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The Decomposability of Z_2-Manifolds in Cut-and-Paste Equivalence

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Introduction

All manifolds considered here are unoriented compact smooth manifolds with or without boundary. G denotes a finite abelian group, and G-manifolds mean manifolds with smooth G-action.

Let $m \geq 0$ be an integer. Let P and Q be m-dimensional compact G-manifolds with boundary, and $\varphi : \partial P \to \partial Q$ be a G-diffeomorphism. Pasting P and Q along the boundary by φ, we obtain a closed G-manifold $P \cup_\varphi Q$ after rounding a corner. If $\psi : \partial P \to \partial Q$ is a second G-diffeomorphism, we obtain a second closed G-manifold $P \cup_\psi Q$. The two closed G-manifolds $P \cup_\varphi Q$ and $P \cup_\psi Q$ are said to be *obtained from each other by cutting and pasting* (*S*chneiden und *K*leben in German). Two m-dimensional closed G-manifolds M and N are said to be *cut-and-paste equivalent*, or *SK-equivalent* to each other, if there is an m-dimensional closed G-manifold L such that the disjoint union $M + L$ is obtained from $N + L$ by a finite sequence of cuttings and pastings. This is an equivalence relation on $\mathfrak{M}_m^G$, the set of m-dimensional closed G-manifolds. Denote by $[M]$ the equivalence class represented by M, and by $\mathfrak{M}_m^G/SK$ the quotient set of $\mathfrak{M}_m^G$ by the SK-equivalence. $\mathfrak{M}_m^G/SK$ becomes a semi-group with the addition induced from the disjoint union of G-manifolds. The Grothendieck group of $\mathfrak{M}_m^G/SK$ is called the *SK-group* of m-dimensional closed G-manifolds and is denoted by SK_m^G. The direct sum $SK_*^G = \bigoplus_{m \geq 0} SK_m^G$ becomes a graded ring with multiplication induced from cartesian product, with diagonal G-action, of G-manifolds.

In Komiya [13] we dealt with the case in which G is of odd order, and obtained a necessary and sufficient condition for that, for a given $u \in SK_m^G$ and an integer $t \geq 0$, u is divisible by t, i.e., $u = tv$ for some $v \in SK_m^G$.

In the present paper we will deal with the case of $G = \mathbf{Z}_2$, the cyclic group of order 2. Using a result in Komiya [12], we will obtain a condition for a closed $\mathbf{Z}_2$-manifold M to decompose in the sense of SK-equivalence into the product $N \times L$ of two closed $\mathbf{Z}_2$-manifolds N and L. In fact, for given $u \in SK_m^{\mathbf{Z}_2}$ and $v \in SK_n^{\mathbf{Z}_2}$ with $n \leq m$, we will obtain a necessary

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and sufficient condition for the existence of an element $w \in SK_{m-n}^{\boldsymbol{Z}_2}$ such that $u = vw$ in $SK_*^{\boldsymbol{Z}_2}$.

NOTE. The SK-group of (nonequivariant) closed manifolds was introduced and observed by Karras, Kreck, Neumann and Ossa [8]. We refer to this book for basic properties and general results on the SK-group. The notion of this group naturally extends to equivariant manifolds for any compact Lie group. For the case of finite abelian group we also refer to Kosniowski's book [16]. Hara [1], [2], [3], Hara and Koshikawa [4], [5], [6], Hermann and Kreck [7], Komiya [9], [10], [11], Koshikawa [14], [15] are also relevant to our present work.

1. Linear equations

Since $SK_n^{\boldsymbol{Z}_2}$ is the Grothendieck group of $\mathfrak{M}_n^{\boldsymbol{Z}_2}/SK$, any element $v \in SK_n^{\boldsymbol{Z}_2}$ is written in the form $v = [M] - [N]$ for some M and $N \in \mathfrak{M}_n^{\boldsymbol{Z}_2}$. Let $M^{\boldsymbol{Z}_2}$ denote the fixed point set of M, and $M_i^{\boldsymbol{Z}_2}$ the i-dimensional component of $M^{\boldsymbol{Z}_2}$ for $0 \le i \le n$. Then $M^{\boldsymbol{Z}_2}$ is the disjoint union of $M_i^{\boldsymbol{Z}_2}$, i.e., $M^{\boldsymbol{Z}_2} = \coprod_{0 \le i \le n} M_i^{\boldsymbol{Z}_2}$. Define $\chi(v) = \chi(M) - \chi(N)$, where $\chi(\ )$ denotes the Euler characteristic. For any integer i, define

$$\chi_i(v) = \begin{cases} \chi(M_i^{\boldsymbol{Z}_2}) - \chi(N_i^{\boldsymbol{Z}_2}) & 0 \le i \le n \\ 0 & \text{otherwise}. \end{cases}$$

