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On Homogeneous Almost Kähler Einstein Manifolds of Negative Curvature

Wakako OBATA

Tohoku University

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Abstract. A homogeneous almost Kähler manifold M of negative curvature can be identified with a solvable Lie group G with a left invariant metric g and a left invariant almost complex structure J. We prove that if g is an Einstein metric and G is of Iwasawa type, then J is integrable so that M is Kähler, and hence is holomorphically isometric to a complex hyperbolic space of the same dimension.

1. Introduction

An almost Hermitian manifold $M = (M, g, J)$ is called homogeneous if the group of almost complex isometries acts transitively on M. If the fundamental 2-form Φ of M defined by $\Phi(X, Y) = g(X, JY)$ is closed, then we call M an almost Kähler manifold. The purpose of this paper is to study the geometry of homogeneous almost Kähler manifolds of negative curvature.

It is proved by Heintze [4] that if M is in particular a homogeneous Kähler manifold of negative curvature, then M is holomorphically isometric to a complex hyperbolic space $\mathbb{C}H^n$ of the same dimension. On the other hand, it has been known, for instance in [1], that there are many examples of homogeneous almost Kähler manifold which are not Kähler. However, in conjunction with the Goldberg conjecture [2], it seems plausible that a homogeneous almost Kähler Einstein manifold of negative curvature is necessarily Kähler, and hence is holomorphically isometric to a complex hyperbolic space.

By a result of Heintze [4] we know that a homogeneous almost Kähler manifold M of negative curvature can be identified with a connected solvable Lie group G with a left invariant metric $\langle\ ,\ \rangle$ and a left invariant almost complex structure J on G. Also, recall that a simply connected solvable Lie group G is said to be *of Iwasawa type* if its Lie algebra $\mathfrak{g}$ with inner product $\langle\ ,\ \rangle$ satisfies the conditions: (i) the orthogonal complement $\mathfrak{a}$ of $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ is abelian, (ii) for any $A \in \mathfrak{a}$, the adjoint representation $\operatorname{ad} A : \mathfrak{g} \to \mathfrak{g}$ is symmetric with respect to $\langle\ ,\ \rangle$, and (iii) for some $A_0 \in \mathfrak{a}$, $\operatorname{ad} A_0|\mathfrak{n} : \mathfrak{n} \to \mathfrak{n}$ is positive definite.

Then we can prove the following

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THEOREM 1. *Let* $(G, \langle\ ,\ \rangle, J)$ *be a homogeneous almost Kähler Einstein manifold of negative curvature. If* G *is of Iwasawa type, then* $(G, \langle\ ,\ \rangle, J)$ *is Kähler, and in fact is holomorphically isometric to a complex hyperbolic space* $(\mathbb{C}H^n, g_0, J_0)$.

It should be remarked that by a result of Heber [3, Theorem 4.10], there exists, under the asumption of Theorem 1, a simply connected solvable Lie group G' of Iwasawa type such that $(G', \langle\ ,\ \rangle)$ is isometric to $(G, \langle\ ,\ \rangle)$. However, we do not know in general if $(G', \langle\ ,\ \rangle, J)$ is to be almost Kähler.

2. Preliminaries

Let $M = (M, g)$ be a homogeneous Riemannian manifold of (strictly) negative curvature $K < 0$. Since M is simply connected ([5]) and admits a solvable Lie group of isometries acting simply transitively on M ([4]), we can identify (M, g) with a simply connected solvable Lie group G with a left invariant metric $\langle\ ,\ \rangle$. We denote by $\mathfrak{g}$ the Lie algebra of G defined by left invariant vector fields on G, and by $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ the derived algebra of $\mathfrak{g}$. Note that $\mathfrak{n}$ is nilpotent, since $\mathfrak{g}$ is solvable.

