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On Some Branched Surfaces Which Admit Expanding Immersions

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Abstract. We deal with the class of branched surfaces K such that 1) the branch set S of K is an embedded circle, 2) all connected components of $K\setminus S$ are orientable and their number is two or three. We show that in this class only two topological types admit expanding immersions. In the proof of the result, the Euler class of the tangent bundle of K plays an important role.

§0. Introduction.

R. Williams [1], [2], [3] introduced the concept of branched manifolds and expanding immersions in order to study the dynamics of expanding attractors. Using his own tools, he succeeded in classifying 1 dimensional expanding attractors. Our final aim is to study the topological conjugacy classes of 2 dimensional expanding attractors. As the first step toward it we propose the following problem:

> Find some topological invariants of branched surfaces which admit expanding immersions.

As an approach to solve this problem, we consider the simplest class of them, i.e., the class of branched surfaces with branch sets a circle.

First of all let us give two examples of expanding immersions. First take a rectangle $[0, 1]\times[0, 2]$ in the coordinate plane, and take two disks D_1 and D_2 whose radii are $1/10$ and centers are $(4/5, 4/5)$ and $(4/5, 4/5+1)$ respectively. We define the equivalence relation among the points in the rectangle; $(s, t)\sim(s', t') \Leftrightarrow 1)$ (s, t) and (s', t') do not belong to D_1 and D_2, and $(s-t)\equiv 0$, $(s'-t')\equiv 0 \bmod 1$. 2) $(s, t)=(s', t')\in D_1$ or D_2. We denote the quotient space by this equivalence relation by T^*. Then T^* is a branched surface whose branch set is homeomorphic to a circle. Notice that there exists a canonical projection $p: T^*\to T^2$. The dilation by 2

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yields a map $f: T^2 \to T^2$. Clearly f lifts to a map $\bar{f}: T^* \to T^*$ in a way that $\bar{f}$ is surjective. Thus T^* admits an expanding immersion.

The second example is as follows. We regard T^2 as a rectangle $[0, 1]\times[0, 1]$, and take two disks D_1 and D_2 in it whose radii are $1/10$ and centers are $(1/2, 1/4)$ and $(1/2, 3/4)$ respectively. We define the following equivalence relation in T^2; $(s, t)\sim(s', t') \Leftrightarrow$ 1) $(s, t)\in D_1$ and $(s', t')\in D_2$, or $(s, t)\in D_2$ and $(s', t')\in D_1$, and $2t\equiv 2t'$, $s\equiv s' \bmod 1$. 2) $(s, t)=(s', t')$. We consider the quotient space by this equivalence relation and denote it by T_*. T_* is a branched surface whose branch set is homeomorphic to a circle, too. The dilation by 2, $f: T^2\to T^2$, projects down to a map $\underline{f}: T_*\to T_*$ via the natural projection $T^2\to T_*$. This shows that T_* admits an expanding immersions.

Suppose a branched surface K has a branch set S homeomorphic to a circle. Then a neighborhood of S is homeomorphic to one of the following N_0 and N_1. Take two copies of a rectangle $I\times I$, where $I=[-1, 1]$, and identify the subsets $I\times[-1, 0]$ of them. (See Figure 1.)

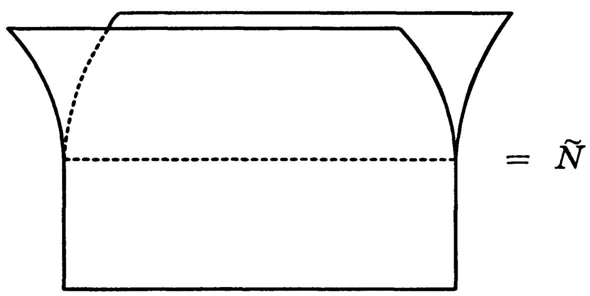


FIGURE 1

$\tilde{N}$ denotes the quotient space. We take subsets I_a and I'_a in $\tilde{N}$ which are the images of $\{-1\}\times I$ and $\{1\}\times I$, contained in one of two copies, respectively, and let I_b and I'_b be the images of $\{-1\}\times I$ and $\{1\}\times I$, contained in the other of them, respectively. Then N is obtained by connecting I_a with I'_a and I_b with I'_b, or connecting I_a with I'_b and I_b with I'_a. We denote the former by N_0 and the latter by N_1. We define subsets of N_0 and N_1 as follows. Let J_1^+ and J_2^+ be the images in N_0 of two copies of $I\times\{1\}$ in two copies of $I\times I$ respectively, and let J^- be the image in N_0 of $I\times\{-1\}$. In N_1, let J^+ and J^- be the images of $I\times\{1\}$ and $I\times\{-1\}$.

