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Compact Homomorphisms on Algebras of Continuous Functions

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Introduction.

The purpose of this note is to study compact and weakly compact homomorphisms between algebras of continuous functions. For a completely regular Hausdorff space S, we denote by $C(S)$ the algebra of all complex-valued continuous functions on S endowed with its compact-open topology. M. Lindström and J. Llavana [4] gave characterizations of compact and weakly compact homomorphisms from $C(S)$ to $C(T)$, where T and S are completely regular Hausdorff spaces. Let A and B be closed subalgebras of $C(S)$ and $C(T)$ respectively. Here we study compact and weakly compact homomorphisms φ from A to B.

After some preliminaries in §1, we introduce in §2 closed subalgebras of some type which are called function algebras induced by uniform algebras. These subalgebras contain $C(S)$ and algebras of analytic functions. We discuss in §2 compactness and weak compactness of φ in the case A is a function algebra induced by a uniform algebra and φ is a composition operator. We give conditions under φ is compact or weakly compact and establish the relationship between compactness and weak compactness of φ.

§1. Preliminaries.

For a completely regular Hausdorff space X, we denote by $C(X)$ the algebra of all complex-valued continuous functions on X endowed with its compact-open topology. Throughout this note we let S and T denote completely regular Hausdorff spaces.

Let A and B be subalgebras of $C(S)$ and $C(T)$ respectively. Then we easily have the following (cf. [6], [8]).

(a) Let φ be a continuous linear operator from A to B. Then there is a continuous mapping τ from T to the dual space A' of A with respect to the w^*-topology $\sigma(A', A)$ such that

$$(*) \qquad [\varphi(f)](y)=\tau(y)(f), \qquad f\in A \text{ and } y\in T.$$

(b) Let φ be a continuous homomorphism from A to B. Then there is a continuous

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mapping τ from T to A' with respect to the topology $\sigma(A', A)$ such that $(*)$ is satisfied and $\tau(y)$ is a continuous homomorphism from A to the complex field $\boldsymbol{C}$ for any $y \in T$.

Let X be a completely regular Hausdorff space. We say that A is a *function algebra* on X if it is a closed subalgebra of $C(X)$ separating points in X and containing constant functions.

Now let A and B be function algebras on S and T respectively and let φ be a continuous homomorphism from A to B. The homomorphism φ is called a *composition operator* if there is a continuous mapping θ from T to S such that

$$[\varphi(f)](y) = f(\theta(y)), \qquad f \in A \text{ and } y \in T.$$

Let A be a function algebra on S. We consider the following conditions on A:

(1) For any non-trivial continuous homomorphism ψ from A to $\boldsymbol{C}$, there is an $\alpha \in S$ such that $\psi(f) = f(\alpha)$ for any $f \in A$.

(2) For any $x_0 \in S$ and any open neighborhood V of x_0, we can find a finite number of functions $f_1, f_2, \cdots, f_n$ in A and $\delta > 0$ such that $f_i(x_0) = 0$ $(i = 1, 2, \cdots, n)$ and $\bigcap_{i=1}^{n} \{x \in S : |f_i(x)| < \delta\} \subset V$.

The conditions (1) and (2) guarantee that any continuous homomorphism φ from A to B with $\varphi(1) = 1$ is a composition operator. We easily see the following.

LEMMA 1.1. *Let A and B be function algerbas on completely regular Hausdorff spaces S and T respectively. Suppose that A satisfies conditions* (1) *and* (2). *If φ is a continuous homomorphism from A to B with $\varphi(1) = 1$, then φ is a composition operator.*