$\chi(v)$ and $\chi_i(v)$ are well-defined, namely independent of representatives M and N.

For given two elements $u \in SK_m^{\boldsymbol{Z}_2}$ and $v \in SK_n^{\boldsymbol{Z}_2}$ $(n \le m)$, we consider the problem: When does v divide u, i.e., $u = vw$ for some element $w \in SK_{m-n}^{\boldsymbol{Z}_2}$? To consider this problem, define the following $(m+2)$-tuples of integers:

$$\begin{aligned} \boldsymbol{a}(v) &= (\chi(v), 0, 0, \ldots, 0), \\ \boldsymbol{a}_j(v) &= (\underbrace{0, \ldots, 0}_{j+1}, \chi_0(v), \chi_1(v), \ldots, \chi_{m-j}(v)) \end{aligned}$$

for $0 \le j \le m-n$. These vectors give an $(m+2) \times (m-n+2)$-matrix

$$A(v) = (\boldsymbol{a}(v)^t, \boldsymbol{a}_0(v)^t, \boldsymbol{a}_1(v)^t, \ldots, \boldsymbol{a}_{m-n}(v)^t),$$

where $\boldsymbol{a}(v)^t$, $\boldsymbol{a}_j(v)^t$ denote the column vectors corresponding to $\boldsymbol{a}(v)$, $\boldsymbol{a}_j(v)$, respectively. Then we get a system of linear equations with integer coefficients and with indeterminates $x, x_0, x_1, \ldots, x_{m-n}$:

$$A(v) \begin{pmatrix} x \\ x_0 \\ x_1 \\ \vdots \\ x_{m-n} \end{pmatrix} = \begin{pmatrix} \chi(u) \\ \chi_0(u) \\ \chi_1(u) \\ \vdots \\ \chi_m(u) \end{pmatrix} \tag{$*$}$$

A solution of this system of linear equations, $(x, x_0, x_1, \dots, x_{m-n}) = (b, b_0, b_1, \dots, b_{m-n})$, is called *admissible*, if the following (i)–(iv) are satisfied:

(i) $b, b_0, b_1, \dots, b_{m-n}$ are all integers,

(ii) $b = 0$ if $m - n$ is odd,

(iii) $b_i = 0$ if i is odd $(0 \leq i \leq m-n)$, and

(iv) $b \equiv \sum_{i=0}^{m-n} b_i \mod 2$.

2. Lemma

In this section we will recall from Komiya [12] the definition of the SK-group of families of submanifolds.

Let P be an m-dimensional compact manifold. For any i with $0 \leq i \leq m$, let P_i be an i-dimensional compact submanifold of P such that $\partial P_i = P_i \cap \partial P$ and $P_i \cap P_j = \emptyset$ if $i \neq j$. We write $\tilde{P} = (P; P_m, P_{m-1}, \dots, P_0)$ for a family of such submanifolds, and call this an *m-dimensional family*. This is modeled on a family of a $\boldsymbol{Z}_2$-manifold and its fixed point sets. For another such family $\tilde{Q} = (Q; Q_m, Q_{m-1}, \dots, Q_0)$, let $\varphi : \partial P \to \partial Q$ be a diffeomorphism which restricts to a diffeomorphism $\varphi_i = \varphi|\partial P_i : \partial P_i \to \partial Q_i$ for any i. Then we obtain a family of submanifolds of a closed manifold

$$\tilde{P} \cup_\varphi \tilde{Q} = (P \cup_\varphi Q; P_m \cup_{\varphi_m} Q_m, \dots, P_0 \cup_{\varphi_0} Q_0).$$