Using N_0 and N_1, we define the types of S. S is called *untwisted* (or *twisted*) if S has a neighborhood homeomorphic to N_0 (or N_1).

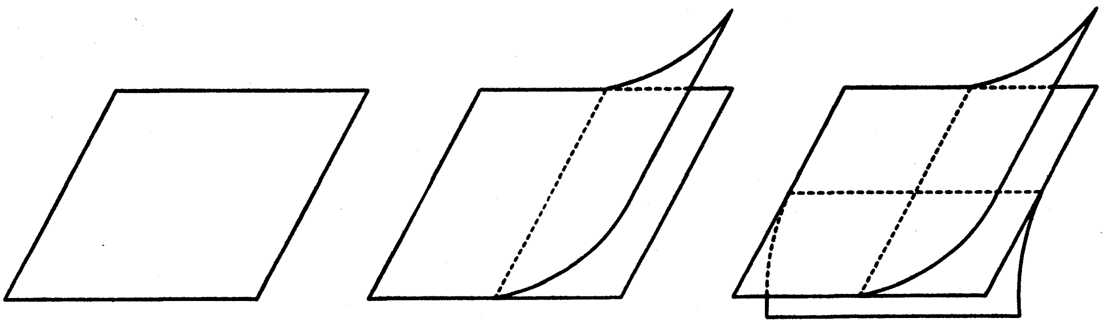
The main result of this paper is as follows. We consider the class of branched surfaces K such that 1) the branch set S of K is an embedded circle, 2) all connected components of $K\backslash S$ are orientable and their number is two or three. In this class, only T^* and T_* admit expanding immersions.

In §1, after giving definitions of branched surfaces and expanding immersions, a precise statement of our result is described. §2 and §3 are devoted to its proof.

The author thanks the referee for suggesting the use of the Euler class of the tangent bundle of K. It makes the proof of the theorem clear and simple.

§1. Definitions and the statement of the result.

In order to define branched surfaces, three types of local neighborhoods are needed. Let us define:
1) $U_{(1)}=I\times I$, where I is an open interval $(-1, 1)$.
2) $U_{(2)}=U^1_{(1)}\amalg U^2_{(1)}/\sim$, which means a quotient space of two copies of $U_{(1)}$, $U^1_{(1)}$ and $U^2_{(1)}$, by the equivalence relation generated by $(t, s)\sim(t', s')\Leftrightarrow (t, s)\in U^1_{(1)}$, $(t', s')\in U^2_{(1)}$ and $-1<t=t'\leqq 0$, $s=s'$.
3) $U_{(3)}=U_{(2)}\amalg U^3_{(1)}/\sim$, which means a quotient space of the copy $U^3_{(1)}$ of $U_{(1)}$ and $U_{(2)}$ by the equivalence relation generated by $(t, s)\sim(t', s')\Leftrightarrow (t, s)\in U^2_{(1)}\subset U_{(2)}$, $(t', s')\in U^3_{(1)}$ and $t=t'$, $-1<s=s'\leqq 0$.



$U_{(1)}$ $U_{(2)}$ $U_{(3)}$

FIGURE 2

Here we have natural maps $\pi_2: U_{(2)}\to U_{(1)}$ and $\pi_3: U_{(3)}\to U_{(1)}$ such that $\pi_i|U^j_{(i)}$ is a natural identification of the copy $U^j_{(i)}$ with $U_{(1)}$ itself, where $i=2$ and $j=1$ or 2, or $i=3$ and $j=1$, 2 or 3.

