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On the Uniqueness of Semistable Embedding and Domain of Semistable Attraction for Probability Measures on p-adic Groups

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Abstract. We show that on a p-adic Lie group, any normal semistable measure has a unique semistable embedding. This, in particular, implies the uniqueness of semistable embedding of any (operator-)semistable measure on a finite dimensional p-adic vector space. We compare two classes of probability measures on a unipotent p-adic algebraic group: the class of semistable measures and that of measures whose domain of semistable attraction is nonempty.

1. Introduction

Let G be a locally compact (Hausdorff) group with identity e and let $M^1(G)$ denote the topological semigroup of probability measures on G with weak topology and convolution '$*$' as the semigroup operation. Let Aut(G) denote the group of continuous automorphisms of G (with compact-open topology).

A probability measure μ on G is said to be (τ, c)-*semistable* for $\tau \in \mathrm{Aut}(G)$ and $c \in]0, 1[$ if μ is embeddable in a continuous (real) one-parameter semigroup $\{\mu_t\}_{t\geq 0} \subset M^1(G)$ as $\mu = \mu_1$ such that $\tau(\mu_t) = \mu_{ct}$ for all $t \geq 0$; we also call $\{\mu_t\}_{t\geq 0}$ (τ, c)-*semistable*. A measure μ (resp. $\{\mu_t\}_{t\geq 0}$) in $M^1(G)$ is said to be *semistable* if it is (τ, c)-semistable for some $\tau \in \mathrm{Aut}(G)$ and some $c \in]0, 1[$. Note that in case of (finite dimensional) p-adic (resp. real) vector spaces, this definition corresponds to that of operator-semistable (resp. strictly operator-semistable) measures.

It is well-known that if any locally compact group G admits a semistable measure μ embeddable in a (τ, c)-semistable $\{\mu_t\}_{t\geq 0} \subset M^1(G)$ for τ and c as above, then μ is supported on the closure of the K-contraction group of τ, namely $C_K(\tau) = \{x \in G \mid \tau^n(x)K \to K \text{ in } G/K\}$, where K is a compact subgroup such that $\mu_0 = \omega_K$, the normalised Haar measure of K. The structure of $C(\tau) = C_{\{e\}}(\tau)$, the contraction group of τ, is well-known. If G admits a contracting automorphism τ; i.e. $C(\tau) = G$, then it is a direct product $G^0 \times D$, where G^0 is a simply connected nilpotent contractible group and D is a totally disconnected contractible group (cf. [Si]). Semistable measures on real vector spaces, or more generally,

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on simply connected nilpotent groups, have been studied in details (see [HSi2] and references cited therein). In this article, we are interested in investigating some aspects of semistability for measures on p-adic Lie groups, which form a significant subclass of totally disconnected groups, (see [V] or [S] for exposition on p-adic Lie groups).

In case G is a p-adic Lie group and $\tau \in \mathrm{Aut}(G)$, by Theorem 3.5 of [W], $C(\tau)$ is a unipotent p-adic algebraic group (see section 2 for some details). Also, if $\tau(K) = K$ then $C_K(\tau)$ is closed and $C_K(\tau) = C(\tau)\cdot K$, a semidirect product. Moreover, any (τ, c)-semistable one-parameter semigroup $\{\mu_t\}_{t\geq 0}$ on G can be expressed as $\mu_t = \mu_t^{(0)} * \omega_K = \omega_K * \mu_t^{(0)}$ for all t, where $\{\mu_t^{(0)}\}_{t\geq 0}$ is a (τ, c)-semistable one-parameter semigroup supported on $C(\tau)$ (cf. [DSh1]; a similar result is true for real Lie groups also, see [HSi1]). For a survey of results on semistable measures on locally compact groups, the reader is referred to [HSi2] and for semistable measures on p-adic groups in particular to [DSh1], [Sh1], and also [MSh1]–[MSh2] for more recent results.

For any $x \in G$, let δ_x denote the dirac measure supported on x. Let $\mu \in M^1(G)$. Let $\tilde{\mu} \in M^1(G)$ be defined as $\tilde{\mu}(B) = \mu(B^{-1})$ for all Borel subsets B of G. μ is said to be *normal* if $\mu * \tilde{\mu} = \tilde{\mu} * \mu$. Let $G(\mu)$ denote the closed subgroup generated by $\mathrm{supp}\,\mu$, the support of μ and let $\mathcal{I}(\mu) = \{x \in G \mid \delta_x * \mu = \mu * \delta_x = \mu\}$ which is a compact subgroup of G. We also define the invariance group of μ as $\mathrm{Inv}(\mu) = \{\tau \in \mathrm{Aut}(G) \mid \tau(\mu) = \mu\}$; it is a closed subgroup of $\mathrm{Aut}(G)$.

We say that a semistable measure on G has a *unique semistable embedding* if the following holds: if μ is embeddable in (τ, c)-semistable and (ψ, d)-semistable one-parameter semigroups $\{\mu_t\}_{t\geq 0}$ and $\{\nu_t\}_{t\geq 0}$ in $M^1(G)$ respectively as $\mu_1 = \mu = \nu_1$, for some $\tau, \psi \in \mathrm{Aut}(G)$ and $c, d \in]0, 1[$, then $\mu_t = \nu_t$ for all $t \geq 0$.

In section 2, we discuss the uniqueness of semistable embedding on a p-adic Lie group G under certain conditions and show that any normal semistable measure on G has a unique semistable embedding (see Theorem 2.1). In particular, this implies the uniqueness of semistable embedding of any (operator-)semistable measure on any p-adic vector space. In section 3, on a unipotent p-adic algebraic group, we compare semistable measures with measures whose domain of semistable attraction is nonempty, (in particular, see Theorem 3.1).

2. On the uniqueness of semistable embedding on p-adic groups

In this section, we discuss the uniqueness of semistable embedding on a p-adic Lie group G under certain conditions. The reader is referred to [C] for generalities on p-adic vector spaces and to [V] and [S] for p-adic Lie groups.

For a prime p, let $\mathbf{Q}_p$ denote the field of p-adic numbers with the usual p-adic absolute value $|\cdot|_p$. Let $GL_m(\mathbf{Q}_p)$ be the group of $m \times m$ non-singular matrices with entries in $\mathbf{Q}_p$, with the topology as a subset of $\mathbf{Q}_p^{m^2}$. Then $GL_m(\mathbf{Q}_p)$ is a p-adic Lie group. Let $\tilde{G}$ be a p-adic algebraic group and let $G = \tilde{G}(\mathbf{Q}_p)$ be the $\mathbf{Q}_p$-rational points of $\tilde{G}$. i.e. $\tilde{G}$ (resp. G)

is the set of common zeros in $GL_m(\overline{\mathbf{Q}_p})$ (resp. in $GL_m(\mathbf{Q}_p)$) of finitely many polynomials with coeffifients in $\mathbf{Q}_p$, where $\overline{\mathbf{Q}_p}$ denotes the algebraic closure of $\mathbf{Q}_p$. Then G is a closed subgroup of $GL_m(\mathbf{Q}_p)$ for some $m \in \mathbf{N}$ (the set of natural numbers), in particular, it is a p-adic Lie group; We will occasionally call G itself a p-adic algebraic group. A subgroup H of $G = \tilde{G}(\mathbf{Q}_p)$ is said to be algebraic if $H = \tilde{H}(\mathbf{Q}_p)$ for some algebraic group $\tilde{H} \subset \tilde{G}$; H is closed in G. An algebraic group $\tilde{G}$ is said to be unipotent if it consists of unipotent elements. If $\tilde{G}$ is unipotent, then $G = \tilde{G}(\mathbf{Q}_p)$ is a subgroup of $U_m(\mathbf{Q}_p)$ (for some $m \in \mathbf{N}$), the group of $m \times m$ upper triangular matrices with all diagonal entries equal to 1, (see [B], [Ho] and [Hu] for generalities on algebraic groups).