PROOF. For any $y \in T$, $\tau(y)$ in $(*)$ is a non-trivial continuous homomorphism from A to $\boldsymbol{C}$ by (b). Since A satisfies condition (1), there is an $\alpha \in S$ such that $\tau(y)(f) = f(\alpha)$ for any $f \in A$. This α is uniquely determined since A separates points of S. If we put $\theta(y) = \alpha$, θ is a mapping from T to S and $[\varphi(f)](y) = f(\theta(y))$ for $f \in A$ and $y \in T$. It remains only to show that θ is continuous. If $y_\lambda \to y$ in T, then $\tau(y_\lambda) \to \tau(y)$ in $\sigma(A', A)$. So $f(\theta(y_\lambda)) = \tau(y_\lambda)(f) \to \tau(y)(f) = f(\theta(y))$ for $f \in A$. From condition (2), for any open neighborhood V of $\theta(y)$, there are $f_1, f_2, \cdots, f_n \in A$ and $\delta > 0$ such that $f_i(\theta(y)) = 0$ $(i = 1, 2, \cdots, n)$ and $\bigcap_{i=1}^{n} \{x \in S : |f_i(x)| < \delta\} \subset V$. Since $f_i(\theta(y)) = 0$, $|f_i(\theta(y_\lambda))| < \delta$ for any $\lambda \geq \lambda_0$ for some λ_0 $(i = 1, 2, \cdots, n)$. Hence $\theta(y_\lambda) \in V$ for $\lambda \geq \lambda_0$ and θ is continuous.

EXAMPLES. (i) Let S be a completely regular Hausdorff space. Then $A = C(S)$ satisfies condition (1) (cf. [3]). A evidently satisfies condition (2).

(ii) Let G be an open subset in $\boldsymbol{C}$ and let $A = H(G)$ be the algebra of all analytic functions on G. Then $H(G)$ satisfies conditions (1) and (2).

(iii) Let G be an open polydisc in $\boldsymbol{C}^2$ and let $H(G)$ be the algebra of all analytic functions on G. Then $H(G)$ has condition (1). The algebra $H(G)$ also satisfies condition (2).

(iv) Let $X = D \times J$, where $D = \{z \in \boldsymbol{C} : |z| < 1\}$ and $J = (0, 1)$. We put $A = \{f \in C(X)$: f can be approximated uniformly by polynomials in z and t on any compact subset of $X\}$. Let φ be a non-trivial continuous homomorphism from A to $\boldsymbol{C}$. If $\varphi(z) = \alpha$ and

$\varphi(t)=\beta$, then β is a real number and $(\alpha, \beta)\in X$, where z, t are the coordinate functions. This shows that A satisfies condition (1). One sees that A satisfies condition (2).

Let S be a completely regular Hausdorff space. We say that S is a k_R*-space* if a complex-valued function f on S is continuous whenever $f|F$ is continuous on F for any compact subset F of S. The space $C(S)$ is complete if and only if S is a k_R-space (cf. [9]). A subset H of $C(S)$ is called *equicontinuous* if for any $x\in S$ and any $\varepsilon>0$, there is an open neighborhood V of x such that $|f(y)-f(x)|<\varepsilon$ for any $f\in H$ and any $y\in V$.

In order to give a characterization of compact homomorphisms from A to B we will need the following compactness criteria of Arzela-Ascoli type (cf. [4]).

LEMMA 1.2. *Let S be a completely regular Hausdorff k_R-space. Then a subset H of $C(S)$ is relatively compact if and only if* (i) *H is equicontinuous on S and* (ii) *$\{f(x): f\in H\}$ is bounded in $\mathbf{C}$ for any $x\in S$.*

For given two locally convex spaces E and F, we call a continuous linear operator φ from E to F is *compact* (resp. *weakly compact*) if it maps bounded subsets of E to relatively compact (resp. relatively weakly compact) subsets of F.

Let A and B be function algebras on S and T respectively, and let φ be a composition operator from A to B, that is, φ is of the form $\varphi(f)=f\circ\theta$, where θ is a continuous mapping from T to S. It is not hard to see the following.