For $X, Y \in \mathfrak{g}$ the covariant derivative $\nabla_X Y \in \mathfrak{g}$ is given by

$$\begin{aligned} \nabla_X Y &= \frac{1}{2}[X, Y] + U(X, Y)\,, \\ U(X, Y) &= -\frac{1}{2}((\operatorname{ad} X)^* Y + (\operatorname{ad} Y)^* X)\,, \end{aligned} \tag{1}$$

where ad is the adjoint representation of $\mathfrak{g}$ and * denotes transpose with respect to $\langle\ ,\ \rangle$. As a result, the curvature tensor $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ is determined by the bracket product, so that we have

$$\begin{aligned} \langle R(X, Y)Y, X\rangle =& \|U(X, Y)\|^2 - \langle U(X, X), U(Y, Y)\rangle - \frac{3}{4}\|[X, Y]\|^2 \\ & - \frac{1}{2}\langle [X, [X, Y]], Y\rangle - \frac{1}{2}\langle [Y, [Y, X]], X\rangle\,. \end{aligned} \tag{2}$$

The curvature condition $K < 0$, where $K(X, Y) = \langle R(X, Y)Y, X\rangle$, implies that the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}$ is one-dimensional, that is,

$$\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}\{A_0\}$$

with a unit vector $A_0 \in \mathfrak{g}$ orthogonal to $\mathfrak{n}$. Moreover, if we denote by D_0 and S_0 the symmetric and the skew-symmetric part of $\operatorname{ad} A_0|_{\mathfrak{n}} : \mathfrak{n} \to \mathfrak{n}$, respectively, then D_0 and $D_0^2 + [D_0, S_0]$ are both positive definite (see Heintze [4] for details). Note that, since A_0 is orthogonal to $\mathfrak{n}$, $(\operatorname{ad} X)^* A_0 = 0$ for all $X \in \mathfrak{g}$. Then it follows from (1) that if $X \in \mathfrak{n}$, then

$$\nabla_{A_0} A_0 = 0\,, \quad \nabla_{A_0} X = S_0 X\,, \quad \nabla_X A_0 = -D_0 X\,. \tag{3}$$

Let J be a left invariant almost complex structure on G, and suppose J is skew-symmetric with respect to $\langle\ ,\ \rangle$, that is, $\langle\ ,\ \rangle$ is a Hermitian metric with respect to J. Then

$(G, \langle\ ,\ \rangle, J)$ is called a homogeneous almost Kähler manifold if the fundamental 2-form Φ defined by $\Phi(X, Y) = \langle X, JY\rangle$, $X, Y \in \mathfrak{g}$, is closed, that is,

$$\langle [X, Y], JZ\rangle - \langle [X, Z], JY\rangle + \langle [Y, Z], JX\rangle = 0 \tag{4}$$

for $X, Y, Z \in \mathfrak{g}$. Furthermore, if J is integrable, then $(G, \langle\ ,\ \rangle, J)$ is called a homogeneous Kähler manifold.

Finally, $(G, \langle\ ,\ \rangle)$ is called an Einstein manifold if the Ricci curvature Ric of $(G, \langle\ ,\ \rangle)$ is proportional to the metric, that is, $\mathrm{Ric}(X, Y) = c\langle X, Y\rangle$ for some constant c and any $X, Y \in \mathfrak{g}$.

3. Proof of Theorem

Let $(G, \langle\ ,\ \rangle, J)$ be a homogeneous almost complex manifold of negative curvature, and $\mathfrak{g}$ the Lie algebra of G. Then the left invariant metric $\langle\ ,\ \rangle$ and the left invariant almost complex structure J of G induce the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ and the skew-symmetric operator $J : \mathfrak{g} \to \mathfrak{g}$, respectively. Recall that $\mathfrak{g}$ is decomposed into the direct sum $\mathfrak{g} = \mathbb{R}\{A_0\} \oplus \mathfrak{n}$, where $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ is the derived algebra of $\mathfrak{g}$ and $\mathbb{R}\{A_0\}$ denotes its one-dimensional orthogonal complement.