DEFINITION 1 [3]. A compact Hausdorff space K is called a C^r *branched surface* if it has a finite family $\{(U_j, \varphi_j)\}$ satisfying
1) $K=\cup_j U_j$,
2) For each j there exists a homeomorphism $g_j: U_j\to U_{(i)}$ $(i=1, 2$ or $3)$ such that $\varphi_j=\pi_{(i)}\circ g_j$,

3) For j and j' such that $U_j \cap U_{j'} \neq \emptyset$, there exists a C^r map $\pi_{j'j}$: $\varphi_j(U_j \cap U_{j'}) \to \varphi_{j'}(U_{j'} \cap U_j)$ such that $\pi_{j'j} \circ \varphi_j = \varphi_{j'}$.
We call (U_j, φ_j) a *coordinate neighborhood* and $\{(U_j, \varphi_j)\}$ a *coordinate neighborhood system of K*.

$S = \{x \in K$; x does not have a neighborhood homeomorphic to an open disk $\mathring{D}^2\}$ is called the *branch set of K*.

As in the case of ordinary manifolds, we define the *tangent bundle* TK of K as the quotient space of $\amalg_j \varphi_j^* TU_{(1)}$ by the natural identification induced by the coordinate change, where $\varphi_j^* TU_{(1)}$ denotes the pull back of the tangent bundle $TU_{(1)}$ by φ_j. (For detail, see [3].) For $x \in K$, $p^{-1}(x)$ is called the *tangent space at x* and is denoted by $T_x K$, where $p: TK \to K$ denotes the projection map, which is induced by $p_j: \varphi_j^* TU_{(1)} \to U_j$ naturally.

A *Riemannian metric on K* is defined as a positive definite symmetric bilinear form on TK.

Next, we define a C^r map from a branched surface to a branched surface, a C^r immersion and an expanding immersion.

DEFINITION 2. Let K and L be C^r branched surfaces, and $\{(U_j, \varphi_j)\}$ and $\{(V_k, \psi_k)\}$ be their coordinate neighborhood systems respectively.

1) A map $f: K \to L$ is called a *C^r map* if for any i, j and k with $f^{-1}(V_k) \cap U_j^i \neq \emptyset$, the composite

$$U_{(1)} \xrightarrow{(\varphi_j | U_j^i)^{-1}} f^{-1}(V_k) \cap U_j^i \xrightarrow{f} V_k \xrightarrow{\psi_k} U_{(1)}$$

is C^r, where $U_j^i = g_j^{-1}(U_{(m)}^i)$.

For a C^r map $f: K \to L$, we can define the differential of f, $df: TK \to TL$, by using the above local representation of f (See [3]). We denote $df | T_x K$ by df_x.

2) A map $f: K \to L$ is called a *C^r immersion* if f is a C^r map and $df_x: T_x K \to T_{f(x)} L$ is injective for any $x \in K$.

3) A map $f: K \to K$ is called a *C^r expanding immersion* if it satisfies
i) f is a C^r immersion,
ii) there exist numbers $\alpha > 0$ and $\nu > 1$ such that for any positive integer n and $v \in T_x K$, $\|df_x^n(v)\| \geqq \alpha \nu^n \|v\|$, where $\|\cdot\|$ means a Riemannian metric,
iii) there exists a positive integer $\bar{n}$ such that for any $x \in K$ and some neighborhood U of x, $f^{\bar{n}}(U)$ is homeomorphic to an open disk,
iv) the nonwandering set $\Omega(f)$ of f is equal to K.

Our branched surfaces are more restrictive than Williams'. His original definition admits more varied types of neighborhoods. But

Williams himself showed that ours are sufficiently general to study expanding immersions.

THEOREM. *Suppose K is a C^1 branched surface such that*
1) *K admits an expanding immersion,*
2) *The branch set S of K is homeomorphic to a circle,*
3) *All connected components of $K\backslash S$ are orientable and their number is 2 or 3.*

Then K is homeomorphic to T^* or T_*.

§2. Proof of Theorem (1).

In this section we deal with the case when the number of connected components of $K\backslash S$ is equal to 3. We show that in this case only T^* admits expanding immersions.