LEMMA 1.3. *Let S and T be completely regular Hausdorff spaces and assume that T is a k_R-space. Let A and B be function algebras on S and T respectively. Then a composition operator φ from A to B is compact if and only if for any $y\in T$ any net $y_\alpha\to y$ in T implies that $\sup_{f\in F}|f(\theta(y_\alpha))-f(\theta(y))|\to 0$ for any bounded set F in A.*

PROOF. M $(\subset B)$ is equicontinuous on T if and only if for any y in T, any net $y_\alpha\to y$ in T implies that $\sup_{g\in M}|g(y_\alpha)-g(y)|\to 0$. Hence the lemma is clear by Lemma 1.2 since B is closed in $C(T)$.

§2. Compact and weakly compact composition operators.

Let A_0 be a uniform algebra on the maximal ideal space M_{A_0} of A_0. Let $\mathfrak{P}_0$ be the family of all Gleason parts for A_0 and let S be the union of members of a subfamily $\mathfrak{P}$ of $\mathfrak{P}_0$, that is, $S=\bigcup_{P\in\mathfrak{P}}P$. Here S is a completely regular Hausdorff space as a subspace of M_{A_0}. We set $A=\{f\in C(S): f$ can be approximated uniformly on F by functions in $A_0|S$ for any compact subset F of $S\}$. Such an A is called the *function algebra on S induced by A_0*. It is the smallest closed subalgebra in $C(S)$ containing $A_0|S$. We say that P is a *non-trivial part* for A if it is a non-trivial Gleason part for A_0 which is in $\mathfrak{P}$, and P is a *trivial part* for A if it is a trivial Gleason part for A_0 and $P\in\mathfrak{P}$.

EXAMPLES. (i) Let S be a completely regular Hausdorff space and let βS be the Čech compactification of S. Then $C(S)$ is the function algebra on S induced by the uniform algebra $C(\beta S)$. Any part for $C(S)$ is trivial.

(ii) Let $D=\{z\in \boldsymbol{C}: |z|<1\}$ and let A_0 be the disc algebra on $\bar{D}$. Then $A=H(D)$ is the function algebra on D induced by A_0 and D is the unique part for A.

(iii) Let $G=\{z=(z_1, z_2)\in \boldsymbol{C}^2 : |z_1|<1, |z_2|<1\}$ and let A_0 be the polydisc algebra on $\bar{G}$. Then $A=H(G)$ is the function algebra on G induced by A_0 and G is the unique part for A.

(iv) Let $X=D\times J$, where $D=\{z\in \boldsymbol{C}: |z|<1\}$ and $J=(0, 1)$. Let A_0 be the cylinder algebra, that is, $A_0=\{f\in C(\bar{D}\times[0, 1]) : f$ can be approximated uniformly by polynomials in z and t on $\bar{D}\times[0, 1]\}$. Then A on $D\times J$ of Example (iv) in §1 is the function algebra on X induced by A_0 and any part for A is of the form $D\times\{a\}$ for $a\in J$.

Let A be the function algebra on S induced by a uniform algebra A_0 and let B be a function algebra on T. We consider a continuous homomorphism φ from A to B. Here we confine our attention to the case φ is a composition operator. We are now in a position to discuss compactness of φ. First we have the following.

THEOREM 2.1. *Let A_0 be a uniform algebra on M_{A_0} and let A be the function algebra on S induced by A_0. Let B be a function algebra on a completely regular Hausdorff k_R-space T. Let φ be a compact composition operator from A to B. Then for any $y\in T$, there is an open neighborhood U of y such that $\theta(U)$ is contained in a part for A.*