Let $\mathfrak{z}$ be the center of $\mathfrak{n}$, $\mathfrak{b} = \mathfrak{z}^{\perp}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. If $(G, \langle\ ,\ \rangle, J)$ is Kähler and of negative curavature, then it is known that $(\mathfrak{g}, \langle\ ,\ \rangle, J)$ satisfies the condtion

$$\begin{aligned} &\mathfrak{z} = \mathbb{R}\{JA_0\}\,,\\ &[A_0, X] = \lambda X + S_0 X\,, \quad [A_0, JA_0] = 2\lambda JA_0\,,\\ &[X, Y] = 2\lambda\langle JX, Y\rangle JA_0\,, \quad [X, JA_0] = 0\,, \end{aligned} \tag{5}$$

for any $X, Y \in \mathfrak{b}$ and some $\lambda \in \mathbb{R}$ (see Heintze [4]). On the other hand, in the case when $(G, \langle\ ,\ \rangle, J)$ is almost Kähler, we have the following

PROPOSITION 2. *Let $(G, \langle\ ,\ \rangle, J)$ be a homogeneous almost Kähler manifold of negative curvature. Suppose that $(G, \langle\ ,\ \rangle)$ is Einstein and G is of Iwasawa type. Then $(\mathfrak{g}, \langle\ ,\ \rangle, J)$ also satisfies Condition* (5) (*with* $S_0 = 0$).

PROOF. Take a non-zero vector $Z \in \mathfrak{z}$. By Condition (4) with Z, JZ and A_0, we obtain

$$\langle [Z, JZ], JA_0\rangle = \langle \mathrm{ad}\, A_0(Z), Z\rangle + \langle \mathrm{ad}\, A_0(JZ), JZ\rangle = \langle D_0 Z, Z\rangle + \langle D_0 JZ, JZ\rangle\,. \tag{6}$$

Since D_0 is positive definite, the right hand side of (6) is positive so that $[Z, JZ] \neq 0$. Since $Z \in \mathfrak{z} \subset \mathfrak{n}$, this implies that $JZ \notin \mathfrak{n}$. Hence $\langle JZ, A_0\rangle \neq 0$, implying that $\mathfrak{z}$ is a 1-dimensional subspace of $\mathfrak{n}$. Moreover, $\mathfrak{z}$ is ad A_0-invariant, since $\mathfrak{z}$ is the center of $\mathfrak{n}$ and ad A_0 is a derivation. Hence there exists a positive real number $\lambda > 0$ such that $\mathrm{ad}\, A_0(Z) = 2\lambda Z$ for any $Z \in \mathfrak{z}$.

From Condition (4) with $A_0 \in \mathfrak{g}$, $Y \in \mathfrak{n}$ and $Z \in \mathfrak{z}$, we have

$$\langle (\mathrm{ad}\, A_0 + 2\lambda\, \mathrm{id})Y, JZ\rangle = 0\,,$$

where id denotes the identity map of $\mathfrak{g}$. Since $(\mathrm{ad}\, A_0 + 2\lambda\, \mathrm{id})|_{\mathfrak{n}}$ is non-degenerate, this implies that $[\mathfrak{n}, JZ] = 0$ so that $JZ \in \mathbb{R}\{A_0\}$. Hence $Z \in \mathbb{R}\{JA_0\}$. Since $Z \in \mathfrak{z}$ is arbitrary, we have $\mathfrak{z} = \mathbb{R}\{JA_0\}$.

Since G is assumed of Iwasawa type, $\mathrm{ad}\, A_0|_{\mathfrak{n}}$ is symmetric with respect to $\langle\ ,\ \rangle$, that is, $\mathrm{ad}\, A_0|_{\mathfrak{n}} = D_0$. Then it follows from Condition (4) with $X, Y \in \mathfrak{b}$ and A_0 that

$$\langle [X, Y], JA_0\rangle = \langle (D_0 J + JD_0)X, Y\rangle\,. \tag{7}$$

Recall that $D_0 = \mathrm{ad}\, A_0|_{\mathfrak{n}}$ leaves $\mathfrak{z} = \mathbb{R}\{JA_0\}$ invariant, and hence also its orthogonal complement $\mathfrak{b} = \mathfrak{z}^{\perp}$ of $\mathfrak{z}$ in $\mathfrak{n}$. Multiplying 2λ to this equation, we then obtain