Assume that K admits an expanding immersion f. Let $\mathring{K}_0$, $\mathring{K}_1$ and $\mathring{K}_2$ be connected components of $K\backslash S$ such that $\mathring{K}_0 \supset J^-$, $\mathring{K}_1 \supset J_1^+$ and $\mathring{K}_2 \supset J_2^+$. For $i=0$, 1 or 2, we attach $\partial\mathring{K}_i$ to $\mathring{K}_i$, and denote the obtained space by K_i. (Below, generally for an open subspace $X \subset Y$, we denote the one obtained by attaching the copies of boundary ∂X to X as $X^\wedge$. For example, $K_i = \mathring{K}_i^\wedge$.)

We construct manifolds M_1 and M_2 from K_0 and K_1, and K_0 and K_2 by identifying their boundaries respectively. M_1 and M_2 are embedded in K by natural inclusions $\iota_1: M_1 \to K$ and $\iota_2: M_2 \to K$. By easy calculation, we know that $H_2(K; \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$ and is generated by $m_1 = (\iota_1)_*[M_1]$ and $m_2 = (\iota_2)_*[M_2]$, where $[M_1]$ and $[M_2]$ are the fundamental homology classes of M_1 and M_2 such that they induce the same orientation on K.

LEMMA 1. *Let*

$$(f^{2n})_* m_1 = \alpha_n m_1 + \beta_n m_2\ , \quad (f^{2n})_* m_2 = \gamma_n m_1 + \delta_n m_2\ .$$

Then α_n, β_n, γ_n and $\delta_n \geqq 0$, and both $\alpha_n + \beta_n$ and $\gamma_n + \delta_n$ become large as n becomes large.

PROOF. Since f^{2n} is orientation preserving, α_n, β_n, γ_n and $\delta_n \geqq 0$.

Let ω be the volume form on K whose local representation is $\sqrt{\det(g_{ij})}\, dx_1 \wedge dx_2$ when the local representation of the Riemannian metric is $\sum_{0 \leq i, j \leq 2} g_{ij} dx_i \otimes dx_j$. Let us denote the areas of M_1, M_2 and K by $a(M_1)$, $a(M_2)$ and $a(K)$ respectively.

We calculate the Kronecker product of $(f^{2n})_* m_1$ and ω:

$$\langle\omega, (f^{2n})_* m_1\rangle = \alpha_n\langle\omega, m_1\rangle + \beta_n\langle\omega, m_2\rangle$$
$$= \alpha_n\int_{M_1}(\iota_1)^*\omega + \beta_n\int_{M_2}(\iota_2)^*\omega = \alpha_n\cdot a(M_1) + \beta_n\cdot a(M_2) .$$

On the other hand, we have

$$\langle\omega, (f^{2n})_* m_1\rangle = \langle(f^{2n})^*\omega, m_1\rangle = \int_{M_1}(\det Df^{2n})\cdot(\iota_1)^*\omega$$
$$\geqq \min_{p\in M_1} \det(Df^{2n})_p\cdot a(M_1) ,$$

where $\det(Df^{2n})_p$ denotes the determinant of $(Df^{2n})_p$ for the orthonormal bases of T_pK and $T_{f^{2n}(p)}K$. Hence we obtain the following inequality:

$$\alpha_n\cdot a(M_1) + \beta_n\cdot a(M_2) \geqq \min_{p\in M_1} \det(Df^{2n})_p\cdot a(M_1) .$$

By Definition 2, 3), ii), the right-hand side of the above inequality becomes large as n becomes large. Hence we have the desired result for $\alpha_n+\beta_n$.

For $\gamma_n+\delta_n$, we can show the lemma in the same way as for $\alpha_n+\beta_n$. □

Let $e(K)$ be the Euler class of the tangent bundle of K. We calculate the Kronecker product of $e(K)$ and m_1:

$$\langle e(K), m_1\rangle = \langle e(K), (\iota_1)_*[M_1]\rangle = \langle(\iota_1)^* e(K), [M_1]\rangle = \langle e(M_1), [M_1]\rangle = \chi(M_1) .$$

On the other hand, since $(f^{2n})^* e(K) = e(K)$,

$$\langle e(K), m_1\rangle = \langle(f^{2n})^* e(K), m_1\rangle = \langle e(K), (f^{2n})_* m_1\rangle = \alpha_n\chi(M_1) + \beta_n\chi(M_2) .$$

Hence we obtain for any n:

$$\chi(M_1) = \alpha_n\chi(M_1) + \beta_n\chi(M_2) . \tag{1}$$

Calculating $\langle e(K), m_2\rangle$, we also have:

$$\chi(M_2) = \gamma_n\chi(M_1) + \delta_n\chi(M_2) . \tag{2}$$

By Lemma 1, for sufficiently large n, $\alpha_n+\beta_n$ and $\gamma_n+\delta_n$ are large. Then from the equalities (1) and (2), we have only the following two cases: 1° $\chi(M_1)=0$ or $\chi(M_2)=0$, 2° $\chi(M_1)>0$ and $\chi(M_2)<0$.

