On maximal Albanese dimensional varieties

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Abstract: We prove that every projective kawamata log terminal pair with maximal Albanese dimension has a good minimal model. We also give an affirmative answer to Ueno's problem on subvarieties of Abelian varieties.

Key words: Minimal model; abundance theorem; rational curves; abelian varieties; maximal Albanese dimensional varieties.

1. Introduction. In this short paper, we prove that every projective klt pair with maximal Albanese dimension has a good minimal model.

Theorem 1.1 (Theorem 4.3)**.** *Let (X, Δ) be a projective kawamata log terminal pair. Assume that X has maximal Albanese dimension. Then (X, Δ) has a good minimal model.*

We also give an affirmative answer to Ueno's problem on subvarieties of Abelian varieties.

Theorem 1.2 (Corollary 5.2)**.** *Let W be a submanifold of a complex torus T with $\kappa(W, K_W) = \dim W$. Then K_W is ample.*

A key ingredient of this paper is a simple fact that there are no rational curves on an Abelian variety. Our arguments depend on an observation that there exists a $(K_X + \Delta)$-negative rational curve on X whenever $K_X + \Delta$ is not nef.

We will work over $\mathbf{C}$, the field of complex numbers, throughout this paper. We will freely use the standard notations in [F3] and [U].

2. Preliminaries.

Notation. For a proper birational morphism $f : X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

Let us recall the definition of maximal Albanese dimensional varieties.

Definition 2.1. Let X be a smooth projective variety. Let $\mathrm{Alb}(X)$ be the Albanese variety of X and let $\alpha : X \to \mathrm{Alb}(X)$ be the corresponding Albanese map. We say that X has *maximal Albanese dimension* if $\dim \alpha(X) = \dim X$.

Remark 2.2. A smooth projective variety X has maximal Albanese dimension if and only if the cotangent bundle of X is generically generated by its global sections, that is,

$$H^0(X, \Omega^1_X) \otimes \mathcal{O}_X \to \Omega^1_X$$

is surjective at the generic point of X. It can be checked without any difficulties.

We note that the notion of maximal Albanese dimension is birationally invariant. So, we can define the notion of maximal Albanese dimension for singular varieties as follows:

Definition 2.3. Let X be a projective variety. We say that X has *maximal Albanese dimension* if there is a resolution $\pi : \overline{X} \to X$ such that $\overline{X}$ has maximal Albanese dimension.

The following lemma is almost obvious by the definition of maximal Albanese dimensional varieties and the basic properties of Albanese mappings. We leave the proof for the reader's exercise.

Lemma 2.4. *Let X be a projective variety with maximal Albanese dimension. Let $\pi : \overline{X} \to X$ be a resolution and let $\alpha : \overline{X} \to \mathrm{Alb}(\overline{X})$ be the Albanese mapping. Let Y be a subvariety of X. Assume that $Y \not\subset \pi(\mathrm{Exc}(\pi) \cup \mathrm{Exc}(\beta))$, where $\beta : \overline{X} \to V$ is the Stein factorization of $\alpha : \overline{X} \to \alpha(\overline{X})$. Then Y has maximal Albanese dimension.*

Let us recall some basic definitions.

Definition 2.5 (Iitaka's D-dimension and numerical D-dimension)**.** Let X be a normal projective variety and let D be a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor. Assume that mD is Cartier for a positive integer m. Let

$$\Phi_{|tmD|} : X \dashrightarrow \mathbf{P}^{\dim |tmD|}$$

be rational mappings given by linear systems $|tmD|$

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for positive integers t. We define *Iitaka's D-dimension* as follows:

$$\kappa(X, D) = \max_{t>0} \dim \Phi_{|tmD|}(X)$$

if $|tmD| \neq \emptyset$ for some $t > 0$. When $|tmD| = \emptyset$ for every $t > 0$, we put $\kappa(X, D) = -\infty$. In case D is nef, we can also define the *numerical D-dimension*

$$\nu(X, D) = \max\{e \mid D^e \not\equiv 0\},$$

where $\equiv$ denotes *numerical equivalence*. We note that $\nu(X, D) \geq \kappa(X, D)$ holds.

In this paper, we adopt the following definition of *minimal models* for klt pairs.

Definition 2.6 (Minimal models for klt pairs)**.** Let $f : X \to S$ be a projective morphism of normal quasi-projective varieties. Suppose that (X, Δ) is klt and let $\phi : X \dashrightarrow X'$ be a birational map of normal quasi-projective varieties over S, where X' is projective over S. We put $\Delta' = \phi_*\Delta$. In this case, (X', Δ') is a *minimal model* of (X, Δ) over S if the following conditions hold.