PROOF. Suppose that there is a $y_0\in T$ such that $\theta(U)$ is not contained in the part P for A containing $\theta(y_0)$ for any open neighborhood U of y_0. Let now I be any fixed open neighborhood base at y_0 in T. Then the order relation $U\leq V$ if and only if $U\subset V$ directs I. If we take $y_U\in U$ such that $\theta(y_U)\notin P$ for any $U\in I$, then $\theta(y_U)$ is in a different part P_1 for A from P. Hence $\sup\{|f(\theta(y_U))-f(\theta(y_0))| : f\in A_0, \|f\|\leq 1\}=2$, and so for any $U\in I$, there is an $f_U\in A_0$, $\|f_U\|\leq 1$ such that $1\leq|f_U(\theta(y_U))-f_U(\theta(y_0))|$. Since $A_0|S\subset A$, $F=\{f_U|S : U\in I\}$ is a bounded set in A. From this,

$$1\leq|f_U(\theta(y_U))-f_U(\theta(y_0))|\leq\sup_{g\in F}|g(\theta(y_U))-g(\theta(y_0))|$$

for any $U\in I$. Since φ is compact and $y_U\to y_0$, this is a contradiction from Lemma 1.3.

REMARK. In Theorem 2.1, if T is connected, there is a uniquely determined part P for A which is independent of y in T.

Next we will give the converse of Theorem 2.1 under a suitable assumption. We begin with explanation of the condition that A has some analytic property. Let P be a non-trivial part for the function algebra A induced by a uniform algebra. We consider the following condition:

(α) For any $x\in P$, there are an open neighborhood V of x in P and a homeomorphism ρ from the open unit polydisc D^n (disc if $n=1$, n depends on x) onto V such that $f\circ\rho$ is analytic on D^n for all $f\in A$.

If any non-trivial part P for A satisfies this condition, then we say A has (α). This property (α) can be found in [6].

REMARK. All of function algebras $C(S)$, $H(D)$, $H(G)$ and A on $D\times J$ of Examples in §2 have (α).

Here we wish to show the converse of Theorem 2.1.

THEOREM 2.2. *Suppose that the function algebra A on S induced by a uniform algebra A_0 has (α) and S is a locally compact Hausdorff space. Let φ be a continuous composition operator from A to a function algebra B on a completely regular Hausdorff k_R-space T. If for any y in T there is an open neighborhood U of y such that $\theta(U)$ is contained in a part for A, then φ is compact.*

PROOF. From Lemma 1.3, in order to prove the theorem, it suffices to show that if $y_\alpha\to y$ in T, then $\sup_{f\in F}|f(\theta(y_\alpha))-f(\theta(y))|\to 0$ for any bounded set F in A. By the hypothesis, for any $y\in T$, there is an open neighborhood U of y such that $\theta(U)$ is contained in a part P for A. Let $y_\alpha\to y$ in T and let F be a bounded set in A. If $\{\theta(y)\}$ is a trivial part for A, there is an open neighborhood U of y such that $\theta(U)=\{\theta(y)\}$. It is simple to check that $\sup_{f\in F}|f(\theta(y_\alpha))-f(\theta(y))|\to 0$ as $y_\alpha\to y$. Next, let the part P containing $\theta(y)$ be non-trivial. From (α), there is a homeomorphism ρ from D^n onto an open neighborhood V of $\theta(y)$ in P such that $f\circ\rho$ is analytic on D^n for any $f\in A$. Since S is locally compact, there is an open neighborhood W of $\rho^{-1}(\theta(y))$ in D^n such that $\rho(W)$ is an open neighborhood of $\theta(y)$ in P and the closure $\overline{\rho(W)}$ of $\rho(W)$ in S is compact. Since F is a bounded set in A, $\sup_{f\in F}\|f\|_{\overline{\rho(W)}}<\infty$. Since $y_\alpha\to y$, $\theta(y_\alpha)\to\theta(y)$ in S. It implies that for some α_0 $\theta(y_\alpha)\in\rho(W)$ for every $\alpha\geq\alpha_0$ since $\theta(U)\subset P$. Here for any $\varepsilon>0$ there is an open neighborhood W_1 ($W_1\subset W$) of $\rho^{-1}(\theta(y))$ in D^n such that for any $z\in W_1$ and any $f\in F$

