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On the Cartier Duality of Certain Finite Group Schemes of Order p^n

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Abstract. In this paper we study the Cartier duality of certain finite subgroup schemes of $\mathcal{G}^{(\lambda)}$ in positive characteristic, where $\mathcal{G}^{(\lambda)}$ denotes the form of $\mathbf{G}_m$ determined by λ. To establish the Cartier duality of these group schemes, we use certain deformations of Artin-Hasse exponential series.

1. Introduction

Throughout the paper, p denotes a prime number. Let A be a commutative ring with unit element and λ an element of A. T. Sekiguchi, F. Oort and N. Suwa [3] have introduced a group scheme $\mathcal{G}^{(\lambda)} = \mathrm{Spec}\, A[X, 1/(1+\lambda X)]$ over A, which is a deformation of the additive group scheme $\mathbf{G}_a$ (in the case $\lambda = 0$) to the multiplicative group scheme $\mathbf{G}_m$ (in the case $\lambda \in A^*$). (We recall the group structure of $\mathcal{G}^{(\lambda)}$ in section 3 below.) The group scheme $\mathcal{G}^{(\lambda)}$ is useful for studying the deformation of Artin-Schreier theory to Kummer theory. More precisely the following surjective homomorphism

$$\psi : \mathcal{G}^{(\lambda)} \to \mathcal{G}^{(\lambda^p)};\ x \mapsto \lambda^{-p}((1+\lambda x)^p - 1)$$

plays a key role in the unified Kummer-Artin-Schreier theory.

If A is of characteristic p, then $\psi(x) = x^p$. Put $N = \mathrm{Ker}\,\psi$. Let $F : \mathbf{G}_{a,A} \to \mathbf{G}_{a,A}$ be the Frobenius endomorphism. Y. Tsuno [6] showed the following:

THEOREM 1 ([6]). *Assume that A is of characteristic p. Then the Cartier dual of N is canonically isomorphic to* $\mathrm{Ker}[F - \lambda^{p-1} : \mathbf{G}_{a,A} \to \mathbf{G}_{a,A}]$.

Our purpose in this paper is to generalize Tsuno's theorem as follows. For a group scheme G, let $\widehat{G}$ be the formal completion along the zero section. The homomorphism ψ induces the natural homomorphism $\psi : \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathcal{G}}^{(\lambda^p)}$. Let l be a positive integer. We consider the following surjective homomorphism

$$\psi^{(l)} : \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathcal{G}}^{(\lambda^{p^l})};\ x \mapsto \lambda^{-p^l}((1+\lambda x)^{p^l} - 1)$$

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which is clearly a generalization of $\psi : \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathcal{G}}^{(\lambda^p)}$. If A is of characteristic p, then $\psi^{(l)}(x) = x^{p^l}$. Set $N_l = \operatorname{Ker} \psi^{(l)}$. The essential point in our argument is that the formal scheme N_l is nothing but the finite subgroup scheme Spec $A[X]/(X^{p^l})$ of $\mathcal{G}^{(\lambda)}$. Let $W_{l,A}$ be the Witt ring scheme of length l over A. Let $F : W_{l,A} \to W_{l,A}$ be the Frobenius endomorphism of $W_{l,A}$ and $[\lambda] : W_{l,A} \to W_{l,A}$ the Teichmüller lifting of $\lambda \in A$. Set $F^{(\lambda)} = F - [\lambda^{p-1}]$. (We analogously define an endomorphism $F^{(\lambda)} : W_A \to W_A$.) Then the result of this paper is the following:

THEOREM 2. *Assume that A is of characteristic p. Then the Cartier dual of N_l is canonically isomorphic to* $\operatorname{Ker}[F^{(\lambda)} : W_{l,A} \to W_{l,A}]$.

The case $l = 1$ of Theorem 2 is nothing but Tsuno's Theorem 1. Tsuno proved his theorem by skillful calculations. Our proof is different from Tsuno's proof even in the case $l = 1$. To prove Theorem 2, we make use of the deformations of Artin-Hasse exponential series introduced by Sekiguchi and Suwa [4] and a duality between $\operatorname{Ker}[F^{(\lambda)} : W(A) \to W(A)]$ with $\widehat{\mathcal{G}}^{(\lambda)}$ proved by them [Ibid.].

The contents of this paper is as follows. The next two sections are devoted to the definitions and the some reviews of properties of the Witt scheme and the deformation $E_p(\boldsymbol{v}, \lambda; x)$ of Artin-Hasse exponential series ($\boldsymbol{v} \in W(A),\ x \in \widehat{\mathcal{G}}^{(\lambda)}$). In section 4 we give a proof of Theorem 2.