Let $\psi : \partial P \to \partial Q$ be another diffeomorphism which restricts to a diffeomorphism $\psi_i : \partial P_i \to \partial Q_i$ for any i. We obtain another family

$$\tilde{P} \cup_\psi \tilde{Q} = (P \cup_\psi Q; P_m \cup_{\psi_m} Q_m, \dots, P_0 \cup_{\psi_0} Q_0).$$

The two families $\tilde{P} \cup_\varphi \tilde{Q}$ and $\tilde{P} \cup_\psi \tilde{Q}$ are said to be *obtained from each other by cutting and pasting*. Let $\mathfrak{M}_m^{\mathcal{F}}$ be the set of m-dimensional family of submanifolds of closed manifolds. Two families $\tilde{M}, \tilde{N} \in \mathfrak{M}_m^{\mathcal{F}}$ are said to be *SK-equivalent* to each other, if there is an $\tilde{L} \in \mathfrak{M}_m^{\mathcal{F}}$ such that $\tilde{M} + \tilde{L}$ is obtained from $\tilde{N} + \tilde{L}$ by a finite sequence of cuttings and pastings, where $\tilde{M} + \tilde{L}$ is the disjoint union of $\tilde{M}$ and $\tilde{L}$, i.e.,

$$\tilde{M} + \tilde{L} = (M + L; M_m + L_m, \dots, M_0 + L_0)$$

for $\tilde{M} = (M; M_m, \dots, M_0)$ and $\tilde{L} = (L; L_m, \dots, L_0)$. The quotient set $\mathfrak{M}_m^{\mathcal{F}}/SK$ by this SK-equivalence becomes a semigroup with the addition induced from the disjoint union of families. The *SK-group of m-dimensional families of submanifolds* is defined as the Grothendieck group of $\mathfrak{M}_m^{\mathcal{F}}/SK$ and is denoted by $SK_m^{\mathcal{F}}$. Any element $x \in SK_m^{\mathcal{F}}$ is written in the form $x = [\tilde{M}] - [\tilde{N}]$ for some $\tilde{M} = (M; M_m, \dots, M_0)$, $\tilde{N} = (N; N_m, \dots, N_0) \in \mathfrak{M}_m^{\mathcal{F}}$. Define $\chi(x) = \chi(M) - \chi(N)$ and $\chi_i(x) = \chi(M_i) - \chi(N_i)$ for $0 \leq i \leq m$.

We have a natural correspondence $\mathfrak{M}_m^{\boldsymbol{Z}_2} \to \mathfrak{M}_m^{\mathcal{F}}$ which assigns to a $\boldsymbol{Z}_2$-manifold $M \in \mathfrak{M}_m^{\boldsymbol{Z}_2}$ the family $(M; M_m^{\boldsymbol{Z}_2}, \ldots, M_0^{\boldsymbol{Z}_2}) \in \mathfrak{M}_m^{\mathcal{F}}$. This induces a homomorphism $\eta : SK_m^{\boldsymbol{Z}_2} \to SK_m^{\mathcal{F}}$.

The following lemma is proved in Komiya [12, Theorem 4.2].

LEMMA 2.1. *An element $x \in SK_m^{\mathcal{F}}$ is in the image of η if and only if $\chi(x) \equiv \sum_{i=0}^m \chi_i(x) \mod 2$*

3. Main result

The main result in this paper is the following:

THEOREM 3.1. *For $u \in SK_m^{\boldsymbol{Z}_2}$ and $v \in SK_n^{\boldsymbol{Z}_2}$ $(n \leq m)$, there exists $w \in SK_{m-n}^{\boldsymbol{Z}_2}$ such that $u = vw$ in $SK_*^{\boldsymbol{Z}_2}$, if and only if the system of linear equations $(*)$ has an admissible solution.*

PROOF. Assume $u = vw$. Then we see $\chi(u) = \chi(v)\chi(w)$ and $\chi_i(u) = \sum_{i=j+k} \chi_j(v)\chi_k(w)$. This implies

$$(x, x_0, x_1, \ldots, x_{m-n}) = (\chi(w), \chi_0(w), \chi_1(w), \ldots, \chi_{m-n}(w))$$

is an integral solution for the equation $(*)$. Moreover, this is admissible from the facts that the Euler characteristic of an odd-dimensional closed manifold is zero and that $\chi(M) \equiv \sum_{i\geq 0} \chi(M_i^{\boldsymbol{Z}_2})(= \chi(M^{\boldsymbol{Z}_2})) \mod 2$ for a closed $\boldsymbol{Z}_2$-manifold M.