$$\begin{aligned} 2\lambda\langle (D_0 J + JD_0)X, Y\rangle &= 2\lambda\langle [X, Y], JA_0\rangle \\ &= \langle 2\lambda [X, Y]_{\mathfrak{z}}, JA_0\rangle = \langle [A_0, [X, Y]_{\mathfrak{z}}], JA_0\rangle \\ &= \langle [A_0, [X, Y]], JA_0\rangle = \langle [[A_0, X], Y], JA_0\rangle + \langle [X, [A_0, Y]], JA_0\rangle \\ &= \langle (D_0 J + JD_0)D_0 X, Y\rangle + \langle (D_0 J + JD_0)X, D_0 Y\rangle\,, \end{aligned} \tag{8}$$

where $[X, Y]_{\mathfrak{z}}$ denotes the component of $[X, Y]$ in $\mathfrak{z}$.

We now consider the restriction $D_0|_{\mathfrak{b}}$ of $\mathrm{ad}\, A_0|_{\mathfrak{n}} = D_0$ to $\mathfrak{b}$, and let $\mu_i > 0$ be the eigenvalues of $D_0|_{\mathfrak{b}}$ with eigenspace $\mathfrak{b}_i$ for $i = 1, \cdots, s$. Without loss of generality, we may suppose $\mu_1 < \mu_2 < \cdots < \mu_s$. Note that if $X \in \mathfrak{b}_i$, then for any $Y \in \mathfrak{b}_j$ we obtain from (8) that

$$\{2\lambda - (\mu_i + \mu_j)\}(\mu_i + \mu_j)\langle JX, Y\rangle = 0\,, \tag{9}$$

which implies that for each μ_i there exists a unique eigenvalue μ_{i^*} that satisfies

$$2\lambda - (\mu_i + \mu_{i^*}) = 0\,. \tag{10}$$

Indeed, for a given μ_i, if we have no eigenvalue μ_j satisfying $2\lambda - (\mu_i + \mu_j) = 0$, then we see from (9) that $JX = 0$ for any $X \in \mathfrak{b}_i$, contradicting that J is non-degenerate.

It follows from (9) together with (10) that $J(\mathfrak{b}_i) = \mathfrak{b}_{i^*}$. Moreover, by our choice of the order of μ_i's, we conclude from (10) that $i^* = s - i + 1$ and hence

$$\mu_i = 2\lambda - \mu_{s-i+1}\,, \quad i = 1, \cdots, s\,. \tag{11}$$

Now, note that we obtain from (3)

$$\nabla_{A_0} A_0 = 0\,, \quad \nabla_{A_0} X = 0\,, \quad \nabla_X A_0 = -D_0 X$$

for $X \in \mathfrak{n}$, since $\mathrm{ad}\, A_0$ is symmetric. Let $\{X_1^k, \cdots, X_{l_k}^k\}$ be an orthonomal basis of $\mathfrak{b}_k$ for $k = 1, \cdots, s$. Then the Ricci curvature $\mathrm{Ric}(A_0, A_0)$ of $(G, \langle\ ,\ \rangle)$ in the direction A_0 is given by

$$\mathrm{Ric}(A_0, A_0) = \sum_{k=1}^{s}\sum_{i=1}^{l_k}\langle R(X_i^k, A_0)A_0, X_i^k\rangle + \langle R(JA_0, A_0)A_0, JA_0\rangle$$

$$
\begin{aligned}
&= \sum_{k=1}^{s}\sum_{i=1}^{l_k}\langle -\nabla_{[X_i^k, A_0]}A_0, X_i^k\rangle + \langle -\nabla_{[JA_0, A_0]}A_0, JA_0\rangle \\
&= -\sum_{k=1}^{s}\sum_{i=1}^{l_k}\langle D_0^2 X_i^k, X_i^k\rangle - \langle D_0^2 JA_0, JA_0\rangle \\
&= -\operatorname{Tr} D_0^2|_{\mathfrak{b}} - \langle D_0^2 JA_0, JA_0\rangle \,.
\end{aligned}
$$