Let $\tilde{G}$ be a p-adic algebraic group and let $G = \tilde{G}(\mathbf{Q}_p)$. For a probability measure μ on G, we will denote by $\tilde{G}(\mu)$ the smallest (closed) algebraic subgroup of G containing $\operatorname{supp}\mu$. A probability measure μ on G is said to be *full* (resp. *S-full*) if $\tilde{G}(\mu) = G$ (resp. $\tilde{G}(\mu * \tilde{\mu}) = G$). Note that these definitions are consistent with the definitions on (p-adic) vector spaces as all its algbraic subgroups are subspaces.

We note that any symmetric semistable measure on any locally compact group has a unique semistable embedding. This is because if a symmetric measure is embeddable in a continuous one-parameter semigroup which consists of symmetric measures then such an embedding is unique and if $\{\mu_t\}_{t\geq 0}$ is (τ, c)-semistable for some automorphism τ and some $c \in]0, 1[$ where μ_1 is symmetric then $\tau^n(\mu_1)^m = \mu_{mc^n}$ is symmetric for $m, n \in \mathbf{N}$, and since $\{\mu_{mc^n} \mid m, n \in \mathbf{N}\}$ is a dense subset of $\{\mu_t\}_{t\geq 0}$, each μ_t is symmetric. Here, we show that any normal semistable measure on a p-adic Lie group G has a unique semistable embedding. In particular, any (operator-)semistable measure on a p-adic vector space (or more generally) on an abelian p-adic Lie group has a unique semistable embedding. Note that on a simply connected nilpotent group, the uniqueness of semistable embedding has been shown (see [H]).

THEOREM 2.1. *Any normal semistable measure on a p-adic Lie group has a unique semistable embedding.*

The following corollary follows easily from the above theorem. In particular, it holds for any (operator-)semistable measure on any p-adic vector space.

COROLLARY 2.2. *Any semistable measure on an abelian p-adic Lie group has a unique semistable embedding.*

Before proving the above theorem, we state and prove several results, some of which will be of independent interest. We also state a specific case of convergence-of-types theorem which we will use often.

THEOREM 2.3 ([Sh2]). *Let G be a unipotent p-adic algebraic group. Let $\{\mu_n\} \subset M^1(G)$ and $\{\tau_n\} \subset \operatorname{Aut}(G)$ be such that $\mu_n \to \mu$ and $\tau_n(\mu_n) \to \lambda$ for some full measures μ and λ in $M^1(G)$. Then $\{\tau_n\}$ is relatively compact in* $\operatorname{Aut}(G)$. *Moreover, for any limit τ of $\{\tau_n\}$ in* $\operatorname{Aut}(G)$, $\tau(\mu) = \lambda$. *In particular,* $\operatorname{Inv}(\mu)$ *is compact in* $\operatorname{Aut}(G)$.

THEOREM 2.4. *Let G be a p-adic Lie group and let $\{\mu_t\}_{t\geq 0}$ and $\{\nu_t\}_{t\geq 0}$ be respectively (τ, c)-semistable and (ψ, d)-semistable one-parameter semigroups in $M^1(G)$ for some $\tau, \psi \in \mathrm{Aut}(G)$ and $c, d \in]0, 1[$ such that $\mu_1 = \nu_1 = \mu \neq \delta_e$ and $\mu_0 = \delta_e$. Then $\log c$ and $\log d$ are commensurable. Moreover, If τ and ψ commute with each other, then $\mu_t = \nu_t$ for all $t \geq 0$.*

PROOF. Since G is totally disconnected, it is well known that μ_t and ν_t are supported on $G(\mu)$ and, in fact, $G(\mu_t) = G(\mu) = G(\nu_t)$, $t > 0$. Therefore, $\tau(G(\mu)) = G(\mu)$ and $\psi(G(\mu)) = G(\mu)$. Hence without loss of any generality, we may assume that $G = G(\mu)$. Since $\mu_0 = \delta_e$, $G = C(\tau)$ (cf. [Si]). This implies that $G = \tilde{G}(\mathbf{Q}_p)$, where $\tilde{G}$ is a unipotent p-adic algebraic group, (cf. [W], Theorem 3.5), and τ contracts G. Also, τ and ψ are $\mathbf{Q}_p$-rational automorphisms of G (cf. [Sh2], Theorem 2.1).

Step 1: If possible, suppose $\log c$ and $\log d$ are incommensurable. Then c and d generate a dense subsemigroup (say) B of $\mathbf{R}_+^*$ (the semigroup of positive real numbers with the usual topology). Hence, for any $t \in \mathbf{R}_+^*$, there exist sequences $\{l_n\}, \{m_n\} \subset \mathbf{Z}$ such that $l_n \to -\infty$ and $m_n \to \infty$ and $c^{m_n} d^{l_n} \to t$. For any $r \in \mathbf{R}_+$, let $[r]$ denote the largest integer less than or equal to r. We have, for $n \in \mathbf{N}$,

$$\psi^{l_n}(\mu) = \mu^{[d^{l_n}]} * \nu_{b_n}\,, \quad \text{where} \quad b_n = d^{l_n} - [d^{l_n}]\,, \quad 0 \leq b_n < 1\,.$$

Here, $\{\nu_{b_n}\}$ is relatively compact and since τ is contracting, we get $\tau^{m_n}(\nu_{b_n}) \to \delta_e$. Also, $c^{m_n} b_n \to 0$ and hence $c^{m_n}[d^{l_n}] \to t$. Now we get

$$\tau^{m_n}\psi^{l_n}(\mu) = \tau^{m_n}(\mu^{[d^{l_n}]} * \nu_{b_n}) = \mu_{c^{m_n}[d^{l_n}]} * \tau^{m_n}(\nu_{b_n}) \to \mu_t\,.$$

Since each μ_t is full, the above implies, by the convergence-of-types theorem (cf. Theorem 2.3), that $\{\tau^{m_n}\psi^{l_n}\}$ is relatively compact and for any limit point σ of it, we have that $\sigma(\mu) = \mu_t$. That is, all limit points of $\{\tau^{m_n}\psi^{l_n}\}$ are contained in $\tau_t \,\mathrm{Inv}(\mu)$ for some $\tau_t \in \mathrm{Aut}(G)$, $t \in \mathbf{R}_+^*$. Moreover, by the convergence-of-types theorem, we get that $\{\tau_t \,\mathrm{Inv}(\mu)\}_{t\in\mathbf{R}_+^*}$ form a continuous image of $\mathbf{R}_+^*$ in the quotient space $\mathrm{Aut}(G)/\,\mathrm{Inv}(\mu)$ and hence it is a connected set. Since $\mathrm{Aut}(G)$ is totally disconnected and locally compact, $\mathrm{Aut}(G)/\,\mathrm{Inv}(\mu)$ is totally disconnected. Hence $\tau_t \,\mathrm{Inv}(\mu) = \tau_1 \,\mathrm{Inv}(\mu) = \mathrm{Inv}(\mu)$ for all $t > 0$. This implies that $\mu = \mu_t$ for all t and hence $\mu = \delta_e$, a contradiction. Therefore, $\log c$ and $\log d$ are commensurable.

