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Lissajous Curves as A'Campo Divides, Torus Knots and Their Fiber Surfaces

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Abstract. We present a new form of the fiber surface F for links arising from isolated complex plane curve singularities, in particular, the torus knot $T(p,q)$, where F is a smoothing of a long thin band which has as many clasp-singularities as the unknotting number of $T(p,q)$. As an application, we give a visual proof that the Lissajous curve $(\cos p\theta, \cos q\theta)$ regarded as a divide corresponds to the torus link $T(p,q)$.

1. Introduction.

The *divide* is a relative, generic immersion of a 1-manifold in a unit disk in $\mathbf{R}^2$. N. A'Campo formulated the way to associate to each divide a link in S^3. The class of links of divides properly contains the class of the links arising from isolated singularities of complex hypersurfaces. (See [1], [2] and [3].) The link $L(P)$ of the divide P is defined as follows:

$$L(P) = \{(u, v) \in TD | u \in P,\ v \in T_u P,\ |u|^2 + |v|^2 = 1\} \subset S^3$$

In [7], the second author gave a visualized method to draw the links of divides together with their fiber surfaces.

A *Lissajous curve* is an immersed curve in a disk in $\mathbf{R}^2$, described by $(x, y) = (\cos p\theta, \cos q\theta)$. We may regard it as a divide and call it *Lissajous divide* of type (p, q), denoting it by $\mathcal{L}(p, q)$.

The purpose of this note is to give a visual proof of the following fact.

THEOREM 1.1. *Any torus knot $T(p,q)$ is represented by the Lissajous curve $(\cos p\theta, \cos q\theta)$ regarded as a divide.*

In this case, (p, q) are assumed to be coprime. In Section 3, we deal with non-coprime cases.

There is another way to understand Theorem 1.1 from the view point of theory of real morsifications of complex plane curve singularities by using Chebyshev polynomials. (cf. [4, P. 117 Example (i)], and [6]). Our approach is via constructing the diagrams of links with fiber surfaces. In the course of it, we find a new form of the fiber surface for the torus

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knots: a smoothing of a long thin band where we can see the minimal sequence of unknotting operations (Remark 2.2). We will claim that such surfaces are always available for knots of isolated complex plane curve singularities (Remark 2.3). As another application, we can see the relationship between the Murasugi decomposition (cf. [9, Definition 4.2.1]) of the fiber and that of the Lissajous divide (Remark 2.4).

The straightening of a Lissajous curve as in Figure 1 (see Lemma 2.1) can be regarded as the natural projection of the half of the *'pillow case'* for the 2-bridge link $B(p, q)$. Then by the same argument in the proof of Theorem 1.1, we have the following:

THEOREM 1.2. *Suppose* (p, q) *is a coprime pair. The natural projection of a 'pillow case of slope* q/p*'* (*for the* 2-*bridge link* $B(p, q)$) *regarded as a divide gives the torus link* $T(2p, 2q)$.

2. Proof of Theorem 1.1.

In this paper, we deal with divides combinatrially as in [7] so that divides are not necessarily in a "unit"disk and may have an acute bent as in Figure 1. Since deformations of divides by isotopies of $\mathbf{R^2}$ do not change the isotopy class of the represented links, it is convenient to draw Lissajous divides with broken straight lines. Therefore, we introduce the notion of '*billiard divides*'. A *billiard divide* of type (p, q), denoted by $\mathcal{B}(p, q)$, is the trace of a ray in a square in $\mathbf{R^2}$, where the ray starts from the right-up corner with slope q/p and perfectly reflects whenever it hits the wall.

Figure 1 gives some examples of billiard divides and Lissajous divides, which we will identify by the following Lemma.