We show that the case 2° cannot occur. In the case 2°, M_1 is a sphere S^2 and M_2 is the Riemann surface Σ_g of genus $g\geqq 2$. Assume the case 2° occurs. First we show that $f(M_1)$ is not equal to M_2. If $f(M_1)=M_2$, then $f|M_1$ is a covering map from S^2 to Σ_g. But it is impossible. So $f(M_1)\supset M_1$, and it is easy to show $(f|M_1)^{-1}(M_1)=M_1$. Hence $f(M_1)=M_1$. But, since $M_1=S^2$, the degree of the covering map $f|M_1$ is equal to 1. This contradicts Definition 2, 3), ii).

In the case 1°, first, we consider the case (a): $\chi(M_1)=0$ and $\chi(M_2)=0$. Next we deal with the case (b): $\chi(M_1)\neq 0$ and $\chi(M_2)=0$.

In the case (a), we can consider two cases: i) $K_0\approx D^2$ and $K_1\approx K_2\approx T^2-\mathring{D}^2$. ii) $K_0\approx T^2-\mathring{D}^2$ and $K_1\approx K_2\approx D^2$. We show that the case i) cannot occur. Assume the case i) occurs. As $(f|M_i)^{-1}(K_0)$, for $i=1$ or 2, are mutually disjoint disks embedded in M_i, $M_i\backslash(f|M_i)^{-1}(K_0)$ is connected. So we have that $f^2(M_1)=M_1$ and $f^2(M_2)=M_2$, because f is surjective. Hence $f^2(K_0)\subset K_0$. This contradicts Definition 2, 3), ii). In the case ii) K becomes T^*.

In the case (b), set $f_*m_2=\gamma m_1+\delta m_2$. Then, by the same calculation as above, we have $\chi(M_2)=\gamma\chi(M_1)+\delta\chi(M_2)$. As $\chi(M_1)\neq 0$ and $\chi(M_2)=0$, we know $\gamma=0$. Hence $f_*m_2=\delta m_2$, and this means that $f(M_2)=M_2$. If, for $x\in \mathring{K}_1$, $f(x)\in M_2$, then x is not a nonwandering point, because $f^n(f(x))\in M_2$ for any integer $n\geqq 1$. Hence $f(\mathring{K}_1)\subset \mathring{K}_1$. Moreover, by the equality (1), we know $\alpha_n=1$. So $f^{2n}|\mathring{K}_1$ is injective. This contradicts Definition 2, 3), ii), and so, in the case (b), we have no branched surface which admits expanding immersions. This completes the proof.

§ 3. Proof of Theorem (2).

In this section, we consider the case when the number of connected components of $K\backslash S$ is two. In this case, there are three types of branched surfaces, two of which have untwisted branch sets and one of which has a twisted branch set.

First we consider branched surfaces K which have untwisted branch sets. Let $\mathring{K}_1$ and $\mathring{K}_2$ be connected components of $K\backslash S$. Two types of them are as follows: 1) $\mathring{K}_1\supset J_1^+$ and J_2^+, and $\mathring{K}_2\supset J^-$, 2) $\mathring{K}_1\supset J_1^+$ and J^-, and $\mathring{K}_2\supset J_2^+$.

