# Two congruences involving Andrews-Paule's broken 3-diamond partitions and 5-diamond partitions

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**Abstract:** In this note, we will prove two congruences involving broken 3-diamond partitions and broken 5-diamond partitions. The two congruences were conjectured by Peter Paule and Silviu Radu in 2009.

**Key words:** Broken diamond partitions; congruences; modular forms.

**1. Introduction.** In 2007 George E. Andrews and Peter Paule [1] introduced a new class of combinatorial objects called broken k-diamond partitions. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of $n$, they showed that

$$\sum_{n=0}^{\infty}\Delta_k(n)q^n=\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}.$$

In 2008 Song Heng Chan [3] proved an infinite family of congruences when $k=2$. In 2009 Peter Paule and Silviu Radu [10] gave two non-standard infinite families of broken 2-diamond congruences. Moreover they stated four conjectures related to broken 3-diamond partitions and 5-diamond partitions. In this note we show that their first conjecture and the third conjecture are true:

**Theorem 1.1** (Conjecture 3.1 of [10])**.**

$$\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^6\equiv 6\sum_{n=0}^{\infty}\Delta_3(7n+5)q^n \pmod 7.$$

**Theorem 1.2** (Conjecture 3.3 of [10])**.**

$$E_4(q^2)\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^2 \equiv 8\sum_{n=0}^{\infty}\Delta_5(11n+6)q^n \pmod{11}.$$

The techniques in [7,8] are adapted here to prove Theorem 1.1 and Theorem 1.2.

**2. Preliminaries.** Let $\mathbf{H}$ denote the upper half of the complex plane, for a positive integer $N$, the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbf{Z})$ is defined by

$$\Gamma_0(N):=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\,\middle|\, ad-bc=1, c\equiv 0 \pmod N\right\}.$$

$\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})$ acts on the upper half of the complex plane by the linear fractional transformation $\gamma z:=\frac{az+b}{cz+d}$. If $f(z)$ is a function on $\mathbf{H}$, which satisfies $f(\gamma z)=\chi(d)(cz+d)^k f(z)$, where $\chi$ is a Dirichlet character modulo $N$, and $f(z)$ is holomorphic on $\mathbf{H}$ and meromorphic at all the cusps of $\Gamma_0(N)$, then we call $f(z)$ a weakly holomorphic modular form of weight $k$ with respect to $\Gamma_0(N)$ and character $\chi$. Moreover, if $f(z)$ is holomorphic on $\mathbf{H}$ and at all cusps of $\Gamma_0(N)$, then we call $f(z)$ a holomorphic modular form of weight $k$ with respect to $\Gamma_0(N)$ and character $\chi$. The set of all holomorphic modular forms of weight $k$ with respect to $\Gamma_0(N)$ and character $\chi$ is denoted by $\mathcal{M}_k(\Gamma_0(N),\chi)$.

Dedekind's eta function is defined by $\eta(z):=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$, where $q=e^{2\pi iz}$ and $\mathrm{Im}(z)>0$. A function $f(z)$ is called an eta-product if it can be written in the form of $f(z)=\prod_{\delta|N}\eta^{r_\delta}(\delta z)$, where $N$ and $\delta$ are natural numbers and $r_\delta$ is an integer. The following Proposition 2.1 obtained by Gordon-Hughes [4] and Newman [11] is useful to verify whether an eta-product is a weakly holomorphic modular form.

**Proposition 2.1** ([9], p.18 Thm 1.64)**.** *If* $f(z)=\prod_{\delta|N}\eta^{r_\delta}(\delta z)$ *is an eta-product with* $k:=\frac{1}{2}\sum_{\delta|N}r_\delta\in\mathbf{Z}$ *satisfying the conditions*:

$$\sum_{\delta|N}\delta r_\delta\equiv 0 \pmod{24},\qquad \sum_{\delta|N}\frac{N}{\delta}r_\delta\equiv 0 \pmod{24},$$

*then* $f(z)$ *is a weakly holomorphic modular form of weight* $k$ *with respect to* $\Gamma_0(N)$ *with the character* $\chi$*,*

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*here* $\chi$ *is defined by* $\chi(d) = (\frac{(-1)^k s}{d})$ *and* $s$ *is defined by* $s := \prod_{\delta|N} \delta^{r_\delta}$.