(i) ϕ^{-1} contracts no divisors.
(ii) (X', Δ') is a $\mathbf{Q}$-factorial klt pair.
(iii) $K_{X'} + \Delta'$ is nef over S.
(iv) $a(E, X, \Delta) < a(E, X', \Delta')$ for all ϕ-exceptional divisors $E \subset X$.

3. Minimal model. First, we recall the following elementary but very important lemma.

Lemma 3.1 (Negative rational curves)**.** *Let (X, Δ) be a projective log canonical pair. Assume that $K_X + \Delta$ is not nef. Then there exists a rational curve C on X such that $(K_X + \Delta) \cdot C < 0$.*

Proof. It is obvious by the cone theorem for log canonical pairs. See, for example, [F2, Proposition 3.21] and [F3, Section 18]. □

By Lemma 3.1, we obtain the next lemma.

Lemma 3.2. *Let (X, Δ) be a log canonical pair and let $f : X \to S$ be a surjective morphism between projective varieties. Assume that $K_X + \Delta$ is f-nef and S contains no rational curves. Then $K_X + \Delta$ is nef.*

Proof. If $K_X + \Delta$ is not nef, then there exists a rational curve C on X such that $(K_X + \Delta) \cdot C < 0$ by Lemma 3.1. Since $K_X + \Delta$ is f-nef, $f(C)$ is not a point. On the other hand, S contains no rational curves by the assumption. It is a contradiction. Therefore, $K_X + \Delta$ is nef. □

Therefore, the following lemma is obvious by Definition 2.6 and Lemma 3.2.

Lemma 3.3. *Let (X, Δ) be a klt pair and let $f : X \to S$ be a surjective morphism between projective varieties. Let (X', Δ') be a minimal model of (X, Δ) over S. Assume that S contains no rational curves. Then (X', Δ') is a minimal model of (X, Δ).*

We give an easy consequence of the main theorem of [B+].

Theorem 3.4 (Existence of minimal models)**.** *Let (X, Δ) be a projective klt pair. Assume that X has maximal Albanese dimension. Then (X, Δ) has a minimal model.*

Proof. Let $\pi : \overline{X} \to X$ be a resolution and let $\alpha : \overline{X} \to \mathrm{Alb}(\overline{X})$ be the Albanese mapping of $\overline{X}$. Since X has only rational singularities, $\overline{X} \to S = \alpha(\overline{X})$ decomposes as

$$\alpha : \overline{X} \xrightarrow{\pi} X \xrightarrow{f} S.$$

See, for example, [BS, Lemma 2.4.1]. Since X has maximal Albanese dimension, $f : X \to S$ is generically finite. By [B+, Theorem 1.2], there exists a minimal model $f : (X', \Delta') \to S$ of (X, Δ) over S. By Lemma 3.3, (X', Δ') is a minimal model of (X, Δ). □

The following corollary is obvious by Theorem 3.4.

Corollary 3.5 (Minimal models)**.** *Let X be a smooth projective variety with maximal Albanese dimension. Then X has a minimal model.*

4. Abundance theorem. Let us consider the abundance theorem for maximal Albanese dimensional varieties.

Lemma 4.1. *Let X be a projective variety with only canonical singularities. Assume that X has maximal Albanese dimension. Then $\kappa(X, K_X) \geq 0$.*

Proof. Since X has only canonical singularities, we can assume that X is smooth by replacing X with its resolution. Then this lemma is obvious by the basic properties of the Kodaira dimension (see [U, Theorem 6.10]). Note that every subvariety of an Abelian variety has non-negative Kodaira dimension (see [U, Lemma 10.1]). □

Theorem 4.2 (Abundance theorem)**.** *Let (X, Δ) be a projective klt pair such that X has maximal Albanese dimension. Assume that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semi-ample.*

Proof. By the standard argument based on Shokurov's polytope, we may assume that Δ is a $\mathbf{Q}$-divisor. By taking a suitable crepant pull-back, we may further assume that the pair (X, Δ) is a