Conversely, assume that the equation $(*)$ has an admissible solution $(b, b_0, b_1, \ldots, b_{m-n})$. We define an $(m-n)$-dimensional families $\tilde{L}$ and $\tilde{L}_i$ $(0 \leq i \leq m-n)$ as follows:

$$\begin{aligned}\tilde{L} &= (RP^{m-n}; \emptyset, \emptyset, \ldots, \emptyset)\,,\\ \tilde{L}_i &= (RP^{m-n}; L_{i,m-n}, L_{i,m-n-1}, \ldots, L_{i,0})\,,\end{aligned}$$

where RP^{m-n} is an $(m-n)$-dimensional real projective space, and

$$L_{i,j} = \begin{cases} RP^i & \text{for } j = i \\ \emptyset & \text{otherwise}\,, \end{cases}$$

where RP^i is considered as a canonically imbedded submanifold of RP^{m-n}. These families give classes $[\tilde{L}]$ and $[\tilde{L}_i]$ in $SK_{m-n}^{\mathcal{F}}$. Define $\tilde{w} \in SK_{m-n}^{\mathcal{F}}$ as follows:

$$\tilde{w} = b[\tilde{L}] + \sum_{i=0}^{m-n} b_i[\tilde{L}_i]\,.$$

Then

$$\chi(\tilde{w}) - \sum_{i=0}^{m-n} \chi_i(\tilde{w}) = b\chi(RP^{m-n}) - \sum_{i=0}^{m-n} b_i \chi(RP^i) = b - \sum_{i=0}^{m-n} b_i \equiv 0 \mod 2\,,$$

since $(b, b_0, b_1, \dots, b_{m-n})$ is admissible and $\chi(RP^i) = 0$ or 1. From Lemma 2.1 we have an element $w \in SK_{m-n}^{\boldsymbol{Z}_2}$ such that $\eta(w) = \tilde{w}$. Then we see $\chi(w) = \chi(\tilde{w}) = b$ and $\chi_i(w) = \chi_i(\tilde{w}) = b_i$ $(0 \le i \le m-n)$. Considering the product $vw \in SK_m^{\boldsymbol{Z}_2}$ of v and w, we have

$$\begin{aligned} \chi(vw) &= \chi(v)\chi(w) = \chi(v)b = \chi(u), \text{ and} \\ \chi_i(vw) &= \sum_{i=j+k} \chi_j(v)\chi_k(w) = \sum_{i=j+k} \chi_j(v) b_k = \chi_i(u) \ \ (0 \le i \le m)\,. \end{aligned}$$

This shows from Kosniowski [16, Corollary 5.3.7] that $vw = u$ in $SK_m^{\boldsymbol{Z}_2}$. □

4. Corollaries and remarks

Let SK_n be the SK-group of n-dimensional closed manifolds, i.e., $SK_n = SK_n^{\{1\}}$, where $\{1\}$ is the trivial group. SK_n is canonically identified with a subgroup of $SK_n^{\boldsymbol{Z}_2}$. Under this identification, for $v \in SK_n^{\boldsymbol{Z}_2}$ we see that

$$v \in SK_n \Leftrightarrow \chi_0(v) = \chi_1(v) = \cdots = \chi_{n-1}(v) = 0 \text{ and } \chi_n(v) = \chi(v)\,.$$

Applying Theorem 3.1 to the case of $v \in SK_n (\subset SK_n^{\boldsymbol{Z}_2})$, we obtain

COROLLARY 4.1. *Given $u \in SK_m^{\boldsymbol{Z}_2}$ and $v \in SK_n$ $(n \le m)$, v divides u in $SK_*^{\boldsymbol{Z}_2}$, i.e., there exists $w \in SK_{m-n}^{\boldsymbol{Z}_2}$ such that $u = vw$ in $SK_*^{\boldsymbol{Z}_2}$, if and only if the following conditions* (i)–(iii) *are satisfied*:

(i) $\chi_0(u) = \chi_1(u) = \cdots = \chi_{n-1}(u) = 0$,
(ii) $\chi(u), \chi_n(u), \chi_{n+1}(u), \dots, \chi_m(u)$ *are all multiples of* $\chi(v)$, *and*
(iii) $\chi(u) \equiv \sum_{i=0}^{m} \chi_i(u) \mod 2\chi(v)$.