In the case 1), we show that only T_* admits expanding immersions. Set $K_1=\mathring{K}_1^{\wedge}$ and $K_2=\mathring{K}_2^{\wedge}$. We connect K_1 with two copies of K_2 by identifying their boundaries naturally, and denote the obtained space by M. Then M has a differentiable structure such that the natural projection $\pi\colon M\to K$ becomes an immersion. We construct a lift $\tilde{f}\colon M\to M$ of $f\colon K\to K$ as follows. For $x\in M-\pi^{-1}\circ f^{-1}(K_2)$, set $\tilde{f}(x)=\pi^{-1}\circ f\circ\pi(x)$. For each connected component $\tilde{K}$ of $\pi^{-1}\circ f^{-1}(K_2)$, we take a sufficiently small neighborhood $\tilde{L}$ of $\tilde{K}$. Then $f\circ\pi(\tilde{L})$ is uniquely lifted to M so as to be continuously connected with the image of $M-\pi^{-1}\circ f^{-1}(K_2)$. It is clear that $\tilde{f}$ is an immersion, and then $\tilde{f}: M\to M$ is a covering map whose degree is greater than 2. Hence we conclude that M is a torus, and $K_2\approx D^2$ and $K_1\approx T^2-(D^2\amalg D^2)$. By Definition 2, 3), iii), two copies of K_2 in M have the same image for $\tilde{f}$. Then K is obtained from M by identifying two copies of K_2 by an orientation preserving C^1 diffeomorphism. It follows that $K\approx T_*$.

Next we show that in the case 2) there exists no branched surface which admits expanding immersions. Assume K admits an expanding immersion f, and we will deduce a contradiction. Set $M=K\backslash\mathring{K}_2$. Then M is a manifold. Remark that $\mathring{K}_1$ is orientable, but M is not necessarily orientable.

LEMMA 2. *$f(M)$ is equal to M.*

PROOF. First in the case when M is orientable, we show the lemma. We know easily that $H_2(K;\boldsymbol{Z})\cong\boldsymbol{Z}$ and it is generated by $m=\iota_*[M]$, where ι_* is the induced homomorphism of the inclusion $\iota: M\to K$, and $[M]$ is the fundamental homology class of M. Here we assume that $f(M)\neq M$. Then $f(M)=K$. Take $x\in\mathring{K}_2$, and consider the following commutative diagram:

$$\begin{array}{ccc} H_2(M;\boldsymbol{Z}) & \xrightarrow{\quad f_* \quad} & H_2(K;\boldsymbol{Z}) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ H_2(M, M\backslash f^{-1}(x);\boldsymbol{Z}) & \xrightarrow{\bar{f}_*} & H_2(K, K-\{x\};\boldsymbol{Z}) \end{array}$$

First we have $q\circ f_*(m)=0$. Remark that we can define an orientation on $\mathring{K}_2$ compatible with the orientation of M, and that f is orientation preserving or reversing. Then $\bar{f}_*\circ p(m)=\pm(\sharp f^{-1}(x)\cap M)\cdot O_x$, where O_x is a generator of $H_2(K, K-\{x\};\boldsymbol{Z})$. By the assumption, $\sharp f^{-1}(x)\cap M\neq 0$. This is a contradiction. Hence $f(M)=M$.

Next we assume M is nonorientable. We take the orientation covering of K, $\pi:\tilde{K}\to K$. We can construct it in the same way as for ordinary manifolds. We take a lift $\tilde{f}:\tilde{K}\to\tilde{K}$ of f. Notice that $\tilde{K}$ is a branched surface whose tangent bundle is orientable and $\tilde{f}$ can be taken as an orientation preserving immersion satisfying $\sigma\circ\tilde{f}\circ\sigma=\tilde{f}$, where σ is the nontrivial covering transformation of $\pi:\tilde{K}\to K$. Let $\tilde{M}=\pi^{-1}(M)$. Then $\tilde{M}$ is an orientable manifold.

We know $H_2(\tilde{K};\boldsymbol{Z})\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$, and we take a pair of generators as follows. We take submanifolds $K_1^{(1)}$ and $K_1^{(2)}$ in $\tilde{K}$ such that $\pi(K_1^{(1)})=$