Step 2: Now we assume that τ and ψ commute with each other and show that $\mu_t = \nu_t$ for all t. Since $\log c$ and $\log d$ are commensurable, there exist $a > 0$ and $l, m \in \mathbf{N}$ such that $c = a^l$ and $d = a^m$. Then $c^m = d^l$ and hence $\{\mu_t\}_{t\geq 0}$ (resp. $\{\nu_t\}_{t\geq 0}$) is (τ^m, c^m)-semistable (resp. (ψ^l, d^l)-semistable). If necessary, replacing τ and ψ by τ^m and ψ^l respectively, without loss of any generality, we may assume $c = d$, i.e. $\{\mu_t\}_{t\geq 0}$ and $\{\nu_t\}_{t\geq 0}$ are respectively (τ, c)-semistable and (ψ, c)-semistable on a unipotent p-adic algebraic group G and $\mu = \mu_1 = \nu_1$ is full on G.

Now for the sequence $\{k_n = [c^{-n}]\} \subset \mathbf{N}$, we have that $\tau^n(\mu^{k_n}) \to \mu$ and $\psi^n(\mu^{k_n}) \to \mu$. Moreover, $\tau^n(\mu^{[k_n t]}) \to \mu_t$ and $\psi^n(\mu^{[k_n t]}) \to \nu_t$ for all $t > 0$. From the first assertion,

$(\psi^n\tau^{-n})\tau^n(\mu^{k_n}) = \psi^n(\mu^{k_n}) \to \mu$. Since μ is full, using the convergence-of-types theorem (cf. Theorem 2.3), we get that $\{\sigma_n = \psi^n\tau^{-n}\}$ is relatively compact and all its limit points belong to $\mathrm{Inv}(\mu)$. Also, since τ and ψ commute with each other, we can choose $\sigma_n = \rho_n\alpha_n = \alpha_n\rho_n$, where $\alpha_n \in \mathrm{Inv}(\mu)$, α_n and ρ_n commute with both τ and ψ, and $\rho_n \to I$, the identity in $\mathrm{Aut}\,G$. Now for all $t > 0$,

$$\begin{aligned}\nu_t = \lim_n \psi^n(\mu^{[k_n t]}) &= \lim_n \sigma_n\tau^n(\mu^{[k_n t]})\\ &= \lim_n \tau^n\sigma_n(\mu^{[k_n t]})\\ &= \lim_n \tau^n\rho_n(\mu^{[k_n t]})\\ &= \lim_n \rho_n\tau^n(\mu^{[k_n t]})\\ &= \mu_t\,.\end{aligned}$$

The above also implies that $\nu_0 = \mu_0$. This completes the proof. □

We now state a simple result which will be useful.

LEMMA 2.5. *Let G be a p-adic Lie group, let $\mu \in M^1(G)$ and let $i \in \{1, 2\}$. Suppose μ is embeddable in $\{\mu_t\}_{t\geq 0}$ and $\{\nu_t\}_{t\geq 0}$ in $M^1(G)$, (as $\mu = \mu_1 = \nu_1$), which are (τ_1, c_1)-semistable and (τ_2, c_2)-semistable respectively for some $\tau_i \in \mathrm{Aut}(G)$ and $c_i \in]0, 1[$. Let K_1, K_2 be compact subgroups of G such that $\mu_0 = \omega_{K_1}$ and $\nu_0 = \omega_{K_2}$. Then $C(\tau_1)\cap G(\mu) = C(\tau_2)\cap G(\mu) = U$ is a closed nilpotent normal subgroup of $G(\mu)$, $U = \tilde{U}(\mathbf{Q}_p)$, where $\tilde{U}$ is a unipotent p-adic algebraic group, $G(\mu) = K_i\cdot U$, a semidirect product and, $\tau_i(K_i) = K_i$ and $\tau_i(U) = U$ for each i. Moreover, there exists a (τ_1, c_1)-semistable (resp. (τ_2, c_2)-semistable) one-parameter semigroup $\{\mu_t^{(0)}\}_{t\geq 0}$, (resp. $\{\nu_t^{(0)}\}_{t\geq 0}$) such that $\mu_0^{(0)} = \nu_0^{(0)} = \delta_e$, each $\mu_t^{(0)}$ (resp. $\nu_t^{(0)}$) is supported on U, $\mu_t = \omega_{K_1} * \mu_t^{(0)} = \mu_t^{(0)} * \omega_{K_1}$, $\nu_t = \omega_{K_2} * \nu_t^{(0)} = \nu_t^{(0)} * \omega_{K_2}$, $t \geq 0$, and K_1 and K_2 are isomorphic. Also, if $\pi : G(\mu) \to G(\mu)/[U, U]$ is the natural projection then $\pi(K_1) = \pi(K_2) = \pi(\mathcal{I}(\mu))$ and $\pi(\mu_1^{(0)}) = \pi(\nu_1^{(0)})$.*

PROOF. Let $i \in \{1, 2\}$. As earlier, we have that $G(\mu_t) = G(\mu) = G(\nu_t)$, $t > 0$, and $\tau_i(G(\mu)) = G(\mu)$ and $\tau_i(K_i) = K_i \subset G(\mu)$ for K_i as above. Hence, we assume that $G = G(\mu)$. Let $U_i = C(\tau_i)$. Then $\tau_i(U_i) = U_i$ and $U_i = \tilde{U}_i(\mathbf{Q}_p)$, where $\tilde{U}_i$ is a unipotent p-adic algebraic group (cf. [W], Theorem 3.5). Also, $G = K_i \cdot U_i$, a semidirect product (cf. [DSh1], Theorem 3.1), each U_i is closed and normal in G. We first show that $U_1 = U_2$. This will also imply that K_1 and K_2 are isomorphic as each quotient group G/U_i is isomorphic to K_i. Since each U_i is unipotent, it is a divisible nilpotent group. Let $\pi_1 : G \mapsto G/U_1$ be the natural projection. Then $\pi_1(U_2)$ is a divisible nilpotent group and its closure L, being compact, is also divisible. It is well-known that any compact divisible nilpotent group is connected. But L is totally disconnected, hence it is a trivial subgroup in G/U_1. Therefore, $U_2 \subset U_1$. Similarly, we get that $U_1 \subset U_2$, i.e. $U_1 = U_2 = U$.

From Theorem 4.1 of [DSh1], we have $\mu_t = \omega_{K_1} * \mu_t^{(0)} = \mu_t^{(0)} * \omega_{K_1}$ (resp. $\nu_t = \omega_{K_2} * \nu_t^{(0)} = \nu_t^{(0)} * \omega_{K_2}$), where $\{\mu_t^{(0)}\}_{t\geq 0}$ (resp. $\{\nu_t^{(0)}\}_{t\geq 0}$) is a (τ_1, c_1)-semistable (resp. (τ_2, c_2)-semistable) continuous one-parameter semigroup on U with $\mu_0^{(0)} = \nu_0^{(0)} = \delta_e$ and $\mu = \omega_{K_1} * \mu_1^{(0)} = \omega_{K_2} * \nu_1^{(0)}$.