On the other hand, since from (7) we have $(\operatorname{ad} X)^* JA_0 = (D_0 J + JD_0)X$ for $X \in \mathfrak{b}$, it follows from (1) that

$$
\nabla_{JA_0} X = \nabla_X JA_0 = (-1/2)(D_0 J + JD_0)X \,, \quad \nabla_{JA_0} JA_0 = 2\lambda A_0 \,.
$$

Substituting JX for Y in (8), we also have

$$
2\lambda\langle (D_0 J + JD_0)X, JX\rangle = -\langle (D_0 J + JD_0)^2 X, X\rangle \,.
$$

Therefore, the Ricci curvature $\operatorname{Ric}(JA_0, JA_0)$ in the direction JA_0 is given by

$$
\begin{aligned}
\operatorname{Ric}(JA_0, JA_0) &= \langle R(A_0, JA_0)JA_0, A_0\rangle + \sum_{k=1}^{s}\sum_{i=1}^{l_k}\langle R(X_i^k, JA_0)JA_0, X_i^k\rangle \\
&= \langle -\nabla_{[JA_0, A_0]}A_0, JA_0\rangle + \sum_{k=1}^{s}\sum_{i=1}^{l_k}\langle (\nabla_{X_i^k}\nabla_{JA_0}JA_0 - \nabla_{JA_0}\nabla_{X_i^k}JA_0), X_i^k\rangle \\
&= -\langle D_0^2 JA_0, JA_0\rangle + \sum_{k=1}^{s}\sum_{i=1}^{l_k}\left(\langle \nabla_{X_i^k}(2\lambda A_0) - \frac{1}{4}(D_0 J + JD_0)^2 X_i^k, X_i^k\rangle\right) \\
&= -\langle D_0^2 JA_0, JA_0\rangle + \sum_{k=1}^{s}\sum_{i=1}^{l_k}\left(-2\lambda\langle D_0 X_i^k, X_i^k\rangle + \lambda\frac{1}{2}\langle (D_0 J + JD_0)X_i^k, JX_i^k\rangle\right) \\
&= -\langle D_0^2 JA_0, JA_0\rangle - 2\lambda \operatorname{Tr} D_0|_{\mathfrak{b}} + \lambda\frac{1}{2}\operatorname{Tr} D_0|_{\mathfrak{b}} + \lambda\frac{1}{2}\operatorname{Tr} D_0|_{\mathfrak{b}} \\
&= -\langle D_0^2 JA_0, JA_0\rangle - \lambda \operatorname{Tr} D_0|_{\mathfrak{b}} \,.
\end{aligned}
$$

Hence we have

$$
\operatorname{Ric}(JA_0, JA_0) - \operatorname{Ric}(A_0, A_0) = \operatorname{Tr}(D_0^2|_{\mathfrak{b}}) - \lambda \operatorname{Tr}(D_0|_{\mathfrak{b}}) \,. \tag{12}
$$

Recall that $\mathfrak{b}_k$ is the eigenspace of D_0 with eigenvalue μ_k, so that we have $D_0 X_j^k = \mu_k X_j^k$. Hence, noting (10) and (11), the right hand side of (12) reads as

$$
\operatorname{Tr}(D_0^2|_{\mathfrak{b}}) - \lambda\operatorname{Tr}(D_0|_{\mathfrak{b}}) = \sum_{k=1}^{s}\sum_{i=1}^{l_k}(\langle D_0^2 X_i^k, X_i^k\rangle - \lambda\langle D_0 X_i^k, X_i^k\rangle)
$$

$$
\begin{aligned}
&= \sum_{k=1}^{s}\sum_{i=1}^{l_k}(\mu_k^2 - \lambda\mu_k) = \sum_{k=1}^{s}(\mu_k^2 - \lambda\mu_k)l_k \\
&= \frac{1}{2}\sum_{k=1}^{s}(\mu_k^2 - \lambda\mu_k)l_k + \frac{1}{2}\sum_{k=1}^{s}(\mu_k^2 - \lambda\mu_k)l_k \\
&= \frac{1}{2}\sum_{k=1}^{s}(\mu_k^2 - \lambda\mu_k)l_k + \frac{1}{2}\sum_{k=1}^{s}((2\lambda - \mu_{s-k+1})^2 - \lambda(2\lambda - \mu_{s-k+1}))l_{s-k+1} \\
&= \frac{1}{2}\sum_{k=1}^{s}(\mu_k^2 - \lambda\mu_k)l_k + \frac{1}{2}\sum_{k=1}^{s}((2\lambda - \mu_k)^2 - \lambda(2\lambda - \mu_k))l_k \\
&= \frac{1}{2}\sum_{k=1}^{s}(\mu_k^2 - \lambda\mu_k + (2\lambda - \mu_k)^2 - \lambda(2\lambda - \mu_k))l_k \\
&= \sum_{k=1}^{s}(\mu_k - \lambda)^2 l_k\,,
\end{aligned}
$$

