1}^t du_2 \cdots \int_{u_{2n-1}}^t du_{2n}\\
&\times \overline{\hat{T}(\lambda_{2n})\cdots\hat{T}(\lambda_1)\hat{\rho}_\varepsilon(\lambda_{2n})\cdots\hat{\rho}_\varepsilon(\lambda_1)}\\
&\times e^{-\frac{|\lambda_{2n}|^2}{2}(u_{2n}-u_{2n-1}) - \frac{|\lambda_{2n}+\lambda_{2n-1}|^2}{2}(u_{2n-1}-u_{2n-2}) - \cdots - \frac{|\lambda_{2n}+\cdots+\lambda_1|^2}{2}u_1}
\end{aligned}
$$

$$\times \overline{(e^{i\lambda_{2n}\cdot a}-1)(e^{i\lambda_{2n-1}\cdot a}-1)\cdots(e^{i\lambda_1\cdot a}-1)}\,.$$

Then we have

$$\begin{aligned}|I_a| \leq &\frac{(2n)!}{(2\pi)^{2nd}}(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho_\varepsilon}\|_\infty)^{2n}\\ &\times \sup_{\lambda_1,\dots,\lambda_{2n}}(1+|\lambda_1|^2)^{-\frac{\beta}{2}}\cdots(1+|\lambda_{2n}|^2)^{-\frac{\beta}{2}}\\ &\quad\times|e^{i\lambda_{2n}\cdot a}-1||e^{i\lambda_{2n-1}\cdot a}-1|\cdots|e^{i\lambda_1\cdot a}-1|\\ &\quad\times\int_0^t du_1\int_{u_1}^t du_2\cdots\int_{u_{2n-1}}^t du_{2n}\\ &\quad\times e^{-\frac{|\lambda_{2n}|^2}{2}(u_{2n}-u_{2n-1})-\frac{|\lambda_{2n}+\lambda_{2n-1}|^2}{2}(u_{2n-1}-u_{2n-2})-\cdots-\frac{|\lambda_{2n}+\cdots+\lambda_1|^2}{2}u_1}\,.\end{aligned}$$

We change the variables λ_i $(1 \leq i \leq 2n)$ to μ_j $(1 \leq j \leq 2n)$ as follows:

$$\begin{aligned}\mu_{2n} &= \lambda_{2n}\\ \mu_{2n-1} &= \lambda_{2n}+\lambda_{2n-1}\\ &\cdots\\ \mu_1 &= \lambda_{2n}+\lambda_{2n-1}+\cdots+\lambda_1\,.\end{aligned}$$

Then we get

$$\begin{aligned}|I_a| \leq &\frac{(2n)!}{(2\pi)^{2nd}}(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho_\varepsilon}\|_\infty)^{2n}\\ &\times \sup_{\mu_1,\dots,\mu_{2n}}(1+|\mu_1-\mu_2|^2)^{-\frac{\beta}{2}}\cdots(1+|\mu_{2n-1}-\mu_{2n}|^2)^{-\frac{\beta}{2}}(1+|\mu_{2n}|^2)^{-\frac{\beta}{2}}\\ &\quad\times|e^{i\mu_{2n}\cdot a}-1||e^{i(\mu_{2n-1}-\mu_{2n})\cdot a}-1|\cdots|e^{i(\mu_1-\mu_2)\cdot a}-1|\\ &\times\int_0^t du_1\int_{u_1}^t du_2\cdots\int_{u_{2n-1}}^t du_{2n}e^{-\frac{|\mu_{2n}|^2}{2}(u_{2n}-u_{2n-1})-\cdots-\frac{|\mu_2|^2}{2}(u_2-u_1)-\frac{|\mu_1|^2}{2}u_1}\,.\end{aligned}$$

Now we notice that for any $k \in \mathbf{C}(Re(k) \geq 0)$

$$\left|\int_0^t e^{-ks}ds\right| \leq \frac{C_1}{1+|k|}$$

and for any $1 \geq l_1 > 0$

$$|e^{i\mu\cdot a}-1| \leq C_2|a|^{l_1}(1+|\mu|^2)^{\frac{l_1}{2}}, \tag{4}$$

where C_1 and C_2 are positive constants.