Let π be as in the hypothesis. Suppose that U is abelian and π is an identity homomorphism. Then using Fourier transforms, one can show that $\mathcal{I}(\mu_1^{(0)}) = \mathcal{I}(\nu_1^{(0)}) = \{e\}$. Hence $\mathcal{I}(\mu) = K_1 = K_2 = K$ (say). That is, $G = K \cdot U$, a semidirect product and hence $\mu_1^{(0)} = \nu_1^{(0)}$. Now suppose that U is not abelian. We have that $[U, U]$ is closed and characteristic and $U/[U, U]$ is abelian. Also, $\{\pi(\mu_t)\}_{t\geq 0}$ (resp. $\{\pi(\nu_t)\}_{t\geq 0}$) is (τ_1', c_1)-semistable (resp. (τ_2', c_2)-semistable) one-parameter semigroup on $\pi(G(\mu))$, where $\tau_i'(\pi(x)) = \pi(\tau_i(x))$, $x \in G(\mu)$, both τ_1' and τ_2' are automorphisms of $\pi(G(\mu))$. Now the last assertion in the theorem follows from above. □

REMARK 2.6. Theorem 2.4 is valid even without the condition that $\mu_0 = \delta_e$. For $\{\mu_t\}_{t\geq 0}$, $\{\nu_t\}_{t\geq 0}$, τ, ψ, c and d as in the hypothesis of the theorem, suppose $\mu_0 = \omega_{K_1}$ and $\nu_0 = \omega_{K_2}$ for some compact subgroups K_1, K_2 in G and $\mu_1 = \nu_1 = \mu \neq \omega_{K_1}$. Then $G(\mu) = K_i \cdot U$, where $U = C(\tau) = C(\psi)$ as shown above. Here, $U \neq \{e\}$ as $\mu \neq \omega_{K_1}$. Let $\pi : G(\mu) \mapsto G(\mu)/[U, U]$ be as above. Then if $\log c$ and $\log d$ are incommensurable, we can replace $\{\mu_t\}_{t\geq 0}$ (resp. $\{\nu_t\}_{t\geq 0}$) in the proof with $\{\pi(\mu_t^{(0)})\}_{t\geq 0}$ (resp. $\{\pi(\nu_t^{(0)})\}_{t\geq 0}$) which is (τ', c)-semistable (resp. (ψ', d)-semistable) on $\pi(U)$ with $\pi(\mu_0^{(0)}) = \pi(\nu_0^{(0)}) = \delta_{\pi(e)}$, for τ' (resp. ψ') defined on $\pi(G(\mu))$ as $\tau'(\pi(x)) = \pi(\tau(x))$ (resp. $\psi'(\pi(x)) = \pi(\psi(x))$) for all $x \in G(\mu)$, and arrive at a contradiction. For the second assertion, if τ and ψ commute then it is easy to show that $\tau(K_2) = K_2$, $\psi(K_1) = K_1$ and $K_1 = K_2$ and hence $\mu_1^{(0)} = \nu_1^{(0)}$ and again we can work with $\{\mu_t^{(0)}\}_{t\geq 0}$ and $\{\nu_t^{(0)}\}_{t\geq 0}$ in place of $\{\mu_t\}_{t\geq 0}$ and $\{\nu_t\}_{t\geq 0}$ and show for all t, $\mu_t^{(0)} = \nu_t^{(0)}$ and hence $\mu_t = \nu_t$.

LEMMA 2.7. *Let G be a unipotent p-adic algebraic group and let $\mu \in M^1(G)$ be a full semistable measure which is embeddable in a (τ, c)-semistable $\{\mu_t\}_{t\geq 0} \subset M^1(G)$ as $\mu = \mu_1$ for some $\tau \in \mathrm{Aut}(G)$ and some $c \in]0, 1[$. Let $T : \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ be an automorphism defined as $T(\rho) = \tau\rho\tau^{-1}$, $\rho \in \mathrm{Aut}(G)$. Then*

(i) *$\mathcal{K} = \bigcap_{t\geq 0} \mathrm{Inv}(\mu_t)$ is a compact subgroup of $\mathrm{Aut}(G)$ and $T(\mathcal{K}) = \mathcal{K}$,*

(ii) *$\bigcap_{n\in\mathbf{N}} \tau^n(\mathrm{Inv}(\mu))\tau^{-n} = \bigcap_{t\geq 0} \mathrm{Inv}(\mu_t)$ and*

(iii) *$\mathrm{Inv}(\mu) \subset C_{\mathcal{K}}(T)$.*

PROOF. Here, $\mathcal{K}$ is obviously a group. Let $t > 0$. Since $\mu_1 = \mu$ is full so is each μ_t, and hence $\mathrm{Inv}(\mu_t)$ is compact. (cf. Theorem 2.3). Therefore $\mathcal{K}$ is compact. Since $\tau(\mu_t) = \mu_{ct}$, $\tau(\mathrm{Inv}(\mu_t))\tau^{-1} = \mathrm{Inv}(\mu_{ct})$. Therefore, $T(\mathcal{K}) = \mathcal{K}$. Thus (i) holds. Here, $\tau^n(\mu) = \tau^n(\mu_1) = \mu_{c^n}$ and hence $\tau^n \mathrm{Inv}(\mu)\tau^{-n} = \mathrm{Inv}(\mu_{c^n}) \subset \mathrm{Inv}(\mu_{kc^n})$ for all $n, k \in \mathbf{N}$. In particular, the inclusion '$\supset$' is obvious in (ii). Also, since each μ_t is full and since $\{kc^n \mid k, n \in \mathbf{N}\}$ is dense in $\mathbf{R}_+$, we can easily show, by using the convergence-of-types theorem (cf. Theorem 2.3), that $\bigcap_{n\in\mathbf{N}} \tau^n \operatorname{Inv}(\mu)\tau^{-n} \subset \operatorname{Inv}(\mu_t)$ for each $t > 0$. This proves (ii).

Let $\rho \in \operatorname{Inv}(\mu)$. From above, $T^n(\rho) \in \operatorname{Inv}(\mu_{kc^n})$ for all $n, k \in \mathbf{N}$. For a fixed $t > 0$, let $r_n = [tc^{-n}]$, $n \in \mathbf{N}$. Since $T^n(\rho) \in \operatorname{Inv}(\mu_{r_n c^n})$ and since $\mu_{r_n c^n} \to \mu_t$, by the convergence-of-types theorem, we get that $\{T^n(\rho)\}$ is relatively compact and all its limit points belong to $\operatorname{Inv}(\mu_t)$, since this is true for all $t > 0$, we get that $\rho \in C_{\mathcal{K}}(T)$. Thus the assertion (iii) is proved. □

PROPOSITION 2.8. *Let G be a locally compact group and let $\mu \in M^1(G)$. Suppose μ is embeddable in a (τ, c)-semistable one-parameter semigroup $\{\mu_t\}_{t\geq 0} \subset M^1(G)$ as $\mu = \mu_1 \neq \mu_0$ for some $\tau \in \operatorname{Aut}(G)$ and some $c \in]0, 1[$. Consider the following statements*:

(i) $\operatorname{Inv}(\mu) = \operatorname{Inv}(\mu_c^n)$ *for all* $n \in \mathbf{N}$.

(ii) τ *normalises* $\operatorname{Inv}(\mu)$.

(iii) μ *has a unique semistable embedding.*

Then (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii). *They are all equivalent if $G = \tilde{G}(\mathbf{Q}_p)$, $\tilde{G}$ is a unipotent p-adic algebraic group and $\mu \in M^1(G)$ is full.*

PROOF. *Step* 1: Let G be any locally compact group. Since $\{\mu_t\}_{t\geq 0}$ is (τ, c)-semistable and $\mu_1 = \mu$, $\tau^n(\mu) = \mu_{c^n}$ and hence $\tau^n \operatorname{Inv}(\mu)\tau^{-n} = \operatorname{Inv}(\mu_{c^n})$, $n \in \mathbf{N}$. Thus, it is clear that (i) and (ii) are equivalent.