PROOF. For $u \in SK_m^{\boldsymbol{Z}_2}$ and $v \in SK_n$, the system of equations $(*)$ reduces to

$$\left\{ \begin{aligned} \chi(v)x &= \chi(u) \\ 0 &= \chi_0(u) \\ &\ \vdots \\ 0 &= \chi_{n-1}(u) \\ \chi(v)x_0 &= \chi_n(u) \\ &\ \vdots \\ \chi(v)x_{m-n} &= \chi_m(u)\,. \end{aligned} \right.$$

We see that the conditions (i)–(iii) are necessary and sufficient for the above equations to have an admissible solution. Hence Theorem 3.1 implies Corollary 4.1. □

REMARK 4.2. When G is a finite abelian group of odd order, in Komiya [13, Theorem 4.2] we obtained a necessary and sufficient condition for that $u \in SK_m^G$ is divisible by an integer $t \geq 0$. If we apply Corollary 4.1 to the case of $v = t \in SK_0$, we obtain a corresponding result for the case $G = \boldsymbol{Z}_2$.

REMARK 4.3. Let M be an m-dimensional closed G-manifold, G a finite abelian group of odd order. It is shown in Komiya [13, Theorem 7.1] that M is equivariantly fibred over the circle S^1 within a cobordism class, i.e., M is equivariantly cobordant to the total space of a G-fibration over S^1 such that the G-action takes place within the fibres, if and only if $[M] \in SK_m^G$ is divisible by 2. When $G = \boldsymbol{Z}_2$, for a closed $\boldsymbol{Z}_2$-manifold M to be equivariantly fibred over S^1 within a cobordism class it is not necessary that $[M] \in SK_m^{\boldsymbol{Z}_2}$ is divisible by 2. Indeed, a closed free $\boldsymbol{Z}_2$-manifold M is equivariantly fibred over S^1 within a cobordism class, but Theorem 3.1 (or Corollary 4.1) shows that $[M] \in SK_m^{\boldsymbol{Z}_2}$ is not divisible by 2 if $\chi(M) \not\equiv 0 \mod 4$. See Hara [3] for a necessary and sufficient condition for a closed $\boldsymbol{Z}_{2^r}$-manifold to be equivariantly fibred over S^1 within a cobordism class. Also see Hermann and Kreck [7] for oriented $\boldsymbol{Z}_2$-manifolds.

Finally we consider the SK-group of n-dimensional closed free $\boldsymbol{Z}_2$-manifolds, which is denoted by $SK_n^{\boldsymbol{Z}_2}(free)$. This is regarded as the subgroup of $SK_n^{\boldsymbol{Z}_2}$ consisting of elements $v \in SK_n^{\boldsymbol{Z}_2}$ such that $\chi_0(v) = \chi_1(v) = \cdots = \chi_n(v) = 0$. Applying Theorem 3.1 to the case of $v \in SK_n^{\boldsymbol{Z}_2}(free)$ $(\subset SK_n^{\boldsymbol{Z}_2})$, we obtain

COROLLARY 4.4. *Given $u \in SK_m^{\boldsymbol{Z}_2}$ and $v \in SK_n^{\boldsymbol{Z}_2}(free)$ $(\subset SK_n^{\boldsymbol{Z}_2})$, v divides u in $SK_*^{\boldsymbol{Z}_2}$, i.e., there exists $w \in SK_{m-n}^{\boldsymbol{Z}_2}$ such that $u = vw$ in $SK_*^{\boldsymbol{Z}_2}$, if and only if $\chi_0(u) = \chi_1(u) = \cdots = \chi_m(u) = 0$ and $\chi(u)$ is a multiple of $\chi(v)$.*

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