LEMMA 2.1. *The Lissajous divide* $\mathcal{L}(p, q)$ *and the billiard divide* $\mathcal{B}(p, q)$ *are isotopic to each other.*

PROOF. Set $F(t) = 1 - \frac{2}{\pi}\arccos(t)$. Then the map:

$$\begin{array}{rccc} Q = (F, F): & [-1, 1] \times [-1, 1] & \to & [-1, 1] \times [-1, 1], \\ & (x, y) & \mapsto & (F(x), F(y)) \end{array}$$

extends over a disk as a self-homeomorphism and maps any billiard divides to Lissajous divides of the same types. □

To prove Theorem 1.1, we first recall a standard construction of the fiber surface for the torus knot $T(p, q)$. Recall that $T(p, q)$ is isotopic to $T(q, p)$ and can be represented as a closed braid of q-strings expressed as $(\sigma_1\sigma_2\cdots\sigma_{q-1})^p$ by Artin's standard braid generators. Applying Seifert's algorithm to this closed braid diagram, we obtain a surface which is composed of q parallel disks such that each adjacent pair of disks are connected by p bands all half-twisted once in the same direction. Moreover, for each intermediate disk, the bands are attached alternatingly on one side and the other. Such Seifert surface is well-known to be the fiber surface of $T(p, q)$.

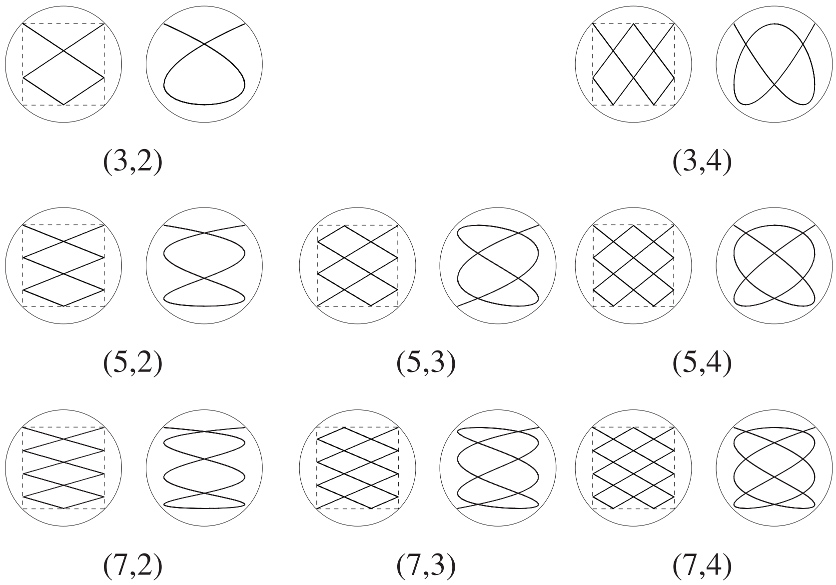


FIGURE 1. Billiard divides and Lissajous divides

We now explicitly construct a Seifert surface for the link $L(P)$ of $\mathcal{L}(p,q)$ and isotope it to the surface recalled in the above. Recall an algorithm in [7, Figure 2.2] to draw link diagrams of divides. In case of Lissajous divides, we may apply [7, Remark 2.1] (cf. Remark 2.3), so that we can construct diagrams and Seifert surfaces for them by the following steps 1–4. We note that Ishikawa [8], and Couture-Perron [5] have other ways of obtaining closed (quasi-positive) braids for links of divides and observe that some class of divides (containing Lissajous) yield positive braids.

Step 1: For a given billiard divide $\mathcal{B}(p,q)$, draw a partial link diagram at each of double points and maximal or minimal points as in Figure 2 (a)–(b). For other parts, naturally extend the diagram to obtain a diagram of the link $L(\mathcal{B}(p,q))$.

Step 2: Draw a long thin band spanned by $L(\mathcal{B}(p,q))$ with clasp singularities.

Step 3: Smooth each clasp singularities in the orientable way as in Figure 4 so that we obtain Figure 2 (c).

Step 4: Isotope the above-obtained as in Figure 2 (d).

See Figure 3 for example. The Seifert surface obtained in Figure 2 (d) is exactly the fiber surface of the torus knot $T(p,q)$. Theorem 1.1 is completed.