Step 2: Suppose (iii) holds. It is enough to show that (ii) holds. If possible, suppose $\tau \operatorname{Inv}(\mu)\tau^{-1} \neq \operatorname{Inv}(\mu)$. Suppose that there exists $\rho \in \operatorname{Inv}(\mu)$ such that $\rho \notin \tau \operatorname{Inv}(\mu)\tau^{-1}$. Since $\rho(\mu) = \mu$, we get that μ is embeddable in $\{\rho(\mu_t)\}_{t\geq 0}$ which is $(\rho\tau\rho^{-1}, c)$-semistable. But $\mu_c \neq \rho(\mu_c)$ as $\rho \notin \tau \operatorname{Inv}(\mu)\tau^{-1} = \operatorname{Inv}(\mu_c)$.

Now suppose there exists $\rho' \in \tau \operatorname{Inv}(\mu)\tau^{-1}$ such that $\rho' \notin \operatorname{Inv}(\mu)$. Let $\rho = \tau^{-1}\rho'\tau \in \operatorname{Inv}(\mu)$. Then $\rho(\mu) = \mu$ and $\rho \notin \tau^{-1} \operatorname{Inv}(\mu)\tau$. Now arguing as above, we get that μ is embeddable in $\{\rho(\mu_t)\}_{t\geq 0}$ which is $(\rho\tau\rho^{-1}, c)$-semistable. But $\rho(\mu_{c^{-1}}) \neq \mu_{c^{-1}}$ as $\rho \notin \tau^{-1} \operatorname{Inv}(\mu)\tau = \operatorname{Inv}(\mu_{c^{-1}})$. Thus, in both cases we get that μ is embeddable in two different semistable one-parameter semigroups, a contradiction. Therefore, $\operatorname{Inv}(\mu) = \tau \operatorname{Inv}(\mu)\tau^{-1}$.

Step 3: Let $G = \tilde{G}(\mathbf{Q}_p)$, where $\tilde{G}$ is a p-adic algebraic group and let $\mu \in M^1(G)$ be full, then each μ_t is also full, $t > 0$. Suppose (i) holds. Then $\operatorname{Inv}(\mu) = \operatorname{Inv}(\mu_{c^n}) = \tau^n \operatorname{Inv}(\mu)\tau^{-n}$ for all $n \in \mathbf{N}$. Now from Lemma 2.7(ii), $\operatorname{Inv}(\mu) = \bigcap_{t\geq 0} \operatorname{Inv}(\mu_t)$. We show that μ has a unique semistable embedding. If possible, suppose there exists a (ψ, d)-semistable one-parameter semigroup $\{\nu_t\}_{t\geq 0}$ such that $\mu = \nu_1$. Then by Theorem 2.4 and Remark 2.6 we get that $\log c$ and $\log d$ are commensurable. Also, if τ normalises $\operatorname{Inv}(\mu)$, so does its power. Replacing τ and ψ by its suitable powers if necessary, we may assume that μ is (τ, c)-semistable and (ψ, c)-semistable. Then as in Step 2 of proof of Theorem 2.4, we get that for $k_n = [c^{-n}]$, $n \in \mathbf{N}$,

$$\tau^n(\mu)^{[k_n t]} \to \mu_t \quad \text{and} \quad \psi^n(\mu)^{[k_n t]} \to \nu_t\,, \quad t > 0\,.$$

Since $\mu_1 = \nu_1 = \mu$, by the convergence-of-types theorem, $\{\psi^n\tau^{-n}\}$ is relatively compact and all its limit points belong to $\mathrm{Inv}(\mu) = \bigcap_{t\geq 0}\mathrm{Inv}(\mu_t)$. Therefore, from the above equation, we get that $\mu_t = \nu_t$ for each $t > 0$ and hence for $t = 0$. This completes the proof. □

REMARK 2.9. It follows from the proposition that if μ is a full semistable measure on a unipotent p-adic algebraic group G such that $\mathrm{Inv}(\mu)$ is trivial then μ has a unique semistable embedding. This also holds for non-full semistable measures on G if $A = \{\alpha \in \mathrm{Aut}(G(\mu)) \mid \alpha(\mu) = \mu\}$ is trivial. More generally, if μ is a semistable measure on any p-adic Lie group G with $\mathcal{I}(\mu) = \{e\}$ and A as above is trivial then μ has a unique semistable embedding. In view of the above, it will be interesting to find conditions under which $\mathrm{Inv}(\mu)$ is trivial.

PROOF OF THEOREM 2.1. Let μ be a normal semistable measure on a p-adic Lie group G. As in the proof of Lemma 2.5, we may assume that $G = G(\mu)$. Suppose there exist $\tau, \psi \in \mathrm{Aut}(G)$ and $c, d \in]0,1[$ such that μ is both (τ, c)-semistable and (ψ, d)-semistable. That is, there exist continuous one-parameter semigroups $\{\mu_t\}_{t\geq 0}$ and $\{\nu_t\}_{t\geq 0}$ in $M^1(G)$ such that $\mu_1 = \mu = \nu_1$ and $\tau(\mu_t) = \mu_{ct}$ and $\psi(\nu_t) = \nu_{dt}$ for all $t \in \mathbf{R}_+$. We have to show that $\mu_t = \nu_t$ for all t.

For $k_n = [c^{-n}]$, $\tau(\mu^{[k_n t]}) \to \mu_t$, $t \in \mathbf{R}_+^*$. Hence since μ is normal, each μ_t is also normal. Similarly, each ν_t is normal. This implies that $\mu * \tilde{\mu}$ is embeddable in $\{\mu_t * \tilde{\mu}_t\}_{t\geq 0}$ and $\{\nu_t * \tilde{\nu}_t\}_{t\geq 0}$. But since both the one-parameter semigroups consist of symmetric measures and $\mu * \tilde{\mu} = \mu_1 * \tilde{\mu}_1 = \nu_1 * \tilde{\nu}_1$, we have by the uniqueness of symmetric embedding, $\mu_t * \tilde{\mu}_t = \nu_t * \tilde{\nu}_t$ for all t. In particular, $\mu_0 = \nu_0 = \omega_K$, (where $K = \mathcal{I}(\mu)$). If $\mu = \omega_K$, then $\mu_t = \nu_t = \omega_K$ for all t. Suppose, $\mu \neq \omega_K$. By Lemma 2.5, we have $G = K \cdot U$, a semidirect product, where $U = C(\tau) = C(\psi)$, $U = \tilde{U}(\mathbf{Q}_p)$, $\tilde{U}$ is a unipotent p-adic algebraic group, $\tau^0 = \tau|_U$ and $\psi^0 = \psi|_U$ are $\mathbf{Q}_p$-rational morphisms and $\mu_t = \omega_K * \mu_t^{(0)}$ and $\nu_t = \omega_K * \nu_t^{(0)}$, where $\mu_1^{(0)} = \nu_1^{(0)}$ and $\{\mu_t^{(0)}\}_{t\geq 0}$ is (τ^0, c)-semistable and $\{\nu_t^{(0)}\}_{t\geq 0}$ is (ψ^0, d)-semistable with $\mu_0^{(0)} = \nu_0^{(0)} = \delta_e$. Also, $\mu_1^{(0)} = \nu_1^{(0)} \neq \delta_e$ is a normal measure. Now it is enough to prove that $\mu_t^{(0)} = \nu_t^{(0)}$ for all t. Therefore, replacing μ by $\mu_1^{(0)} = \nu_1^{(0)}$ we may assume that $\mu \neq \delta_e$, $G = \tilde{G}(\mathbf{Q}_p)$ is a unipotent p-adic algebraic group and $\tau, \psi \in \mathrm{Aut}(G)$ are contracting automorphisms which are also $\mathbf{Q}_p$-rational. Also, by Theorem 2.4, $\log c$ and $\log d$ are commensurable.