since $\mu_k = 2\lambda - \mu_{s-k+1}$ and $l_k = l_{s-k+1}$. Consequently, we obtain

$$
\mathrm{Ric}(JA_0, JA_0) - \mathrm{Ric}(A_0, A_0) = \sum_{k=1}^{s}(\mu_k - \lambda)^2 l_k\,. \tag{13}
$$

Since $(G, \langle\ ,\ \rangle)$ is assumed Einstein, we have $\mathrm{Ric}(A_0, A_0) = \mathrm{Ric}(JA_0, JA_0)$. Hence it follows from (13) that

$$
\sum_{k=1}^{s}(\mu_k - \lambda)^2 l_k = 0\,,
$$

from which we have $\lambda = \mu_j$ for $j = 1, \cdots, s$, that is,

$$
D_0|_{\mathfrak{b}} = \lambda\,\mathrm{id}\,|_{\mathfrak{b}}\,.
$$

Since $D_0 = \mathrm{ad}\,A_0|_{\mathfrak{n}}$ is a derivation of $\mathfrak{n}$, for any $X, Y \in \mathfrak{b}$ we have

$$
D_0[X, Y] = [D_0X, Y] + [X, D_0Y] = 2\lambda[X, Y]\,.
$$

Hence $[X, Y] \in \mathfrak{z}$, so that $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{z}$. Therefore it follows from Condition (4) that

$$
[X, Y] = \lambda\langle JX, Y\rangle JA_0\,.
$$

Consequently, $(\mathfrak{g}, \langle\ ,\ \rangle, J)$ satisfies Condition (5) □

In Propsition 2 we assume that $(G, \langle\ ,\ \rangle)$ is Einstein. However, in the proof we only use this assumption to assure that $\mathrm{Ric}(JA_0, JA_0) = \mathrm{Ric}(A_0, A_0)$. As a result, we also have the following

PROPOSITION 3. *Let $(G, \langle\ ,\ \rangle, J)$ be a homogeneous almost Kähler manifold of negative curvature. Suppose that G is of Iwasawa type and the Ricci tensor field of $(G, \langle\ ,\ \rangle)$ is J-invariant, that is,* $\mathrm{Ric}(JX, JY) = \mathrm{Ric}(X, Y)$ *for $X, Y \in \mathfrak{g}$. Then $(\mathfrak{g}, \langle\ ,\ \rangle, J)$ satisfies Condition* (5) (*with* $S_0 = 0$).

Regarding the integrability of the almost complex structure J, we now have

LEMMA 4. *Let $(G, \langle\ ,\ \rangle, J)$ be a homogeneous almost complex manifold of negative curvature. If $(\mathfrak{g}, \langle\ ,\ \rangle, J)$ satisfies Condition* (5), *then J is integrable.*

PROOF. By a straightforward computation, we can see that the Nijenhuis tensor of J

$$N(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$$

vanishes identically, so that J is integrable.