Notation

$\mathbf{G}_{a,A}$: additive group scheme over A

$\mathbf{G}_{m,A}$: multiplicative group scheme over A

$W_{n,A}$: group scheme of Witt vectors of length n over A

W_A : group scheme of Witt vectors over A

$\widehat{\mathbf{G}}_{m,A}$: multiplicative formal group scheme over A

F : Frobenius endomorphism of W_A

V : Verschiebung endomorphism of W_A

R_n : restriction homomorphism of W_A to $W_{n,A}$

$[\lambda]$: Teichmüller lifting $(\lambda, 0, 0, \dots) \in W(A)$ of $\lambda \in A$

$F^{(\lambda)}$: $= F - [\lambda^{p-1}]$

$W(A)^{F^{(\lambda)}}$: $= \operatorname{Ker}[F^{(\lambda)} : W(A) \to W(A)]$

$W(A)/F^{(\lambda)}$: $= \operatorname{Coker}[F^{(\lambda)} : W(A) \to W(A)]$

2. Witt vectors

In this short section we recall necessary facts on Witt vectors for this paper. For details, see [1, Chap. V] or [2, Chap. III].

2.1. Let $\mathbf{X} = (X_0, X_1, \dots)$ be a sequence of variables. For each $n \geq 0$, we denote by $\Phi_n(\mathbf{X}) = \Phi_n(X_0, X_1, \dots, X_n)$ the Witt polynomial

$$\Phi_n(\mathbf{X}) = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n$$

in $\mathbf{Z}[\mathbf{X}] = \mathbf{Z}[X_0, X_1, \dots]$. Let $W_{n,\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[X_0, X_1, \dots, X_{n-1}]$ be the n-dimensional affine space over $\mathbf{Z}$. We define a morphism $\Phi^{(n)}$ by

$$\Phi^{(n)} : W_{n,\mathbf{Z}} \to \mathbf{A}^n_{\mathbf{Z}};\ \boldsymbol{x} \mapsto (\Phi_0(\boldsymbol{x}), \Phi_1(\boldsymbol{x}), \dots, \Phi_{n-1}(\boldsymbol{x}))\,,$$

where $\mathbf{A}^n_{\mathbf{Z}}$ is usual n-dimensional affine space over $\mathbf{Z}$. We call $\Phi^{(n)}$ the phantom map. The scheme $\mathbf{A}^n_{\mathbf{Z}}$ has a natural ring scheme structure. It is well-known that $W_{n,\mathbf{Z}}$ has a unique commutative ring scheme structure over $\mathbf{Z}$ such that the phantom map $\Phi^{(n)}$ is a homomorphism of commutative ring schemes over $\mathbf{Z}$. Then the points of $W_{n,\mathbf{Z}}$ are called Witt vectors of length n over $\mathbf{Z}$.

2.2. The Verschiebung homomorphism V is defined by

$$V : W(A) \to W(A);\ \boldsymbol{x} = (x_0, x_1, \dots) \mapsto \boldsymbol{x}' = (0, x_0, x_1, \dots)\,.$$

The restriction homomorphism R_n is defined by

$$R_n : W(A) \to W_n(A);\ \boldsymbol{x} = (x_0, x_1, \dots) \mapsto \boldsymbol{x}_n = (x_0, x_1, \dots, x_{n-1})\,.$$

We define a morphism $F : W_n(A) \to W_{n-1}(A)$ by

$$\Phi_i(F\boldsymbol{x}) = \Phi_{i+1}(\boldsymbol{x})$$

for $\boldsymbol{x} \in W_n(A)$. If A is of characteristic p, F is nothing but the usual Frobenius endomorphism. For $\lambda \in A$, $[\lambda]$ denotes the Teichmüller lifting $[\lambda] = (\lambda, 0, 0, \dots) \in W(A)$ and $F^{(\lambda)}$ denotes the endomorphism $F - [\lambda^{p-1}]$ of $W(A)$.