Here, since μ is full on G which is unipotent, $\mathrm{Inv}(\mu)$ is a compact subgroup of $\mathrm{Aut}(G)$ (cf. Theorem 2.3). Let $\mathcal{K} = \bigcap_t \mathrm{Inv}(\mu_t)$. Then $\mathcal{K}$ is a compact subgroup of $\mathrm{Aut}(G)$. Let $T : \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ be defined as follows: $T(\rho) = \tau\rho\tau^{-1}$ for all $\rho \in \mathrm{Aut}(G)$. Then T is a continuous automorphism of $\mathrm{Aut}(G)$ which is a p-adic Lie group. Now by Lemma 2.7, $T(\mathcal{K}) = \mathcal{K}$ and $\mathrm{Inv}(\mu) \subset C_{\mathcal{K}}(T)$. We also have $C_{\mathcal{K}}(T) = \mathcal{K} \cdot C(T)$ (cf. [DSh1]). Since μ is embeddable and G is totally disconnected, we have $G(\mu) = G(\mu * \tilde{\mu})$ and since μ is full so is $\mu * \tilde{\mu}$. Hence $\mathrm{Inv}(\mu * \tilde{\mu}) = \mathcal{H}$ (say) is compact. Since $\mu * \tilde{\mu}$ has a unique semistable embedding, we get by Proposition 2.8 that $T(\mathcal{H}) = \mathcal{H}$. Therefore, $\mathcal{H} \cap C(T) = \{I\}$, but $\mathrm{Inv}(\mu) \subset \mathcal{H}$, and hence $\mathcal{K} \subset \mathcal{H}$ and $\mathrm{Inv}(\mu) \subset \mathcal{H} \cap (\mathcal{K} \cdot C(T)) = \mathcal{K}$. This implies that

$\mathrm{Inv}(\mu) = \mathcal{K}$. Now by Proposition 2.8, μ is embeddable in a unique semistable one-parameter semigroup. $\square$

3. Domain of semistable attraction and semistable measures on unipotent p-adic groups

For a probability measure μ on a locally compact group G, we define $\mathrm{DSSA}(\mu)$, the *domain of semistable attraction of* μ, as follows:

$$\mathrm{DSSA}(\mu) = \{\nu \in M^1(G) \mid \text{there exist } \tau_n \in \mathrm{Aut}(G) \text{ and } k_n \in \mathbf{N} \text{ such that } \tau_n(\nu^{k_n}) \to \mu \text{ and } k_n/k_{n+1} \to c \in]0, 1[\}\,.$$

Note that our definition is slightly different from an earlier definition in [T], as we do not assume that $\{\tau_n(\nu)\}$ is infinitesimal. It is easy to see that for any (τ, c)-semistable measure μ, $\mathrm{DSSA}(\mu)$ is nonempty. For, μ itself belongs to $\mathrm{DSSA}(\mu)$, since $\tau^n(\mu)^{k_n} \to \mu$, where $k_n = [c^{-n}]$, $n \in \mathbf{N}$. Conversely, we have the following.

THEOREM 3.1. *Let μ be an S-full probability measure on a unipotent p-adic algebraic group G such that $\mathcal{I}(\mu) = \{e\}$ and $\mathrm{DSSA}(\mu)$ is nonempty. Then μ is semistable.*

Before proving the theorem let us state and prove several results which will be useful.

LEMMA 3.2. *Let G be a unipotent p-adic algebraic group and let $\mu_n, \mu \in M^1(G)$ be such that $\mu_n \to \mu$. If μ is full (resp. S-full) in $M^1(G)$, then so is μ_n for all large n.*

We need following notations for the proof of the lemma. For a p-adic vector space V isomorphic to $\mathbf{Q}_p^m$ ($m \in \mathbf{N}$), let $\{e_1, \ldots, e_m\}$ be a basis of V. For any $x, y \in V$, we have $x = \sum_{i=1}^m x_i e_i$ and $y = \sum_{i=1}^m y_i e_i$ and we define $\langle x, y\rangle = \sum_{i=1}^m x_i y_i$. It is a continuous bilinear map from V^2 to $\mathbf{Q}_p$. Any one-dimensional projection of V is of the form $y \mapsto \langle x, y\rangle$ for some $x \in V$. Also, $x \mapsto \|x\|_p = |\langle x, x\rangle|_p^{1/2}$ on V defines a norm on V. For $x \in V \setminus \{0\}$, let $V_x = \mathrm{Ker}(y \mapsto \langle x, y\rangle)$; it is a subspace of co-dimension 1 in V. Also, any subspace W of co-dimension 1 in V is of this form, i.e. $W = V_x$ for some $x \in V \setminus \{0\}$. For $\mu \in M^1(V)$ and $x \in V$, let (x, μ) denote the image of μ under the map $y \mapsto \langle x, y\rangle$.

PROOF OF LEMMA 3.2. Let G, μ_n and μ be as in the hypothesis. Since $\mu_n \to \mu$, $\mu_n * \tilde{\mu}_n \to \mu * \tilde{\mu}$. Also, μ is S-full if and only if $\mu * \tilde{\mu}$ is full. Hence it is enough to prove that fullness of μ implies that of μ_n for all large n. Let $\pi : G \to G/[G, G]$ be the natural projection. Then $G/[G, G]$ is an abelian unipotent p-adic algebraic group. It is easy to see that any probability measure ν is full on G if and only if $\pi(\nu)$ is full on $G/[G, G]$, which is isomorphic to a p-adic vector space. Now since $\pi(\mu_n) \to \pi(\mu)$, it is enough to prove the assertion in case G is a p-adic vector space.

Now we may assume $G = V$, an m-dimensional p-adic vector space and $\mathrm{supp}\,\mu$ generates V as a vector space. We fix a basis $\{e_1, \ldots, e_m\}$ for V. If possible, suppose μ_n is not full on V for infinitely many n. Passing to a subsequence if necessary, we get that $\mathrm{supp}\,\mu_n \subset V_n$, where V_n is a proper subspace of co-dimension 1 in V for all n. Let $x_n \in V \setminus \{0\}$ be such that $V_n = V_{x_n} = \{y \in V \mid \langle x_n, y\rangle = 0\}$. Replacing x_n by $x_n/\|x_n\|_p$, we may assume that $\|x_n\|_p = 1$ for all n. Then $(x_n, \mu_n) = \delta_0$ on $\mathbf{Q}_p$ for all n. Here, $\{x_n\}$ is relatively compact, it has a limit point x (say), then $\|x\|_p = 1$. Since $\mu_n \to \mu$, $(x, \mu) = \delta_0$ on $\mathbf{Q}_p$. In particular, $\operatorname{supp}\mu \subset V_x$, which is a subspace of co-dimension 1 in V as $x \neq 0$. This leads to a contradiction as μ is full. Hence μ_n is full for all large n. □

PROPOSITION 3.3. *Let $G = GL_m(\mathbf{Q}_p)$ and let U be the subgroup of G consisting of all upper triangular matrices. Let $\{a_n\}$ be a sequence in G and ψ_n be an inner automorphism defined by $\psi_n(x) = a_n x a_n^{-1}$ for all $x \in G$, $n \in \mathbf{N}$. Let $H = \{x \in G \mid \psi_n(x) \to e\}$ and let $C = \{\nu \in M^1(G) \mid \psi_n(\nu) \to \delta_e\}$, where the identity $e = I$, the identity matrix in $GL_m(\mathbf{Q}_p)$. Then*

(i) *H is a (closed) algebraic subgroup of G and there exists $a \in G$ such that $H \subset aUa^{-1}$ and*
(ii) *for any $\nu \in C$, $\operatorname{supp}\nu$ is contained in H.*

The above proposition can be deduced along the same lines as Theorem 2.1 of [DSh2]; this theorem uses Lemma 2.2 of [DSh2], which is also valid for any locally compact first countable group. Also, instead of the polar decomposition of a_n, one has to use the decomposition $a_n = c_n d_n k_n$, where $c_n, k_n \in GL_m(\mathbf{Z}_p)$, which is compact, where $\mathbf{Z}_p$ is the ring of p-adic integers, and d_n are diagonal matrices which can be chosen to have entries whose p-adic absolute values are in the ascending order (cf. [Ma]), subsequently also, one has to use the p-adic absolute value instead of the real absolute value. We will not repeat the proof here.