$\pi(K_1^{(2)})=M$, $K_1^{(1)}\cup K_1^{(2)}=\tilde{M}$ and $K_1^{(1)}\cap K_1^{(2)}=\pi^{-1}(S)$, and take submanifolds $K_2^{(1)}$ and $K_2^{(2)}$ such that $\pi(K_2^{(1)})=\pi(K_2^{(2)})=K_2$. Set $L_1=K_1^{(1)}\cup K_2^{(1)}\cup K_2^{(2)}$ and $L_2=K_1^{(2)}\cup K_2^{(1)}\cup K_2^{(2)}$. We choose a pair of generators l_1 and l_2 of $H_2(L_1; \boldsymbol{Z})$ and $H_2(L_2; \boldsymbol{Z})$ such that $\bar{l}_1+\bar{l}_2=\tilde{m}$, where $\bar{l}_1=(\iota_1)_*l_1$, $\bar{l}_2=(\iota_2)_*l_2$ and $\tilde{m}=\iota_*[\tilde{M}]$, and $\iota_1\colon L_1\to\tilde{K}$, $\iota_2\colon L_2\to\tilde{K}$ and $\iota\colon \tilde{M}\to\tilde{K}$ are inclusions. Then $\bar{l}_1$ and $\bar{l}_2$ are generators of $H_2(\tilde{K}; \boldsymbol{Z})$. Let $\tilde{f}_*\bar{l}_1=\alpha\cdot\bar{l}_1+\beta\cdot\bar{l}_2$ and $\tilde{f}_*\bar{l}_2=\gamma\cdot\bar{l}_1+\delta\cdot\bar{l}_2$. Since $\sigma\circ\tilde{f}\circ\sigma=\tilde{f}$, we have $\alpha=\delta$ and $\beta=\gamma$. Then $\tilde{f}_*\tilde{m}=\tilde{f}_*\bar{l}_1+\tilde{f}_*\bar{l}_2=(\alpha+\beta)\cdot(\bar{l}_1+\bar{l}_2)=(\alpha+\beta)\cdot\tilde{m}$. Hence in the same way as the above case, we obtain that $\tilde{f}(\tilde{M})=\tilde{M}$, and $f(M)=M$. □

By Definition 2, 3), iii), for some positive integer $\bar{n}$ and $x\in\mathring{K}_2$ sufficiently near S, there exists $y\in M$ such that $f^{\bar{n}}(x)=f^{\bar{n}}(y)$. Since $f(M)\subset M$ by Lemma 2, for any positive integer m, $f^{\bar{n}+m}(x)=f^{\bar{n}+m}(y)\in M$. This contradicts Definition 2, 3), iv).

Finally we consider the last type, each of which has a twisted branch set. We also assume that K admits an expanding immersion f. Let $\mathring{K}_1$ and $\mathring{K}_2$ be connected components of $K\backslash S$ such that $\mathring{K}_1\supset J^+$ and $\mathring{K}_2\supset J^-$, and let K_1^N and K_2^N be connected components of $K\backslash\mathring{N}$ such that $K_1^N\subset\mathring{K}_1$ and $K_2^N\subset\mathring{K}_2$, where N is a neighborhood of S homeomorphic to N_1. Easily we have $H_2(K; \boldsymbol{Z})\cong\boldsymbol{Z}$, and denote a generator by $[K]$.

LEMMA 3. *Set $f^{2n}[K]=\alpha_n\cdot[K]$. Then as n becomes large, α_n becomes large.*

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow H_2(K; \boldsymbol{Z}) & \xrightarrow{p} & H_2(K, S; \boldsymbol{Z}) & \xrightarrow{\qquad\partial\qquad} & H_1(S; \boldsymbol{Z}) & \longrightarrow & H_1(K; \boldsymbol{Z}) \\
& & \cong\uparrow r_2 & & \cong\uparrow r_1 & & \\
& & H_2(K, N; \boldsymbol{Z}) & & H_1(N; \boldsymbol{Z}) & & \\
& & \uparrow \iota_2 & & \uparrow \iota_1 & & \\
& & H_2(K_1^N, \partial K_1^N; \boldsymbol{Z})\oplus H_2(K_2^N, \partial K_2^N; \boldsymbol{Z}) & \xrightarrow{\partial_1\oplus\partial_2} & H_1(\partial K_1^N; \boldsymbol{Z})\oplus H_1(\partial K_2^N; \boldsymbol{Z}) & &
\end{array}$$