$$|f(\rho(z))-f(\rho(\rho^{-1}(\theta(y))))|<\varepsilon,$$

since $f\circ\rho$ is analytic on W for any $f\in F$ and $\sup_{f\in F}\|f\|_{\overline{\rho(W)}}<\infty$. Now since $\theta(y_\alpha)\in\rho(W_1)$ for $\alpha\geq$ some α_1, $\theta(y_\alpha)=\rho(z_\alpha)$ for a $z_\alpha\in W_1$ ($\alpha\geq\alpha_1$). Hence

$$\sup_{f\in F}|f(\theta(y_\alpha))-f(\theta(y))|=\sup_{f\in F}|f(\rho(z_\alpha))-f(\rho(\rho^{-1}(\theta(y))))|<\varepsilon$$

for $\alpha\geq\alpha_1$. The proof is completed.

REMARKS. (1) When $A=C(S)$, the hypotheses in Theorem 2.2 that S is locally compact and A has (α) are unnecessary since any part for A is trivial. Hence if $A=C(S)$, we have the following by Theorems 2.1 and 2.2: Assume that S and T are completely regular Hausdorff spaces and T is a k_R-space. A continuous homomorphism φ from $C(S)$ to $C(T)$ is compact if and only if θ is locally constant. This was given in [4, Proposition 3].

(2) The identity mapping φ from $H(D)$ to itself is compact by Theorem 2.2. This

also can be seen in [5]. On the other hand, if A is the function algebra on $S=D\cup\{1\}$ induced by the disc algebra on $\bar{D}$, then the identity mapping from A to itself is not compact by Theorem 2.1.

Our next aim is to find out conditions for a weakly compact composition operator to be compact. We begin with the following theorem. It is proved by using a way similar to the argument in [7, Theorem 1].

THEOREM 2.3. *Let A be the function algebra on S induced by a uniform algebra A_0 and let B be a function algebra on a locally compact Hausdorff space T. Let φ be a weakly compact composition operator from A to B. Then for any $y\in T$, there is an open neighborhood U of y such that $\theta(U)$ is contained in a part for A.*

PROOF. For any $y_0\in T$, let P_0 be the part for A containing $\theta(y_0)$. Put $W=\{z\in T: \theta(z)\in P_0\}$. In order to prove the theorem, it suffices to show that W is a neighborhood of y_0. Assume the contrary. Then W does not contain any open neighborhood of y_0. Since T is locally compact, we can choose an open neighborhood V_1 of y_0 such that $\bar{V}_1$ is compact. Put $\varepsilon_n=1/(n+2)^2$ for any n. Now if we take a y_1 in $V_1\backslash W$, then $\theta(y_1)$ is in a different part for A from P_0. Hence there is an $f_1\in A_0$, $\|f_1\|\le 1$ such that $[\theta(y_1)](f_1)=0$, $[\theta(y_0)](f_1)>1-\varepsilon_1$. Put $V_2=\{z\in V_1 : |[\theta(z)](f_1)|>1-\varepsilon_1\}$. Then V_2 is an open neighborhood of y_0. Take $y_2\in V_2\backslash W$. Then $\theta(y_2)$ is in a different part for A from P_0. So there is a $g_2\in A_0$, $\|g_2\|\le 1$ such that $[\theta(y_2)](g_2)=0$, $[\theta(y_0)](g_2)>1-\varepsilon_2$. Set $f_2=f_1g_2$. Then $[\theta(y_1)](f_2)=[\theta(y_2)](f_2)=0$, $[\theta(y_0)](f_2)>(1-\varepsilon_1)(1-\varepsilon_2)$ and $|[\theta(y_2)](f_1)|>1-\varepsilon_1$. Continuing this process, we obtain a sequence $\{y_n\}$ in V_1 and a sequence $\{f_n\}$ in A_0 such that $\|f_n\|\le 1$ for any n, and

$$\begin{aligned} &[\theta(y_m)](f_n)=0 \qquad (1\le m\le n) \\ (1)\qquad &[\theta(y_0)](f_n)>(1-\varepsilon_1)(1-\varepsilon_2)\cdots(1-\varepsilon_n) \\ &|[\theta(y_n)](f_m)|>(1-\varepsilon_1)\cdots(1-\varepsilon_m) \qquad (1\le m<n). \end{aligned}$$