For $\boldsymbol{a} = (a_0, a_1, \dots) \in W(A)$, we also define a morphism $T_{\boldsymbol{a}} : W(A) \to W(A)$ by

$$\Phi_n(T_{\boldsymbol{a}}\boldsymbol{x}) = a_0{}^{p^n}\Phi_n(\boldsymbol{x}) + pa_1{}^{p^{n-1}}\Phi_{n-1}(\boldsymbol{x}) + \cdots + p^n a_n \Phi_0(\boldsymbol{x})$$

for $\boldsymbol{x} \in W(A)$. Then it is known that this morphism has the equality $T_{\boldsymbol{a}} = \sum_{k\geq 0} V^k \cdot [a_k]$. (cf. [5, Chap. 4, p. 20])

3. Deformed Artin-Hasse exponential series

In this short section we recall necessary facts on the deformed Artin-Hasse exponential series for this paper.

3.1. Let A be a ring and λ an element of A. Put $\mathcal{G}^{(\lambda)} = \operatorname{Spec} A[X, 1/(1 + \lambda X)]$. We define a morphism $\alpha^{(\lambda)}$ by

$$\alpha^{(\lambda)} : \mathcal{G}^{(\lambda)} \to \mathbf{G}_{m,A};\ x \mapsto 1 + \lambda x\,.$$

It is well-known that $\mathcal{G}^{(\lambda)}$ has a unique group scheme structure such that the morphism $\alpha^{(\lambda)}$ is a homomorphism over A. Then the group scheme structure of $\mathcal{G}^{(\lambda)}$ is given as follows:

$$\text{multiplication: } X \mapsto X \otimes 1 + 1 \otimes X + \lambda X \otimes X\,,$$
$$\text{unit: } X \mapsto 0\,,$$
$$\text{inverse: } X \mapsto -X/(1+\lambda X)\,.$$

If λ is invertible in A, $\alpha^{(\lambda)}$ is an A-isomorphism. On the other hand, if $\lambda = 0$, $\mathcal{G}^{(\lambda)}$ is nothing but the additive group scheme $\mathbf{G}_{a,A}$.

3.2. The Artin-Hasse exponential series $E_p(X)$ is given by

$$E_p(X) = \exp\left(\sum_{r \geq 0} \frac{X^{p^r}}{p^r}\right) \in \mathbf{Z}_{(p)}[[X]]\,.$$

We define a formal power series $E_p(U, \Lambda; X)$ in $\mathbf{Q}[U, \Lambda][[X]]$ by

$$E_p(U, \Lambda; X) = (1 + \Lambda X)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} X^{p^k})^{\frac{1}{p^k}((\frac{U}{\Lambda})^{p^k} - (\frac{U}{\Lambda})^{p^{k-1}})}\,.$$

As in [4, Corollary 2.5] or [5, Lemma 4.8], we see that this formal power series $E_p(U, \Lambda; X)$ is integral over $\mathbf{Z}_{(p)}$. Note that $E_p(1, 0; X) = E_p(X)$.

Let A be a $\mathbf{Z}_{(p)}$-algebra. Let $\lambda \in A$ and $\boldsymbol{v} = (v_0, v_1, \dots) \in W(A)$. We define a formal power series $E_p(\boldsymbol{v}, \lambda; X)$ in $A[[X]]$ by

$$\begin{aligned} E_p(\boldsymbol{v}, \lambda; X) &= \prod_{k=0}^{\infty} E_p(v_k, \lambda^{p^k}; X^{p^k}) \\ &= (1 + \lambda X)^{\frac{v_0}{\lambda}} \prod_{k=1}^{\infty} (1 + \lambda^{p^k} X^{p^k})^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)} \boldsymbol{v})}\,. \end{aligned}$$

Moreover we define a formal power series $F_p(\boldsymbol{v}, \lambda; X, Y)$ as follows:

$$F_p(\boldsymbol{v}, \lambda; X, Y) = \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k}(X + Y + \lambda XY)^{p^k}}\right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(\boldsymbol{v})}\,.$$

As in [4, Lemma 2.16] or [5, Lemma 4.9], we see that the formal power series $F_p(\boldsymbol{v}, \lambda; X, Y)$ is integral over $\mathbf{Z}_{(p)}$. For the formal power series $F_p(F^{(\lambda)}\boldsymbol{v}, \lambda; X, Y)$, we have the following equalities:

$$\begin{aligned} F_p(F^{(\lambda)}\boldsymbol{v}, \lambda; X, Y) &= \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k}(X + Y + \lambda XY)^{p^k}}\right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)} \boldsymbol{v})} \\ &= \frac{E_p(\boldsymbol{v}, \lambda; X) E_p(\boldsymbol{v}, \lambda; Y)}{E_p(\boldsymbol{v}, \lambda; X + Y + \lambda XY)}\,. \end{aligned}$$