The following Proposition obtained by Ligozat [6] gives the analytic order of an eta-product at a cusp of $\Gamma_0(N)$.

**Proposition 2.2** ([9], p.18 Thm 1.65)**.** *Let* $c, d$ *and* $N$ *be positive integers with* $d|N$ *and* $(c, d) = 1$. *If* $f(z)$ *is an eta-product satisfying the conditions in Proposition* 2.1 *for* $N$, *then the order of vanishing of* $f(z)$ *at the cusp* $\frac{c}{d}$ *is*

$$\frac{N}{24}\sum_{\delta|N}\frac{(d,\delta)^2 r_\delta}{(d,\frac{N}{d})d\delta}.$$

Let $p$ be a prime, and $f(q) = \sum_{n\geq n_0}^{\infty} a(n)q^n$ be a formal power series, we define $f(q)|U_p = \sum_{pn\geq n_0} a(pn)q^n$. If $f(z) \in \mathcal{M}_k(\Gamma_0(N), \chi)$, then $f(z)$ has an expansion at the point $i\infty$ of the form $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ where $q = e^{2\pi iz}$ and $\mathrm{Im}(z) > 0$. We call this expansion the Fourier series of $f(z)$. Moreover we define $f(z)|U_p$ to be the result of applying $U_p$ to the Fourier series of $f(z)$. When $U_p$ acts on spaces of modular forms and $p|N$, we have

$$U_p : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(N), \chi).$$

The $U_p$ operator has the property that

$$\left[\left(\sum_{n=0}^{\infty} a(n)q^{pn}\right)\sum_{n=0}^{\infty} b(n)q^n\right]|U_p = \left(\sum_{n=0}^{\infty} a(n)q^n\right)\left(\sum_{n=0}^{\infty} b(pn)q^n\right).$$

In [12] Sturm proved the following criterion to determine whether two modular forms are congruent, this reduces the proof of a conjectured congruence to a finite calculation. In order to state his theorem, we introduce the notion of the $M$-adic order of a formal power series. Let $M$ be a positive integer and $f = \sum_{n\geq N} a(n)q^n$ be a formal power series in the variable $q$ with rational integer coefficients. The $M$-adic order of $f$ is defined by

$$\mathrm{Ord}_M(f) = \inf\{n \mid a(n) \not\equiv 0 \bmod M\}.$$

**Proposition 2.3** ([9], p.40 Thm 2.58)**.** *Suppose that* $f(z)$ *and* $g(z)$ *is in* $M_k(\Gamma_0(N), \chi) \bigcap \mathbf{Z}[[q]]$ *and* $M$ *is prime. If*

$$\mathrm{Ord}_M(f(z) - g(z)) \geq 1 + \frac{kN}{12}\prod_p\left(1 + \frac{1}{p}\right),$$

*where the product is over all prime divisors* $p$ *of* $N$. *Then* $f(z) \equiv g(z) \pmod M$.

**Proposition 2.4** ([9], p.19 Theorem 1.67)**.**

$$E_4(z) = \frac{\eta^{16}(z)}{\eta^8(2z)} + 2^8\frac{\eta^{16}(2z)}{\eta^8(z)},$$

*where* $E_4(z)$ *is the Eisenstein series of weight* 4 *for the full modular group.*

**3. Proof of Theorem 1.1.** *Proof.* We define an eta-product

$$F(z) := \frac{\eta(2z)\eta^9(7z)}{\eta^3(z)\eta(14z)},$$

setting $N = 56$, we find that $F(z)$ satisfies the conditions of Proposition 2.1 and $F(z)$ is holomorphic at all cusps of $\Gamma_0(56)$ by using Proposition 2.2, so $F(z)$ is in $\mathcal{M}_3(\Gamma_0(56), \chi)$, where $\chi(d) = (\frac{-1}{d})$ is a Dirichlet character modulo 56. We note that

$$F(z) = q^2\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{7n})^9}{(1-q^n)^3(1-q^{14n})}$$

and

$$\sum_{n=0}^{\infty}\Delta_3(n)q^n = \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{7n})}{(1-q^n)^3(1-q^{14n})}.$$