Take fundamental homology classes $[K_1^N, \partial K_1^N]$ and $[K_2^N, \partial K_2^N]$ of K_1^N and K_2^N such that they induce the same orientation on K induced by $[K]$. Moreover let $[S]$ be a generator of $H_1(S; \boldsymbol{Z})$ such that $[S]=r_1\circ\iota_1\circ\partial_2[K_2^N, \partial K_2^N]$. Since $r_1\circ\iota_1\circ\partial_1\oplus\partial_2([K_1^N, \partial K_1^N]+2[K_2^N, \partial K_2^N])=-2[S]+2[S]=0$, we have $\partial\circ r_2\circ\iota_2([K_1^N, \partial K_1^N]+2[K_2^N, \partial K_2^N])=0$. Hence, $p[K]=r_2\circ\iota_2([K_1^N, \partial K_1^N]+2[K_2^N, \partial K_2^N])$.

For $x\in\mathring{K}_1$ such that $f^{-2n}(x)\cap S=\emptyset$, set $\{y_i^1\}_{i=1}^{k(1)}=f^{-2n}(x)\cap\mathring{K}_1$ and $\{y_j^2\}_{j=1}^{k(2)}=f^{-2n}(x)\cap\mathring{K}_2$. We consider the commutative diagram:

$$\begin{array}{ccc} H_2(K;\boldsymbol{Z}) & \xrightarrow{\qquad\qquad (f^{2n})_* \qquad\qquad} & H_2(K;\boldsymbol{Z}) \\ \downarrow p_1 & & \downarrow p_2 \\ \left(\bigoplus_{i=1}^{k(1)} H_2(K_1, K_1-\{x_i^1\};\boldsymbol{Z})\right)\oplus\left(\bigoplus_{j=1}^{k(2)} H_2(K_2, K_2-\{x_j^2\};\boldsymbol{Z})\right) & \xrightarrow{(\overline{f^{2n}})_*} & H_2(K, K-\{x\};\boldsymbol{Z}) \end{array}$$

Then

$$(\overline{f^{2n}})_*\circ p_1[K]=(\overline{f^{2n}})_*\left(\sum_{i=1}^{k(1)} O_i^1+2\cdot\sum_{j=1}^{k(2)} O_j^2\right)=(k(1)+2\cdot k(2))\cdot O_x\,,$$

since $p[K]=r_2\circ\iota_2([K_1^N, \partial K_1^N]+2[K_2^N, \partial K_2^N])$, where O_i^1 and O_j^2 denote generators of $H_2(K_1, K_1-\{x_i^1\};\boldsymbol{Z})$ and $H_2(K_2, K_2-\{x_j^2\};\boldsymbol{Z})$ respectively, and O_x denotes a generator of $H_2(K, K-\{x\};\boldsymbol{Z})$. On the other hand, $p_2\circ(f^{2n})_*[K]=\alpha_n\cdot O_x$. Hence we have $\alpha_n=k(1)+2k(2)\geqq\#f^{-2n}(x)$. By Definition 2, 3), ii), the right-hand side of the above inequality becomes large as n becomes large. So we complete the proof. $\square$

We calculate the Kronecker product of $[K]$ and $e(K)$. First, since $(f^{2n})^*e(K)=e(K)$, $\langle e(K), [K]\rangle=\langle (f^{2n})^*e(K), [K]\rangle=\langle e(K), (f^{2n})_*[K]\rangle=\alpha_n\langle e(K), [K]\rangle$. By Lemma 3, we have $\langle e(K), [K]\rangle=0$. On the other hand, in the same way as the proof of the index theorem $\langle e[M], [M]\rangle=\chi(M)$ for an ordinary manifold M, we calculate $\langle e(K), [K]\rangle$ by using a vector field X with finite singularities such that the indices of $X|\mathring{K}_1$ and $X|\mathring{K}_2$ are equal to $\chi(K_1)$ and $\chi(K_2)$. As $p[K]=r_2\circ\iota_2([K_1, \partial K_1]+2[K_2, \partial K_2])$, we have $\langle e(K), [K]\rangle=\chi(K_1)+2\chi(K_2)$. Hence $\chi(K_1)+2\chi(K_2)$ must be zero, but $\chi(K_1)$ is odd since K_1 has one boundary circle. It follows that in this case we have no branched surface which admits expanding immersions.

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