We put $[p]E_p(\boldsymbol{v}, \lambda; X) = E_p([p]\boldsymbol{v}, \lambda; X)$, and we define a new formal power series $\widetilde{E}(\boldsymbol{w}, \lambda_2; E)$ as follows:

$$\widetilde{E}(\boldsymbol{w}, \lambda_2; E) = E^{\frac{w_0}{\lambda_2}} \prod_{r=1}^{\infty} ([p]^r E)^{\frac{1}{p^r \lambda_2{}^{p^r}} \Phi_{r-1}(F^{(\lambda_2)}\boldsymbol{w})}$$

where $E = E_p(\boldsymbol{v}, \lambda_1; X)$. Then it is known that the formal power series $\widetilde{E}(\boldsymbol{w}, \lambda_2; E)$ has the equality $\widetilde{E}(\boldsymbol{w}, \lambda_2; E) = E_p(T_{\boldsymbol{v}}\boldsymbol{w}, \lambda_1; X)$. (cf. [5, Chap. 4, p. 26])

4. Proof of Theorem 2

In this section we give a proof of Theorem 2.

Suppose that A is a ring of characteristic p and λ is an element of A. Let $\mathcal{G}^{(\lambda)}$ be the group scheme defined in section 3 and $\widehat{\mathcal{G}}^{(\lambda)}$ the formal completion of $\mathcal{G}^{(\lambda)}$ along the zero section. We consider the following homomorphism:

$$\psi^{(l)} : \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathcal{G}}^{(\lambda^{p^l})};\ x \mapsto \lambda^{-p^l}((1 + \lambda x)^{p^l} - 1)\,.$$

For the kernel of the homomorphism $\psi^{(l)}$, we have

$$N_l = \operatorname{Ker} \psi^{(l)} = \operatorname{Spf} A[[X]]/(X^{p^l}) = \operatorname{Spec} A[X]/(X^{p^l})\,.$$

Note that this is a finite subgroup scheme of order p^l of $\mathcal{G}^{(\lambda)}$. The following exact sequence is induced by the homomorphism $\psi^{(l)}$

$$0 \longrightarrow N_l \xrightarrow{\iota} \widehat{\mathcal{G}}^{(\lambda)} \xrightarrow{\psi^{(l)}} \widehat{\mathcal{G}}^{(\lambda^{p^l})} \longrightarrow 0 \tag{1}$$

where ι is a canonical inclusion. This exact sequence (1) deduces the following long exact sequence:

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbf{G}}_{m,A}) &\xrightarrow{(\psi^{(l)})^*} \operatorname{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A}) \xrightarrow{(\iota)^*} \operatorname{Hom}(N_l, \widehat{\mathbf{G}}_{m,A}) \\ \xrightarrow{\partial} \operatorname{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbf{G}}_{m,A}) &\xrightarrow{(\psi^{(l)})^*} \operatorname{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A}) \longrightarrow \quad \cdots \quad . \end{aligned} \tag{2}$$

On the other hand, as in [4, Theorem 2.19.1.] or the case $n = 1$ of [5, Theorem 5.1.], the following morphisms are isomorphic:

$$W(A)^{F^{(\lambda)}} \to \operatorname{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A});\ \boldsymbol{v} \mapsto E_p(\boldsymbol{v}, \lambda; x) \tag{3}$$

$$W(A)/F^{(\lambda)} \to H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A});\ \boldsymbol{w} \mapsto F_p(\boldsymbol{w}, \lambda; x, y)\,. \tag{4}$$

Here $H_0^2(G, H)$ denotes the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in H for formal group schemes G and H. (c.f. [1, Chap. II.3 and Chap. III.6])

We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W(A) & \xrightarrow{V^l} & W(A) & \xrightarrow{R_l} & W_l(A) & \longrightarrow & 0 \\
 & & {\scriptstyle F^{(\lambda^{p^l})}}\big\downarrow & & {\scriptstyle F^{(\lambda)}}\big\downarrow & & {\scriptstyle F^{(\lambda)}}\big\downarrow & & \\
0 & \longrightarrow & W(A) & \xrightarrow{V^l} & W(A) & \xrightarrow{R_l} & W_l(A) & \longrightarrow & 0\,.
\end{array}$$