REMARK 2.2. Here is a relationship between the argument above and the unknotting number of $T(p,q)$, which is now well-known to be $\frac{(p-1)(q-1)}{2}$. (For example, this number

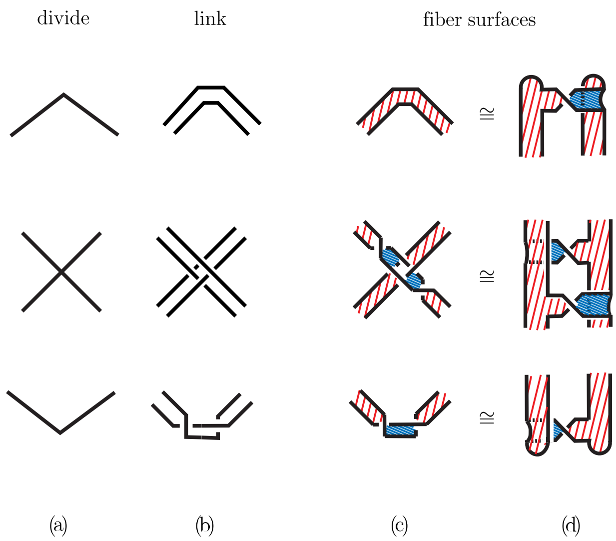


FIGURE 2.

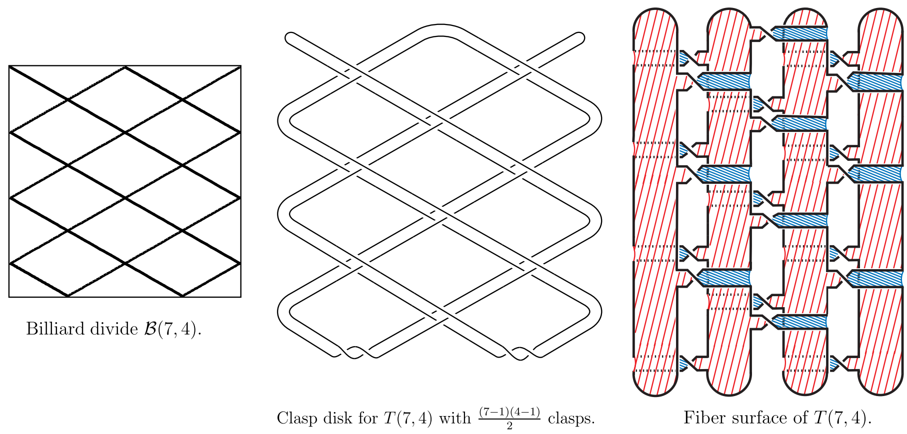


FIGURE 3.

coincides with the number of double points of the divide. See [2].) As in Figure 3, we can see that $T(p,q)$ spans a thin long band which has $\frac{(p-1)(q-1)}{2}$ self-intersections of clasp type. By smoothing the clasp singularities in the orientable way as in Figure 4, we obtain a Seifert surface which is isotopic to the fiber surface of $T(p,q)$. In this form, it is obvious that $\frac{(p-1)(q-1)}{2}$ crossing changes (each at the clasp singularity) are sufficient to trivialize $T(p,q)$.

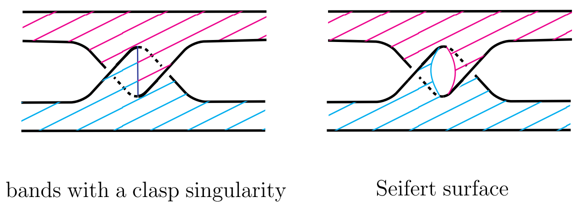


FIGURE 4.

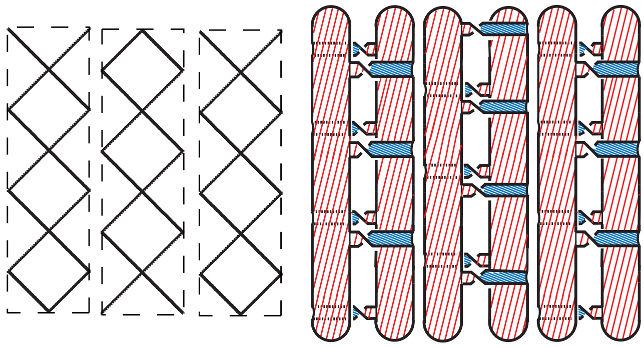


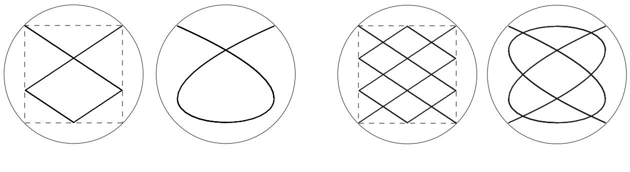
FIGURE 5.