THEOREM 3.4. *Let $G = \tilde{G}(\mathbf{Q}_p)$ be a unipotent p-adic algebraic group and let $\{\tau_n\} \subset \operatorname{Aut}(G)$. Let $H = \{x \in G \mid \tau_n(x) \to e\}$ and let $C = \{\nu \in M^1(G) \mid \tau_n(\nu) \to \delta_e\}$. Then H is a (closed) algebraic subgroup of G such that for any $\nu \in C$, $\operatorname{supp}\nu$ is contained in H. In particular, if there exists a $\nu \in C$ which is full on G, then $H = G$ and $\tau_n(\mu) \to \delta_e$ for all $\mu \in M^1(G)$. That is, if C contains a full measure, then $C = M^1(G)$.*

REMARK 3.5. The above theorem is also valid for any Zariski connected semisimple p-adic algebraic group with trivial center. It can also be shown to hold for any Zariski connected p-adic algebraic group $G = \tilde{G}(\mathbf{Q}_p)$ such that the maximal central torus of $\tilde{G}$ is trivial and $\{\tau_n\} \subset \operatorname{Aut}(G)$ are $\mathbf{Q}_p$-rational automorphisms. In both these cases, a similar proof (as given below) works and H will be a unipotent algebraic subgroup of G.

PROOF OF THEOREM 3.4. Since G is unipotent, any continuous automorphism of G is $\mathbf{Q}_p$-rational and the group $\operatorname{Aut}(G)$ is also a group of rational points of a p-adic algebraic group and the action of $\operatorname{Aut}(G)$ on G is $\mathbf{Q}_p$-rational (cf. [Sh2], Theorems 2.1, 3.1). As in the proof of Main Theorem in [Sh2], we can form a semidirect product $L = \operatorname{Aut}(G)\cdot G$, $L = \tilde{L}(\mathbf{Q}_p)$ is a p-adic algebraic group, with the group operation $(\tau, g)(\tau', g') = (\tau\tau', g\tau(g'))$, for $\tau, \tau' \in \operatorname{Aut}(G)$ and $g, g' \in G$. That is, the action of $\operatorname{Aut}(G)$ on G is given by $\tau g \tau^{-1} = \tau(g)$ for all $\tau \in \operatorname{Aut}(G)$ and $g \in G$. Also, $L \subset GL_m(\mathbf{Q}_p)$ for some $m \in \mathbf{N}$.

We have $H = \{x \in G \mid \tau_n(x) \to e\} = \{x \in G \mid \tau_n x \tau_n^{-1} \to e \text{ in } L\}$. This implies $H = H' \cap G$, where $H' = \{x \in GL_m(\mathbf{Q}_p) \mid \tau_n x \tau_n^{-1} \to e\}$, here $e = I$, the identity matrix. By Proposition 3.3 (i), H' is an algebraic subgroup of $GL_m(\mathbf{Q}_p)$ and hence H is an algebraic subgroup of G. In particular, it is a closed subgroup of G. Since G is closed in $GL_m(\mathbf{Q}_p)$, $C = M^1(G) \cap C'$, where $C' = \{\nu \in M^1(GL_m(\mathbf{Q}_p)) \mid \tau_n(\nu) \to \delta_e\}$. Let $\nu \in C$. Using Proposition 3.3 (ii), we get that $\operatorname{supp} \nu \subset H' \cap G = H$. Now, assume that ν is full on G. Then since H is an algebraic subgroup of G, we get that $G = \tilde{G}(\nu) \subset H$ and hence $G = H$. Moreover, from the definition of H, it is clear that for any $\mu \in M^1(G)$, $\tau_n(\mu) \to \delta_e$, i.e. $C = M^1(G)$. □

For any $\alpha \in M^1(G)$, let $F(\alpha)$ be the set of (two-sided) factors of α, i.e. $F(\alpha) = \{\beta \in M^1(G) \mid \beta * \gamma = \gamma * \beta = \alpha \text{ for some } \gamma \in M^1(G)\}$.

LEMMA 3.6. *Let G be a p-adic algebraic group. Let $\{\nu_n\} \subset M^1(G)$ and $\mu \in M^1(G)$ be such that μ is full and $\nu_n^{k_n} \to \mu$ for some $\{k_n\} \subset \mathbf{N}$. Then we have the following*:

(i) *Let Z be the center of G and let $\pi : G \to G/Z$ be the natural projection. For $\mathcal{A} = \{\nu_n^m \mid m \leq k_n, n \in \mathbf{N}\}$, $\pi(\mathcal{A})$ is relatively compact.*

(ii) *$\{\nu_n * \tilde{\nu}_n\}$ is relatively compact and all its limit points are supported on $\mathcal{I}(\mu)$.*

PROOF. As μ is full, Theorem 4.1 of [Sh3] implies (i), and it also implies that $\{\nu_n * \delta_{z_n}\}$ is relatively compact for some sequence $\{z_n\} \subset Z$. For every $m \in \mathbf{N}$, $\{\nu_n^m * \delta_{z_n^m}\}_{n \in \mathbf{N}}$, and hence $\{\nu_n^m * \tilde{\nu}_n^m\}_{n \in \mathbf{N}}$ is relatively compact. Let λ be a limit point of $\{\nu_n * \delta_{z_n}\}$. Then the above implies that $\lambda^m \in F(\mu)$, $m \in \mathbf{N}$. Again by Theorem 4.1 of [Sh3], we get that $\{\lambda^m * \delta_{z'_m}\}$ is relatively compact for some sequence $\{z'_m\} \subset Z$; let β be any limit point of it. It is clear that $\beta \in F(\mu)$ and $\beta^2 \in F(\beta)$. This implies that $\beta = \omega_H * \delta_x = \delta_x * \omega_H$ for some compact subgroup $H \subset \mathcal{I}(\mu)$. We also have $\lambda \in F(\beta)$. Therefore, $\operatorname{supp} \lambda \subset Hy = yH$ for some $y \in \operatorname{supp} \lambda$ and hence $\operatorname{supp}(\lambda * \tilde{\lambda}) \subset \mathcal{I}(\mu)$. Since any limit point of $\{\nu_n * \tilde{\nu}_n\}$ is of the form $\lambda * \tilde{\lambda}$ for some λ as above, we get that all limit points of $\{\nu_n * \tilde{\nu}_n\}$ are supported on $\mathcal{I}(\mu)$. This proves (ii). □

PROOF OF THEOREM 3.1. Let μ be an S-full probability measure on a unipotent p-adic algebraic group G such that $\mathcal{I}(\mu) = \{e\}$ and $\mathrm{DSSA}(\mu) \neq \emptyset$. Then there exist sequences $\{\tau_n\} \subset \mathrm{Aut}(G)$, $\{k_n\} \subset \mathbf{N}$ and a measure $\nu \in M^1(G)$ such that $\tau_n(\nu)^{k_n} \to \mu$ and $k_n/k_{n+1} \to c \in]0, 1[$.