The exactness of the horizontal sequences are obvious. By the well-known elementary properties on F, V and $[\lambda]$, we have $F^{(\lambda)} \circ V^l = V^l \circ F^{(\lambda^{p^l})}$. Therefore, by the snake lemma for this diagram, we have the following exact sequence:

$$\begin{aligned}
(5) \qquad & 0 \longrightarrow W(A)^{F^{(\lambda^{p^l})}} \xrightarrow{V^l} W(A)^{F^{(\lambda)}} \xrightarrow{R_l} W_l(A)^{F^{(\lambda)}} \\
& \xrightarrow{\partial} W(A)/F^{(\lambda^{p^l})} \xrightarrow{V^l} W(A)/F^{(\lambda)} \xrightarrow{R_l} W_l(A)/F^{(\lambda)} \longrightarrow 0\,.
\end{aligned}$$

Now, by combining the exact sequences (2), (5) and the isomorphisms (3), (4), we have the following diagram:

$$(6)\qquad \begin{array}{ccccc}
\mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbf{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \mathrm{Hom}(N_l, \widehat{\mathbf{G}}_{m,A}) \\
{\scriptstyle \phi_1}\big\uparrow & & {\scriptstyle \phi_2}\big\uparrow & & {\scriptstyle \phi}\big\uparrow \\
W(A)^{F^{(\lambda^{p^l})}} & \xrightarrow{V^l} & W(A)^{F^{(\lambda)}} & \xrightarrow{R_l} & W_l(A)^{F^{(\lambda)}} \\
 & \xrightarrow{\partial} & \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbf{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A}) \\
 & & {\scriptstyle \phi_3}\big\uparrow & & {\scriptstyle \phi_4}\big\uparrow \\
 & \xrightarrow{\partial} & W(A)/F^{(\lambda^{p^l})} & \xrightarrow{V^l} & W(A)/F^{(\lambda)}\,,
\end{array}$$

where ϕ is the following homomorphism induced from the exact sequence (1) and the isomorphism (3):

$$\phi : W_l(A)^{F^{(\lambda)}} \to \mathrm{Hom}(N_l, \widehat{\mathbf{G}}_{m,A});\ \boldsymbol{v}_l \mapsto E_p(\boldsymbol{v}_l, \lambda; x)\,.$$

We remark that ϕ_1 and ϕ_2 are isomorphisms, and that ϕ_3 and ϕ_4 are injective but may be not isomorphisms since $H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A}) \subsetneqq \mathrm{Ext}^1(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A})$ in general. But we can replace the groups of extensions with Hochschild cohomology groups in the diagram (6), since we have $\mathrm{Im}\ \partial \subseteq \mathrm{Im}\ \phi_3$ and $\mathrm{Im}\ (\psi^{(l)})^* \subseteq \mathrm{Im}\ \phi_4$. Thus we get the following diagram whose each row is exact and all vertical morphisms are isomorphisms except ϕ:

$$
\begin{array}{ccccc}
\mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbf{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & \mathrm{Hom}(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \mathrm{Hom}(N_l, \widehat{\mathbf{G}}_{m,A}) \\
\phi_1 \uparrow & & \phi_2 \uparrow & & \phi \uparrow \\
W(A)^{F^{(\lambda^{p^l})}} & \xrightarrow{V^l} & W(A)^{F^{(\lambda)}} & \xrightarrow{R_l} & W_l(A)^{F^{(\lambda)}} \\
& \xrightarrow{\partial} & H_0^2(\widehat{\mathcal{G}}^{(\lambda^{p^l})}, \widehat{\mathbf{G}}_{m,A}) & \xrightarrow{(\psi^{(l)})^*} & H_0^2(\widehat{\mathcal{G}}^{(\lambda)}, \widehat{\mathbf{G}}_{m,A}) \\
& & \phi_3 \uparrow & & \phi_4 \uparrow \\
& \xrightarrow{\partial} & W(A)/F^{(\lambda^{p^l})} & \xrightarrow{V^l} & W(A)/F^{(\lambda)} .
\end{array}
\tag{7}
$$

If the diagram (7) is commutative, then the five lemma shows that ϕ is isomorphism, i.e., $W_l(A)^{F^{(\lambda)}} \simeq \mathrm{Hom}(N_l, \widehat{\mathbf{G}}_{m,A})$. Since $\mathrm{Hom}(N_l, \widehat{\mathbf{G}}_{m,A}) \simeq \mathrm{Hom}(N_l, \mathbf{G}_{m,A})$ and the Cartier duals are characterized by the character groups, we obtain the Theorem 2. Therefore it is sufficient to prove that the diagram (7) is commutative.

LEMMA 1. $(\psi^{(l)})^* \circ \phi_1 = \phi_2 \circ V^l$.