Then we apply these inequalities to I_a:

$$\begin{aligned}|I_a| \leq & \frac{(2n)!K_1}{(2\pi)^{2nd}}(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}|a|^{2nl_1} \\ & \times \sup_{\mu_1,\ldots,\mu_{2n}}(1+|\mu_1-\mu_2|^2)^{-(\frac{\beta}{2}-\frac{l_1}{2})}\cdots(1+|\mu_{2n-1}-\mu_{2n}|^2)^{-(\frac{\beta}{2}-\frac{l_1}{2})} \\ & \qquad \times (1+|\mu_1|^2)^{-1}\cdots(1+|\mu_{2n-1}|^2)^{-1}(1+|\mu_{2n}|^2)^{-1-(\frac{\beta}{2}-\frac{l_1}{2})},\end{aligned}$$

where $K_1 = (C_1 C_2)^{2n}$.

We first estimate the following. We set

$$|I_a^{2n}| = \sup_{\mu_{2n}}(1+|\mu_{2n-1}-\mu_{2n}|^2)^{-(\frac{\beta}{2}-\frac{l_1}{2})}(1+|\mu_{2n}|^2)^{-1-(\frac{\beta}{2}-\frac{l_1}{2})}.$$

Now we apply (2) to this equation. If β satisfies

$$\left(\frac{\beta}{2}-\frac{l_1}{2}\right)+\left(1+\frac{\beta}{2}-\frac{l_1}{2}\right)\geq 0,$$

then we get

$$|I_a^{2n}| \leq C(1+|\mu_{2n-1}|^2)^{-(\frac{\beta}{2}-\frac{l_1}{2})}.$$

Therefore, by induction, we reach the inequality

$$|I_a| \leq \frac{(2n)!K_1 C^{2n-1}}{(2\pi)^{2nd}}(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}|a|^{2nl_1}\sup_{\mu_1}(1+|\mu_1|^2)^{-1-(\frac{\beta}{2}-\frac{l_1}{2})}.$$

For the finiteness of this inequality, we set the following condition:

$$1+\left(\frac{\beta}{2}-\frac{l_1}{2}\right)\geq 0.$$

Thus we obtain the condition

$$\beta \geq l_1 - 1 \tag{5}$$

and

$$|I_a| \leq K_2|a|^{2nl_1}(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n},$$

where K_2 is a positive constant and only depends on n.

Next we estimate I_t in a similar way of I_a. But we notice that for any $l_2 > 0$, $k \in \mathbf{C}(Re(k) > 0)$ and fixed $N > 0$, there exists a positive constant C_3 such that

$$\left|\int_0^s e^{-ku}du\right| \leq C_3\left(\frac{s^{l_2}}{1+|k|}\right)^{\frac{1}{l_2+1}} \quad \text{for } s \in [0, N],$$

because it is easy to see that

$$s^{-\frac{l_2}{l_2+1}}(1+|k|)^{\frac{1}{l_2+1}}\left|\int_0^s e^{-ku}du\right|$$

is a bounded function on $(s,|k|)\in[0,N]\times[0,\infty)$. Then we have

$$\begin{aligned}|I_t| \leq& \frac{(2n)!K_3}{(2\pi)^{2nd}}|t-s|^{2n\frac{l_2}{l_2+1}}(\|T\|_1^{\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}\\ &\times\sup_{\mu_1,\dots,\mu_{2n}}(1+|\mu_1-\mu_2|^2)^{-\frac{\beta}{2}}\cdots(1+|\mu_{2n-1}-\mu_{2n}|^2)^{-\frac{\beta}{2}}\\ &\qquad\times(1+|\mu_1|^2)^{-\frac{1}{l_2+1}}\cdots(1+|\mu_{2n-1}|^2)^{-\frac{1}{l_2+1}}(1+|\mu_{2n}|^2)^{-\frac{\beta}{2}-\frac{1}{l_2+1}},\end{aligned}$$

where $K_3=C_3^{2n}$.