Indeed, for any $X, Y \in \mathfrak{b}$ and A_0, the Jacobi identity together with Condition (4) yield that

$$\begin{aligned} 0 &= [A_0, [X, Y]] + [X, [Y, A_0]] + [Y, [A_0, X]] \\ &= [A_0, 2\lambda\langle JX, Y\rangle JA_0] + [X, -(\lambda Y + S_0 Y)] + [Y, \lambda X + S_0 X] \\ &= 4\lambda^2\langle JX, Y\rangle JA_0 + 2\lambda\langle JX, -(\lambda Y + S_0 Y)\rangle JA_0 + 2\lambda\langle JY, \lambda X + S_0 X\rangle JA_0 \\ &= 2\lambda\langle (S_0 J - JS_0)X, Y\rangle JA_0\,. \end{aligned}$$

Hence we have $S_0 J - JS_0 = 0$. Using this identity, we see that the Nijenhuis tensor $N(A_0, X)$ for $X \in \mathfrak{b}$ and A_0 vanishes as follows.

$$\begin{aligned} N(A_0, X) &= [JA_0, JX] - J[A_0, JX] - J[JA_0, X] - [A_0, X] \\ &= -J(\lambda JX + S_0 JX) - (\lambda X + S_0 X) \\ &= -J(S_0 J - JS_0)X = 0\,. \end{aligned}$$

The vanishing of the other components $N(A_0, JA_0)$, $N(JA_0, X)$ and $N(X, Y)$ for $X, Y \in \mathfrak{b}$ of the Nijenhuis tensor can be seen in a similar manner. □

It is shown in Heintze [4] that a connected homogeneous Kähler manifolds of negative curvature is holomorphically isometric to a complex hyperbolic space . Hence, it follows form Proposition 2 and Lemma 4 that $(G, \langle\ ,\ \rangle, J)$ must be holomorphically isometric to a complex hyperbolic space. This completes the proof of Theorem 1

It should be remarked that, combining Proposition 3 with Lemma 4, we also have the following

THEOREM 5. *Let $(G, \langle\ ,\ \rangle, J)$ be a homogeneous almost Kähler manifold of negative curvature. If G is of Iwasawa type and the Ricci tensor field of $(G, \langle\ ,\ \rangle)$ is J-invariant, then $(G, \langle\ ,\ \rangle, J)$ is Kähler, and in fact is holomorphically isometric to a complex hyperbolic space* $(\mathbb{C}H^n, g_0, J_0)$.

Finally, we give an example of a homogeneous almost Kähler manifold of negative curvature which is neither Einstein nor Kähler.

EXAMPLE 6. Let $\mathfrak{g}$ be a real Lie algebra spanned by A, X, Y, Z, with the bracket operation defined by

$$\begin{aligned} &[A, X] = X\,, \quad [A, Y] = 2Y\,, \quad [A, Z] = 3Z\,, \\ &[X, Y] = 3Z\,, \quad \text{otherwise} = 0\,, \end{aligned} \tag{14}$$

and with an inner product $\langle\ ,\ \rangle$ for which A, X, Y, Z are orthonormal. We define a skew-symmetric endmorphism J on $\mathfrak{g}$ by

$$JA = Z\,, \quad JX = Y\,, \quad JY = -X\,, \quad JZ = -A\,. \tag{15}$$

Let G be the simply connected Lie group associated with $\mathfrak{g}$. By left translations, $\langle\ ,\ \rangle$ and J on $\mathfrak{g}$ extend to G as an left invariant metric $\langle\ ,\ \rangle$ and a left invariant almost complex structure J, respectively. Then it is immediate from (14) that G is of Iwasawa type. Moreover, from (14) and (15), it is easily verified by a straightforward computation that $(G, \langle\ ,\ \rangle, J)$ has negative curvature and is almost Kähler, that is, J satisfies Condition (4). However, $(G, \langle\ ,\ \rangle, J)$ is not Einstein, since we have

$$\mathrm{Ric}(A, A) = -14 = -14\langle A, A\rangle\,, \quad \mathrm{Ric}(X, X) = -21/2 = -(21/2)\langle X, X\rangle$$

for unit vectors $A, X \in \mathfrak{g}$. Also, $(G, \langle\ ,\ \rangle, J)$ is not Kähler, since we have

$$\begin{aligned} N(A, X) &= [JA, JX] - J[A, JX] - J[JA, X] - [A, X] \\ &= [Z, Y] - J[A, Y] - J[Z, X] - X = -3X\,. \end{aligned}$$

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Present Address:
MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY,
SENDAI, 980–8578 JAPAN.