PROOF. By the definition and the results stated in [5, Proposition 4.11.], we have the following equalities for $\boldsymbol{v} \in W(A)^{F^{(\lambda^{p^l})}}$:

$$
\begin{aligned}
E_p(\boldsymbol{v}, \lambda^{p^l}; \psi^{(l)}(x)) &= E_p(\boldsymbol{v}, \lambda^{p^l}; \lambda^{-p^l}((1+\lambda x)^{p^l} - 1)) \\
&= E_p(\boldsymbol{v}, \lambda^{p^l}; \lambda^{-p^l}(E_p([\lambda], \lambda; x)^{p^l} - 1)) \\
&= \widetilde{E}_p(\boldsymbol{v}, \lambda^{p^l}; E_p(p^l[\lambda], \lambda; x)) \cdot G_p(F^{(\lambda^{p^l})}\boldsymbol{v}, \lambda^{p^l}; E_p(p^l[\lambda], \lambda; x)) \\
&= E_p(T_{\lambda^{-p^l} p^l[\lambda]}\boldsymbol{v}, \lambda; x) .
\end{aligned}
$$

Since the third equality is always true for variables, $\boldsymbol{v}$, λ and x as in [5, Chap.4, p.29], the above last equality is true for any element (even nilpotent) $\lambda \in A$. Thus we must show the equality of $V^l = T_{\lambda^{-p^l} p^l[\lambda]}$ in our case.

In order to show the equality, by Lemma 4.2 of [5], it is sufficient that we prove the equality $\lambda^{-p^l} p^l[\lambda] = (0, \dots, 0, 1, 0, \dots)$: all component is 0 except the l-th component 1. By the phantom map we can calculate $p^l[\lambda] = (x_0, x_1, \dots)$ as follows. By the equality $\Phi_i(p^l[\lambda]) = \Phi_i(x_0, x_1, \dots)$ for each i (where Φ_i is the Witt polynomial) we have the following equalities:

$$
x_i = p^{-i}(p^l \lambda^{p^i} - x_0^{p^i} - p x_1^{p^{i-1}} - \dots - p^{i-1} x_{i-1}^p) = p^{-i}\left(p^l \lambda^{p^i} - \sum_{j=0}^{i-1} p^j x_j^{p^{i-j}}\right).
$$

We claim the following

$$x_i \equiv \begin{cases} 0 \pmod p & \text{if } i \neq l \\ \lambda^{p^l} \pmod p & \text{if } i = l\,. \end{cases}$$

We show the claim by induction on i. If $i = 0$, then it is obvious. Now assume that we have the following congruencies for $j < i$:

$$x_j \equiv \begin{cases} 0 \pmod p & \text{if } j \neq l \\ \lambda^{p^l} \pmod p & \text{if } j = l\,. \end{cases}$$

If $l < i$, we see the following equalities:

$$x_i = p^{-i}\left(p^l\lambda^{p^i} - \sum_{j=0}^{i-1} p^j x_j^{p^{i-j}}\right) = p^{-i}\left(p^l\lambda^{p^i} - p^l x_l^{p^{i-l}} - \sum_{\substack{j=0 \\ j\neq l}}^{i-1} p^j x_j^{p^{i-j}}\right).$$

The assumption of the induction gives the following congruencies:

$$p^j x_j^{p^{i-j}} \equiv \begin{cases} 0 \pmod{p^{i+1}} & \text{if } j \neq l \\ p^j\lambda^{p^i} \pmod{p^{i+1}} & \text{if } j = l\,. \end{cases}$$

Therefore we obtain the congruence $x_i \equiv 0 \pmod p$. In the case of $i \leq l$, it is similarly verified. Consequently we have the claim. Hence we obtain the equalities:

$$\lambda^{-p^l} p^l[\lambda] = (0, \dots, 0, 1, 0, \dots) \quad \text{and} \quad T_{\lambda^{-p^l} p^l[\lambda]} = V^l\,.$$

□

LEMMA 2. $(\iota)^* \circ \phi_2 = \phi \circ R_l$.

PROOF. This follows from the definitions of ϕ and $(\iota)^*$. □

LEMMA 3. $\partial \circ \phi = \phi_3 \circ \partial$.