We apply (2) to the inequality with respect to $\mu_1,\dots,\mu_{2n}$ of I_t. Then we obtain the condition

$$\beta \geq -\frac{1}{l_2+1} \tag{6}$$

for the finiteness of this integral and

$$|I_t| \leq K_4|t-s|^{2n\frac{l_2}{l_2+1}}(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n},$$

where K_4 is a positive constant and only depends on n, N.

Therefore by (5) and (6) we make l_1 and l_2 satisfy the following equality:

$$-\frac{1}{l_2+1}=l_1-1$$

Since l_1 is positive, if β satisfies the condition in Theorem 5, then we obtain

$$\begin{aligned}&|E_0[(A_T^\varepsilon(a:t,\omega)-A_T^\varepsilon(0:s,\omega))^{2n}]|\\ &\qquad\leq C_{BM}(|a|^{2n\delta}+|t-s|^{2n\delta})(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}\end{aligned}$$

where we take δ as follows and $C_{BM}=\max(K_2,K_4)$.

For $\beta>-1$ we take δ as $\beta+1\geq\delta$ by (5) or (6).

Thus tending ε to zero, we get (a,t)-joint continuity of $A_T(a:t,\omega)$ by Kolmogorov–Čentsov theorem. □

2.2. The case of stable process with index α. Let P_x be the probability measure of the one-dimensional stable process $\{X_s\}$ with index $\alpha(0<\alpha<2)$ starting from x. We notice that the characteristic function of X_s is

$$E_x[e^{i\lambda X_s}]=\exp\{-s\psi(\lambda)+i\lambda x\}, \tag{7}$$

where $\psi(\lambda)$ is given in the following. For some constants $c > 0$, $-1 \leq \gamma \leq 1$ and $\gamma_0 \in \mathbf{R}$, if $\alpha \neq 1$ then

$$\psi(\lambda) = c|\lambda|^{\alpha}\left(1 - i\gamma(sgn\lambda)\tan\frac{\pi}{2}\alpha\right) + i\gamma_0\lambda \tag{8}$$

and if $\alpha = 1$ then

$$\psi(\lambda) = c|\lambda|\left(1 + i\gamma\frac{2}{\pi}(sgn\lambda)\log|\lambda|\right) + i\gamma_0\lambda\,.$$

We remember the following lemma in [3].

LEMMA 10. *Let $F = |\int_0^t e^{-\psi(\lambda)s}ds|$. Then we get*

$$F \leq \frac{C_4}{(1+|\lambda|^2)^{\frac{\eta}{2}}}\,, \tag{9}$$

where we take $\eta = \alpha$ but if $\alpha < 1$ and $\gamma_0 \neq 0$ then we take $\eta = 1$.

Next we discuss the (a,t)-joint continuity of $A_T(a : t, \omega)$. We get the following in the similar way to the case of Brownian motion.

THEOREM 11. *Let $T \in H_1^{\beta}$, where we take $\beta > -\alpha/2$. Suppose that*

1. *In the case where $\alpha > 1$*

$$\delta = \min\left(1, \beta + \frac{\alpha}{2}\right)$$

2. *In the case where $\alpha \leq 1$*

$$\delta = \min\left(\alpha, \beta + \frac{\alpha}{2}\right).$$

3. *In the case where $\alpha < 1$ and $\gamma_0 \neq 0$*

$$\delta = \min\left(1, \beta + \frac{1}{2}\right).$$

Then $A_T(a : t, \omega)$ has (a,t)-jointly continuous modification, which is locally Hölder-continuous with exponent γ, where $0 < \gamma < \delta$.

PROOF. Without loss of generality, for fixed $N > 0$ we take t and s such that $N > t > s$ and we suppose that the stable process starts from zero and $b = 0$.