Since μ is S-full, μ is also full and by Lemma 3.6, $\tau_n(\nu * \tilde{\nu}) \to \delta_e$ as $\mathcal{I}(\mu) = \{e\}$. By Lemma 3.2, for all large n, $\tau_n(\nu)^{k_n}$ is S-full, and hence so is $\tau_n(\nu)$. This implies that $\tau_n(\nu * \tilde{\nu})$ is full. Also each τ_n is $\mathbf{Q}_p$-rational (cf. [Sh2], Theorem 2.1). Therefore, $\nu * \tilde{\nu}$ is also full. Since $\tau_n(\nu * \tilde{\nu}) \to \delta_e$, by Theorem 3.4, $\tau_n(x) \to e$ for all $x \in G$, and hence $\tau_n(\nu) \to \delta_e$.

Now we have that $\tau_n(\nu)^{k_n} \to \mu$, $\tau_n(\nu) \to \delta_e$, $k_n/k_{n+1} \to c \in]0, 1[$ (this condition has not been used so far), and μ is full on a unipotent p-adic algebraic group. Hence the assertion follows from Theorem 4.6 of [T]. □

REMARK 3.7. 1. The above theorem generalises Theorem 4.6 of [T] in the case of S-full probability measures μ with $\mathcal{I}(\mu) = \{e\}$, as we do not assume that $\{\tau_n(\nu)\}$ is infinitesimal in the hypothesis but derive it as a consequence.

2. In view of Theorem 4.6 of [T] stated for the semisimple group case, we would like to note that there does not exist any nontrivial (non-idempotent) full semistable measure on $G = \tilde{G}(\mathbf{Q}_p)$, where $\tilde{G}$ is any semisimple p-adic algebraic group. That is, if μ is a full semistable measure, on such a group G, embeddable in (τ, c)-semistable $\{\mu_t\}_{t\geq 0} \subset M^1(G)$, with $\mu_0 = \omega_K$, then as mentioned earlier $\mu = \mu_1$ is supported on $C_K(\tau) = K \cdot C(\tau)$ (cf. [DSh1], Theorem 3.1), and hence $C_K(\tau)$ is Zariski dense in G. In particular, this implies that $C(\tau)$ is normal in G, but $C(\tau)$ is a unipotent algebraic subgroup of G (cf. [W], Theorem 3.5). Since G is semisimple, the above implies that $C(\tau) = \{e\}$, hence $G(\mu) = K$ and $\mu = \omega_K$, with $\tau(K) = K$.

We now give examples to show that in the hypothesis of Theorem 3.1, S-fullness of μ can not be replaced by fullness of μ, and also the condition that $\mathcal{I}(\mu) = \{e\}$ is necessary.

EXAMPLE 1. Let $G = \mathbf{Q}_p$. Let $\nu = \delta_x$ for some $x \in G$ such that $x \neq 0$, Take $\tau_n = I$, the identity in $\mathrm{Aut}(G)$, and $k_n = p^n + 1$ for all $n \in \mathbf{N}$. Then $k_n/k_{n+1} \to 1/p$, $\tau_n(\nu)^{k_n} \to \delta_x$, hence $\mathrm{DSSA}(\delta_x) \neq \emptyset$ but δ_x is not semistable. Here $\mathcal{I}(\delta_x) = \{0\}$, δ_x is full but not S-full. □

EXAMPLE 2. Let $G = \mathbf{Q}_p$ and $\nu = \omega_H * \delta_x$ for some compact open subgroup H of G and some $x \in G \setminus H$. So μ is S-full. Let $\{\tau_n\}$ and $\{k_n\}$ be as in Example 1. Then $k_n/k_{n+1} \to 1/p$, $\tau_n(\nu)^{k_n} = \omega_H * \delta_x$ for all large n, hence $\mathrm{DSSA}(\omega_H * \delta_x) \neq \emptyset$ but $\omega_H * \delta_x$ is not semistable as it is not even embeddable. Note that $\mathcal{I}(\omega_H * \delta_x) = H \neq \{0\}$. □

Now we state a result comparing semistable measures and measures with nonempty domain of semistable attraction on any totally disconnected locally compact group.

THEOREM 3.8. *Let G be any totally disconnected locally compact group and $\mu \in M^1(G)$. Then the following are equivalent*:

(i) *μ is (τ, c)-semistable.*

(ii) *There exist $\{\tau_n\} \subset \mathrm{Aut}(G)$, $\nu \in M^1(G)$, $\{k_n\} \subset \mathbf{N}$ such that $\tau_n(\nu^{k_n}) \to \mu$, $k_n/k_{n+1} \to c \in]0, 1[$ (i.e. $\mathrm{DSSA}(\mu) \neq \emptyset$), with the additional properties that $\tau_{n+1}\tau_n^{-1} \to \tau$ and $\tau_n(\nu) \to \omega_H$ for some compact subgroup H of G.*

PROOF. "(i) ⇒ (ii)" is obvious. Now we assume (ii) and show that (i) holds. By Theorem 2.1 of [Sh4], the set $\mathcal{A} = \{\tau_n(\nu)^m \mid m \leq k_n, n \in \mathbf{N}\}$ is relatively compact. Now from Theorem 3.6 of [T], μ is embeddable in a continuous one-parameter semigroup $\{\mu_t\}_{t\geq 0} \subset M^1(G)$ such that, for some sequence $\{n_j\} \subset \mathbf{N}$,

$$\lim_{j\to\infty} \tau_{n_j}(\nu)^{[k_{n_j}t]} \to \mu_t \,, \tag{1}$$

uniformly on compact subsets of $]0, \infty[$.

Let $m \in \mathbf{N}$ be fixed. Clearly, $\tau_n \tau_{n-m}^{-1} \to \tau^m$ and $k_n/k_{n+m} \to c^m$. Let $a_j = k_{n_j - m}/k_{n_j}$, then $a_j \to c^m$. For any fixed $r \in \mathbf{N}$,

$$\lim_{j\to\infty} \tau_{n_j}(\nu)^{rk_{n_j-m}} = \lim_{n\to\infty} \tau_{n_j}(\nu)^{[rk_{n_j}a_j]} = \mu_{rc^m}\,, \tag{2}$$

from the uniform convergence on compact sets of $]0, \infty[$ in (1) above (see the proof of Theorem 4.6 of [T]). Also,

$$\lim_{n\to\infty} \tau_{n_j}(\nu)^{rk_{n_j-m}} = \lim_{n\to\infty} \tau_{n_j}\tau_{n_j-m}^{-1}(\tau_{n_j-m}(\nu)^{rk_{n_j-m}}) = \tau^m(\mu)^r\,. \tag{3}$$

From (2) and (3), for all $m, r \in \mathbf{N}$, we have that $\tau^m(\mu)^r = \mu_{rc^m}$, and hence $\tau(\mu_{rc^m}) = \mu_{rc^{m+1}}$. Let $M = \{rc^m \mid r, m \in \mathbf{N}\}$. Then $\tau(\mu_t) = \mu_{ct}$ for all $t \in M$. Since M is dense in $\mathbf{R}_+$ and τ is continuous we get that $\tau(\mu_t) = \mu_{ct}$ for all $t \in \mathbf{R}_+$, i.e. $\{\mu_t\}_{t\geq 0}$ is (τ, c)-semistable with $\mu_1 = \mu$. $\square$

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