PROOF. We can calculate $\partial E_p(\boldsymbol{v}_l, \lambda; x)$ ($\boldsymbol{v}_l \in W_l(A)^{F^{(\lambda)}}$) by the direct product $\widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})}$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_l & \longrightarrow & \widehat{\mathcal{G}}^{(\lambda)} & \xrightarrow{\psi^{(l)}} & \widehat{\mathcal{G}}^{(\lambda^{p^l})} & \longrightarrow & 0 \\ & & {\scriptstyle E_p(\boldsymbol{v}_l,\lambda;x)}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \widehat{\mathbf{G}}_{m,A} & \longrightarrow & \widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})} & \longrightarrow & \widehat{\mathcal{G}}^{(\lambda^{p^l})} & \longrightarrow & 0\,. \end{array} \tag{8}$$

We choose an inverse image $\boldsymbol{w}$ of $\boldsymbol{v}_l$ for the homomorphism $R_l : W(A) \to W_l(A)$. By the commutativity of the diagram (8), φ should be given by:

$$\varphi : \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})};\ x \mapsto (E_p(\boldsymbol{w}, \lambda; x), \psi^{(l)}(x))\,.$$

(Note that $E_p(\boldsymbol{v}_l, \lambda; x) : \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathbf{G}}_{m,A}$ is not a homomorphism.) We endow $\widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})}$ with a group scheme structure such that $\varphi : \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})}$ is a homomorphism. i.e., the following equality should be satisfied:

$$\varphi(x_1 x_2) = \varphi(x_1) \cdot \varphi(x_2) \quad (x_1, x_2 \in \widehat{\mathcal{G}}^{(\lambda)}),$$

where

$$\varphi(x_1 x_2) = (E_p(\boldsymbol{w}, \lambda; x_1 + x_2 + \lambda x_1 x_2), \psi^{(l)}(x_1 x_2)),$$
$$\varphi(x_1) \cdot \varphi(x_2) = (E_p(\boldsymbol{w}, \lambda; x_1), \psi^{(l)}(x_1)) \cdot (E_p(\boldsymbol{w}, \lambda; x_2), \psi^{(l)}(x_2)).$$

For elements (t_1, y_1) and (t_2, y_2) of $\widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})}$, we choose x_1 and x_2 in the inverse images of y_1 and y_2 for the homomorphism $\psi^{(l)}$, respectively. Then the group structure of $\widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})}$ should be given by

$$(t_1, y_1) \cdot (t_2, y_2) = \left(t_1 t_2 \cdot \frac{E_p(\boldsymbol{w}, \lambda; x_1 + x_2 + \lambda x_1 x_2)}{E_p(\boldsymbol{w}, \lambda; x_1) \cdot E_p(\boldsymbol{w}, \lambda; x_2)},\ y_1 + y_2 + \lambda^{p^l} y_1 y_2 \right).$$

Hence the boundary map ∂ should be given by the following formal power series:

$$\begin{aligned} F_p(F^{(\lambda)}\boldsymbol{w}, \lambda; x_1, x_2) &= \frac{E_p(\boldsymbol{w}, \lambda; x_1) \cdot E_p(\boldsymbol{w}, \lambda; x_2)}{E_p(\boldsymbol{w}, \lambda; x_1 + x_2 + \lambda x_1 x_2)} \\ &= \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} x_1{}^{p^k})(1 + \lambda^{p^k} x_2{}^{p^k})}{1 + \lambda^{p^k}(x_1 + x_2 + \lambda x_1 x_2)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(F^{(\lambda)}\boldsymbol{w})}. \end{aligned}$$

To prove the equality of Lemma 3, we must show the following equality of the formal power series:

$$F_p(F^{(\lambda)}\boldsymbol{w}, \lambda; x_1, x_2) \equiv F_p(z, \lambda^{p^l}; \psi^{(l)}(x_1), \psi^{(l)}(x_2)) \pmod p,$$

where z is an inverse image of the boundary $\partial \boldsymbol{v}_l$ for $W(A) \to W(A)/F^{(\lambda^{p^l})}$. This equality is proved as follows:

$$\begin{aligned} F_p(F^{(\lambda)}\boldsymbol{w}, \lambda; x_1, x_2) &= F_p(V^l z, \lambda; x_1, x_2) \\ &= \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} x_1{}^{p^k})(1 + \lambda^{p^k} x_2{}^{p^k})}{1 + \lambda^{p^k}(x_1 + x_2 + \lambda x_1 x_2)^{p^k}} \right)^{\frac{1}{p^k \lambda^{p^k}} \Phi_{k-1}(V^l z)} \\ &= \prod_{r=1}^{\infty} \left(\frac{(1 + \lambda^{p^{l+r}} x_1{}^{p^{l+r}})(1 + \lambda^{p^{l+r}} x_2{}^{p^{l+r}})}{1 + \lambda^{p^{l+r}}(x_1 + x_2 + \lambda x_1 x_2)^{p^{l+r}}} \right)^{\frac{1}{p^r \lambda^{p^{r+l}}} \Phi_{r-1}(z)} \\ &\equiv \prod_{r=1}^{\infty} \left(\frac{(1 + \lambda^{p^{l+r}} x_1{}^{p^{l+r}})(1 + \lambda^{p^{l+r}} x_2{}^{p^{l+r}})}{1 + \lambda^{p^{l+r}}(x_1^{p^l} + x_2^{p^l} + \lambda^{p^l} x_1^{p^l} x_2^{p^l})^{p^r}} \right)^{\frac{1}{p^r \lambda^{p^{r+l}}} \Phi_{r-1}(z)} \pmod p \end{aligned}$$

$$= F_p(z, \lambda^{p^l}; x_1^{p^l}, x_2^{p^l}) = F_p(z, \lambda^{p^l}; \psi^{(l)}(x_1), \psi^{(l)}(x_2)).$$

$\square$

LEMMA 4. $(\psi^{(l)})^* \circ \phi_3 = \phi_4 \circ V^l$.

PROOF. We can calculate $(\psi^{(l)})^* F_p(\boldsymbol{v}, \lambda; x_1, x_2)$ ($\boldsymbol{v} \in W(A)/F^{(\lambda^{p^l})}$) by the direct product $\widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda)}$ such that the following diagram is commutative:

(9)
$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathbf{G}}_{m,A} \arrow[r] \arrow[d, equal] & \widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda)} \arrow[r] \arrow[d, "\Psi"'] & \widehat{\mathcal{G}}^{(\lambda)} \arrow[r] \arrow[d, "\psi^{(l)}"'] & 0 \\
0 \arrow[r] & \widehat{\mathbf{G}}_{m,A} \arrow[r] & \widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})} \arrow[r] & \widehat{\mathcal{G}}^{(\lambda^{p^l})} \arrow[r] & 0 .
\end{tikzcd}$$

By the commutativity of the diagram (9), Ψ should be given by

$$\Psi : \widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda)} \to \widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda^{p^l})}; \ (t, x) \mapsto (t, \psi^{(l)}(x)).$$

We endow $\widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda)}$ with a group scheme structure such that Ψ is a homomorphism. For local sections (t_1, x_1) and (t_2, x_2) in $\widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda)}$, suppose that the product $(t_1, x_1) \cdot (t_2, x_2)$ is written as $(t_1, x_1) \cdot (t_2, x_2) = (t_1 t_2 G(x_1, x_2), x_1 \cdot x_2)$, where $G(x_1, x_2)$ is a cocycle on $\widehat{\mathbf{G}}_{m,A} \times \widehat{\mathcal{G}}^{(\lambda)}$. Then we have

$$\begin{aligned}
\Psi(t_1, x_1) \cdot \Psi(t_2, x_2) &= (t_1, \psi^{(l)}(x_1)) \cdot (t_2, \psi^{(l)}(x_2)) \\
&= (t_1 t_2 F_p(\boldsymbol{v}, \lambda^{p^l}; \psi^{(l)}(x_1), \psi^{(l)}(x_2)), \psi^{(l)}(x_1) \cdot \psi^{(l)}(x_2)),
\end{aligned}$$

on the other hand, we have

$$\Psi((t_1, x_1) \cdot (t_2, x_2)) = \Psi(t_1 t_2 G(x_1, x_2), x_1 \cdot x_2) = (t_1 t_2 G(x_1, x_2), \psi^{(l)}(x_1) \cdot \psi^{(l)}(x_2)).$$

Hence, in order for Ψ to be a homomorphism, the following equality is necessary:

$$G(x_1, x_2) = F_p(\boldsymbol{v}, \lambda^{p^l}; \psi^{(l)}(x_1), \psi^{(l)}(x_2)).$$

To prove the equality of Lemma 4, we must show the following equality:

$$F_p(\boldsymbol{v}, \lambda^{p^l}; \psi^{(l)}(x_1), \psi^{(l)}(x_2)) = F_p(V^l \boldsymbol{v}, \lambda; x_1, x_2),$$

but this equality has been already proved in Lemma 3. $\square$

These lemmas show that the diagram (7) is commutative. Hence we obtain the Theorem 2.

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