REMARK 2.3. In [7, Remark 2.1], we pointed out that the above construction works for any divide P in the disk D that satisfies the following;

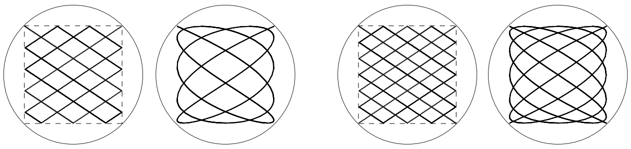
Any maximal point and any minimal point of P directly faces the boundary ∂D.

In [5], they defined *ordered Morse divide* and showed that any link arising from isolated complex plane curve singularities can be represented by an ordered Morse divide. The ordered Morse divides satisfy the above-quoted property, and hence we see that any such link has a Seifert surface obtained by smoothing long thin bands with clasp singularities. Moreover, we see that such a Seifert surface is a fiber surface for it is obtained from a finite number of Hopf bands by Murasugi sum.

REMARK 2.4. Another interesting point is the relationship between the Lissajous divide and a Murasugi decomposition of Seifert surfaces which are obtained in the proof of Theorem 1.1. For example, look at the middle two disks in Figure 3. They are Murasugi-disks, which give a Murasugi-decomposition of the fiber surface of $T(7,4)$ into three copies of fiber surfaces of $T(7,2)$. See Figure 5. In case of the torus knot $T(p,q)$ represented as the closed braid of q-strings, the fiber surface can be decomposed along the intermediate disks into $q-1$ copies of the fiber surface of $T(p,2)$. In billiard divides, we see this corresponds to breaking the divide to $q-1$ copies of billiard divides $\mathcal{B}(p,2)$.



(3,2) (6,4)



(9,6) (12,9)

FIGURE 6. Billiard divides and Lissajous divides of type $(3d, 2d)$.

3. Non-coprime case.

For a given non-coprime pair (p, q), let d be their greatest common divisor and p', q' be the coprime quotients by d, i.e., $p = dp', q = dq'$. Furthermore, we take integers a, b satisfying $bp' - aq' = 1$. Then as a generalization, we define Lissajous divide $\mathcal{L}(dp', dq')$ for non-coprime pair as the union of the images of the curves $\big(\cos(p'\theta + 2\pi \frac{a}{d}k), \cos(q'\theta + 2\pi \frac{b}{d}k)\big)$, for $k = 0, 1, \cdots, d-1$. We also define the billiard divide $\mathcal{B}(dp', dq')$ as the preimages of $\mathcal{L}(dp', dq')$ by the self-homeomorphism Q in Lemma 2.1. See Figure 6. The number of arcs, and loops in $\mathcal{B}(dp', dq')$ and the number $|L|$ of the components of the represented link are as in the following table: According to the parity of d, $\mathcal{B}(dp', dq')$ has one or two immersed arc(s) and $[(d-1)/2]$ loops, where $[\cdot]$ means the integer part. Note that, by the definition of links of divides, $|L| = \#\text{arcs} + \#\text{loops} \times 2$.

d	1	2	3	4	5	6	$\cdots$	$2m-1$	$2m$	$\cdots$
$\sharp$arcs	1	2	1	2	1	2	$\cdots$	1	2	$\cdots$
$\sharp$loops	0	0	1	1	2	2	$\cdots$	$m-1$	$m-1$	$\cdots$
$\|L\|$	1	2	3	4	5	6	$\cdots$	$2m-1$	$2m$	$\cdots$

Numbers of components of a divide and its link

By the same argument as in the proof of Theorem 1.1, we have the following:

THEOREM 3.1. *Any d-component torus link $T(dp', dq')$ is represented by the Lissajous divide of type (dp', dq').*

Note that Theorems 1.1 and 1.2 are special cases of Theorem 3.1, for $d = 1$ and 2.

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