Q-factorial terminal pair. In particular, X has only terminal singularities. Since X has maximal Albanese dimension, $\kappa(X, K_X + \Delta) \geq 0$ always holds by Lemma 4.1. Therefore, it is sufficient to prove that $\nu(X, K_X + \Delta) > 0$ implies $\kappa(X, K_X + \Delta) > 0$ by [Fk, Proposition 3.1] and [F1, Corollary 2.5]. Note that the general fiber W of the Iitaka fibration in the proof of [Fk, Proposition 3.1] has maximal Albanese dimension (cf. Lemma 2.4). So we can use Theorem 3.4 and the induction on dimension in the proof of [Fk, Proposition 3.1]. From now on, we assume that $\nu(X, K_X + \Delta) > 0$. If $\kappa(X, K_X) > 0$, then $\kappa(X, K_X + \Delta) > 0$ obviously holds. So, we may assume that $\kappa(X, K_X) = 0$. In this case, the Albanese map $\alpha : X \to A$ is birational by [K, Theorem 1]. Since $\mathrm{Supp} K_X$ coincides with the α-exceptional locus and K_X is effective, $\kappa(X, K_X + \Delta) \geq \kappa(X, \alpha^*\alpha_*\Delta) = \kappa(A, \alpha_*\Delta)$. If $\alpha_*\Delta = 0$, then $K_X + \Delta$ is effective and α-exceptional. On the other hand, $\nu(X, K_X + \Delta) > 0$ implies $K_X + \Delta \neq 0$. It is a contradiction by the negativity lemma since $K_X + \Delta$ is nef. Therefore, $\alpha_*\Delta \neq 0$. So, we get $\kappa(X, K_X + \Delta) \geq \kappa(A, \alpha_*\Delta) \geq 1$. □

The following theorem is an easy consequence of Theorem 3.4 and Theorem 4.2.

Theorem 4.3 (Existence of good minimal models). *Let (X, Δ) be a projective klt pair such that X has maximal Albanese dimension. Then (X, Δ) has a good minimal model.*

Corollary 4.4 (Good minimal models). *Let X be a smooth projective variety with maximal Albanese dimension. Then X has a good minimal model. This means that there is a normal projective variety X' with only **Q**-factorial terminal singularities such that X' is birationally equivalent to X and $K_{X'}$ is semi-ample.*

Corollary 4.4 is a very special case of Lai's result (see [L, Theorem 4.5]). His method is completely different from ours and is more sophisticated.

5. Ueno's problem. The final theorem is a supplement to [U, Remark 10.13]. It is an easy consequence of Lemma 3.1.

Theorem 5.1. *Let (X, Δ) be a projective dlt pair. Assume that there are no rational curves on X and that $K_X + \Delta$ is log big. Then $K_X + \Delta$ is ample.*

Proof. By Lemma 3.1, $K_X + \Delta$ is nef. Therefore, it is well known that $K_X + \Delta$ is semi-ample because $K_X + \Delta$ is log big. Then there exists a birational morphism $f : X \to S$ such that $f_*\mathcal{O}_X \simeq \mathcal{O}_S$ and $K_X + \Delta \sim_{\mathbf{R}} f^*B$ for some ample **R**-Cartier **R**-divisor B on S. By induction on dimension, we may assume that $(K_X + \Delta)|_W$ is ample for every log canonical center W of (X, Δ). So we may assume that $\mathrm{Exc}(f)$ contains no log canonical centers of (X, Δ). Let H be an ample Cartier divisor on S. Assume that f is not an isomorphism. Let A be an f-ample Cartier divisor. Then there is an effective Cartier divisor D on X such that $\mathrm{Supp} D$ contains no log canonical centers of (X, Δ) and that $D \sim -A + f^*lH$ for some large integer l. Therefore, $(X, \Delta + \varepsilon D)$ is dlt and $K_X + \Delta + \varepsilon D$ is not nef for $0 < \varepsilon \ll 1$ since f is not an isomorphism and $K_X + \Delta + \varepsilon D \sim_{\mathbf{R},S} -\varepsilon A$. By Lemma 3.1, there exists a rational curve C on X such that $(K_X + \Delta + \varepsilon D) \cdot C < 0$. It is a contradiction because there are no rational curves on X. Therefore, f is an isomorphism. Thus, $K_X + \Delta$ is ample. □

Corollary 5.2 (see [U, Remark 10.13]). *Let W be a submanifold of a complex torus T with $\kappa(W, K_W) = \dim W$. Then K_W is ample.*

Proof. By [U, Lemma 10.8], there is an Abelian variety A which is a complex subtorus of T such that $W \subset A$. Thus, W is projective. Then, by Theorem 5.1, we obtain K_W is ample. □

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