Since $\{y_n\}$ is contained in the compact set $\bar{V}_1$, there is a cluster point z_0 of $\{y_n\}$. Let $\{y_\alpha\}$ be a net converging to z_0 which consists of the members of $\{y_n\}$. By weak compactness of φ, the weak closure F of $\{\varphi(f_n|S)\}$ in B is a weakly compact subset in B, since $\{f_n|S\}$ is a bounded set in A. For any $g\in F$, $g(y_\alpha)\to g(z_0)$. Now put $\hat{y}_\alpha(g)=g(y_\alpha)$ and $\hat{z}_0(g)=g(z_0)$ for any $g\in F$. Then $\hat{y}_\alpha$ and $\hat{z}_0$ are continuous functions on F with respect to the weak topology $\sigma(B, B')$. Since $\hat{y}_\alpha(f)\to\hat{z}_0(f)$ for any $f\in F$ and F is a compact Hausdorff space with respect to $\sigma(B, B')$, $\hat{y}_\alpha$ converges quasi-uniformly to $\hat{z}_0$ on F (cf. [1, p. 268]). Hence for $0<\varepsilon<1/4$, and for any α_0, there are $\alpha_1, \alpha_2, \cdots, \alpha_m\ge\alpha_0$ such that

$$\min_{1\le i\le m} |\hat{y}_{\alpha_i}(g)-\hat{z}_0(g)|<\varepsilon$$

for any $g\in F$. In particular the inequality

$$(2) \qquad \min_{1 \le i \le m} |\varphi(f_n|S)(y_{\alpha_i}) - \varphi(f_n|S)(z_0)| < \varepsilon < 1/4$$

holds for any n. Put $N = \max\{n_1, n_2, \cdots, n_m\}$, where n_i is a positive integer such that $y_{\alpha_i} = y_{n_i}$. Then we have by (1)

$$\varphi(f_N|S)(y_{n_i}) = f_N(\theta(y_{n_i})) = [\theta(y_{n_i})](f_N) = 0 \qquad (i = 1, 2, \cdots, m) .$$

Take an $M > N$ such that

$$(3) \qquad |f_N(\theta(y_M)) - f_N(\theta(z_0))| < 1/6 .$$

Hence, from (1) and (3)

$$\begin{aligned} |f_N(\theta(z_0))| &> |f_N(\theta(y_M))| - 1/6 = |[\theta(y_M)](f_N)| - 1/6 \\ &> (1-\varepsilon_1)(1-\varepsilon_2)\cdots(1-\varepsilon_N) - 1/6 > 1/3 . \end{aligned}$$

It follows that,

$$|\varphi(f_N|S)(y_{n_i}) - \varphi(f_N|S)(z_0)| = |\varphi(f_N|S)(z_0)| = |f_N(\theta(z_0))| > 1/3$$

holds for every $1 \le i \le m$. This contradicts (2) and the proof is completed.

Thus, by Theorems 2.2 and 2.3, we can give conditions under any weakly compact composition operator from A to B is compact.

THEOREM 2.4. *Let A be the function algebra on S induced by a uniform algebra A_0. Let B be a function algebra on T. Assume that S and T are both locally compact Hausdorff spaces and A satisfies the condition* (α). *Then any weakly compact composition operator from A to B is compact.*

Finally we remark that if A is even a closed subalgebra in $C(S)$ containing $A_0|S$ in place of a function algebra on S induced by A_0, all of theorems in §2 hold true by defining that a part for A is a member of $\mathfrak{P}$ as before.

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