We set

$$\begin{aligned}
&E_0[(A_T^{\varepsilon}(a : t, \omega) - A_T^{\varepsilon}(0 : s, \omega))^{2n}] \\
&\quad \leq 2^{2n}|E_0[(A_T^{\varepsilon}(a : t, \omega) - (A_T^{\varepsilon}(0 : t, \omega))^{2n}]| + 2^{2n}|E_0[(A_T^{\varepsilon}(0 : t, \omega) - (A_T^{\varepsilon}(0 : s, \omega))^{2n}]| \\
&\quad = 2^{2n}|I_a| + 2^{2n}|I_t|.
\end{aligned}$$

First we estimate I_a. By the similar calculation of the case of Brownian motion we obtain

$$\begin{aligned}|I_a| \leq & \frac{(2n)!}{(2\pi)^{2n}}(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n} \\ & \times \sup_{\lambda_1,\dots,\lambda_{2n}}(1+|\lambda_1|^2)^{-\frac{\beta}{2}}\cdots(1+|\lambda_{2n}|^2)^{-\frac{\beta}{2}} \\ & \times |e^{-i\lambda_{2n}a}-1||e^{-i(\lambda_{2n}+\lambda_{2n-1})a}-1|\cdots|e^{-i(\lambda_{2n}+\cdots+\lambda_1)a}-1| \\ & \times \left|\int_0^t du_1 \int_{u_1}^t du_2 \cdots \int_{u_{2n-1}}^t du_{2n}\right. \\ & \quad \left.\times\, e^{-\psi(\lambda_{2n})(u_{2n}-u_{2n-1})-\psi(\lambda_{2n}+\lambda_{2n-1})(u_{2n-1}-u_{2n-2})-\cdots-\psi(\lambda_{2n}+\cdots+\lambda_1)u_1}\right|.\end{aligned}$$

By the change of variables we have

$$\begin{aligned}|I_a| \leq & \frac{(2n)!}{(2\pi)^{2n}}(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n} \\ & \times \sup_{\mu_1,\dots,\mu_{2n}}(1+|\mu_1-\mu_2|^2)^{-\frac{\beta}{2}}\cdots(1+|\mu_{2n-1}-\mu_{2n}|^2)^{-\frac{\beta}{2}}(1+|\mu_{2n}|^2)^{-\frac{\beta}{2}} \\ & \times |e^{-i\mu_{2n}a}-1||e^{-i(\mu_{2n-1}-\mu_{2n})a}-1|\cdots|e^{-i(\mu_1-\mu_2)a}-1| \\ & \times \int_0^t du_1 \int_{u_1}^t du_2 \cdots \int_{u_{2n-1}}^t du_{2n}|e^{-\psi(\mu_{2n})(u_{2n}-u_{2n-1})-\cdots-\psi(\mu_2)(u_2-u_1)-\psi(\mu_1)u_1}|.\end{aligned}$$

Then we apply (4) and (9) to I_a:

$$\begin{aligned}|I_a| \leq & K_5(\|T\|_{H_1^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}|a|^{2nl_1} \\ & \times \sup_{\mu_1,\dots,\mu_{2n}}(1+|\mu_1-\mu_2|^2)^{-\frac{\beta}{2}+\frac{l_1}{2}}\cdots(1+|\mu_{2n-1}-\mu_{2n}|^2)^{-\frac{\beta}{2}+\frac{l_1}{2}} \\ & \times (1+|\mu_1|^2)^{-\frac{\eta}{2}}\cdots(1+|\mu_{2n-1}|^2)^{-\frac{\eta}{2}}(1+|\mu_{2n}|^2)^{-\frac{1}{2}(\eta-l_1+\beta)}.\end{aligned}$$

Now we apply (2) to the above inequality. Then for the finiteness of I_a, we have

$$\left(\frac{\beta-l_1}{2}\right)+\left(\frac{\eta+\beta-l_1}{2}\right)\geq 0.$$

Thus we get

$$\beta > l_1-\frac{\eta}{2} \tag{10}$$

and

$$|I_a| \leq K_6|a|^{2nl_1}(\|T\|_{H_1^\beta})^{2n}\|\hat{\rho}_\varepsilon\|_\infty^{2n},$$

where K_6 is a positive constant and only depends on n.

Next we estimate I_t in a similar way of I_a. But we notice that for any $l_3 > 0$ and fixed $N > 0$, there exists a positive constant C_5 such that

$$\left| \int_0^s e^{-\psi(\mu)u} du \right| \leq C_5 \left(\frac{s^{l_3}}{(1+|\mu|^2)^{\frac{\eta}{2}}} \right)^{\frac{1}{l_3+1}} \quad \text{for } s \in [0, N] .$$

Then we have

$$\begin{aligned} |I_t| \leq & K_7 |t-s|^{2n\frac{l_3}{l_3+1}} (\|T\|_{H_1^\beta})^{2n} (\|\hat{\rho}_\varepsilon\|_\infty)^{2n} \\ & \times \sup_{\mu_1,\ldots,\mu_{2n}} (1+|\mu_1-\mu_2|^2)^{-\frac{\beta}{2}} \cdots (1+|\mu_{2n-1}-\mu_{2n}|^2)^{-\frac{\beta}{2}} \\ & \times (1+|\mu_1|^2)^{-\frac{\eta}{2(l_3+1)}} \cdots (1+|\mu_{2n-1}|^2)^{-\frac{\eta}{2(l_3+1)}} (1+|\mu_{2n}|^2)^{-\frac{\beta}{2}-\frac{\eta}{2(l_3+1)}} . \end{aligned}$$

We apply (2) to the above inequality. Then we have

$$\beta \geq -\frac{\eta}{2(l_3+1)} \tag{11}$$

and

$$|I_t| \leq K_8 |t-s|^{2n\frac{l_3}{l_3+1}} (\|T\|_{H_1^\beta})^{2n} (\|\hat{\rho}_\varepsilon\|_\infty)^{2n} ,$$

where K_8 is a positive constant and only depends on n and N.

Therefore by (10) and (11) we make l_1 and l_3 satisfy the following equality:

$$-\frac{\eta}{2(l_3+1)} = l_1 - \frac{\eta}{2}$$

That is, $l_3 = 2l_1/(\eta - 2l_1)$. Since l_1 is positive, $\beta > -\alpha/2$ and then we get

$$\begin{aligned} & |E_0[(A_T^\varepsilon(a:t,\omega) - A_T^\varepsilon(0:s,\omega)^{2n}]| \\ & \leq C_{st}(|a|^{2n\delta} + |t-s|^{2n\delta})(\|T\|_{H_1^\beta})^{2n} (\|\hat{\rho}_\varepsilon\|_\infty)^{2n} , \end{aligned} \tag{12}$$

where we denote l_1 by δ and $C_{st} = \max(K_6, K_8)$.

Therefore we get the condition in the theorem.

Then tending ε to zero, we get (a,t)-jointly continuity of $A_T(a:t,\omega)$ by Kolmogorov–Čentsov theorem. □

We can apply the above method to the d-dimensional symmetric stable process. Let $\{X_s\}$ be the d-dimensional symmetric stable process with index α. That is,

$$E_x[e^{i\lambda \cdot X_s}] = \exp\{-c|\lambda|^\alpha s + i\lambda \cdot x\} ,$$

where c is a positive constant and $x \cdot y (x \in \mathbf{R}, y \in \mathbf{R})$ denotes the inner product.

Noting

$$\int_0^t e^{-c|\lambda|^\alpha s} ds \le \frac{C_5}{(1+|\lambda|^2)^{\frac{\alpha}{2}}} .$$

We have the next corollary.

COROLLARY 12. *Let $T \in H_1^\beta$, where we take $\beta > -\alpha/2$. Suppose that $\delta = \min(\alpha/2, \beta + \frac{\alpha}{2})$. Then $A_T(a : t, \omega)$ has (a, t)-jointly continuous modification, which is locally Hölder-continuous with exponent γ, where $0 < \gamma < \delta$.*

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