Applying $U_7$ operator on $F(z)$, we find that

$$\begin{aligned} F(z)|U_7 &= \left(q^2\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{7n})^9}{(1-q^n)^3(1-q^{14n})}\right)|U_7 \\ &= \left(q^2\sum_{n=0}^{\infty}\Delta_3(n)q^n\prod_{n=1}^{\infty}(1-q^{7n})^8\right)|U_7 \\ &= \left(\sum_{n\geq 2}^{\infty}\Delta_3(n-2)q^n\right)|U_7\cdot\prod_{n=1}^{\infty}(1-q^n)^8 \\ &= \sum_{7n\geq 2}^{\infty}\Delta_3(7n-2)q^n\prod_{n=1}^{\infty}(1-q^n)^8 \\ &= q\sum_{7n\geq 2}^{\infty}\Delta_3(7n-2)q^{n-1}\prod_{n=1}^{\infty}(1-q^n)^8 \\ &= q\sum_{n\geq 0}^{\infty}\Delta_3(7n+5)q^n\prod_{n=1}^{\infty}(1-q^n)^8. \end{aligned} \tag{1}$$

We define another eta-product

$$G(z) := \frac{\eta^6(2z)\eta^2(7z)}{\eta^2(z)},$$

by Proposition 2.1 and Proposition 2.2, we find that $G$ is also in $\mathcal{M}_3(\Gamma_0(56), \chi)$, where $\chi(d) = (\frac{-1}{d})$ is a Dirichlet character modulo 56. Moreover, we have

$$G(z) = \frac{\eta^6(2z)\eta^2(7z)}{\eta^2(z)} = \eta^{12}(z)\eta^6(2z)\frac{\eta^2(7z)}{\eta^{14}(z)} \tag{2}$$

$$\equiv \eta^{12}(z)\eta^6(2z) \pmod 7$$
$$\equiv q\prod_{n=1}^{\infty}(1-q^n)^{12}(1-q^{2n})^6 \pmod 7.$$

Where we used the elementary fact

$$\frac{\eta^2(7z)}{\eta^{14}(z)} = \prod_{n=1}^{\infty}\frac{(1-q^{7n})^2}{(1-q^n)^{14}} \equiv 1 \pmod 7.$$

We note that our Theorem 1.1 is equivalent to the congruence:

$$q\prod_{n=1}^{\infty}(1-q^n)^{12}(1-q^{2n})^6$$
$$\equiv 6q\sum_{n\geq 0}\Delta_3(7n+5)q^n\prod_{n=1}^{\infty}(1-q^n)^8 \pmod 7,$$

i.e.

$$G(z) \equiv 6F(z)|U_7 \pmod 7.$$

Using Sturm's theorem 2.3, it suffices to verify the congruence above holds for the first $\frac{3}{12}\cdot[SL_2(\mathbf{Z}):\Gamma_0(56)]+1=25$ terms, which is easily completed by using Mathematica 6.0. □

**4. Proof of Theorem 1.2.** The proof of Theorem 1.2 is similar. The difference is that we need to construct two eta-products to represent the left hand side of the equation in Theorem 1.2 up to a factor by using Proposition 2.4.

*Proof.* Define

$$H(z) := \frac{\eta(2z)\eta^{13}(11z)}{\eta^3(z)\eta(22z)},$$

setting $N=88$, we find that $H(z)$ satisfies the conditions of Proposition 2.1 and $H(z)$ is holomorphic at all cusps of $\Gamma_0(88)$ by Proposition 2.2, so $H(z)$ is in $\mathcal{M}_5(\Gamma_0(88),\chi)$, where $\chi(d)=(\frac{-1}{d})$ is a Dirichlet character modulo 88. We note that

$$H(z) = q^5\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{11n})^{13}}{(1-q^n)^3(1-q^{22n})}.$$

and

$$\sum_{n=0}^{\infty}\Delta_5(n)q^n = \prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{11n})}{(1-q^n)^3(1-q^{22n})}.$$

As before, applying $U_{11}$ operator on $H(z)$, we find that

$$\text{(3)}\quad H(z)|U_{11} = \left(q^5\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{11n})^{13}}{(1-q^n)^3(1-q^{22n})}\right)\Big|U_{11}$$
$$= \left(q^5\sum_{n=0}^{\infty}\Delta_5(n)q^n\prod_{n=1}^{\infty}(1-q^{11n})^{12}\right)\Big|U_{11}$$
$$= \left(\sum_{n\geq 5}^{\infty}\Delta_5(n-5)q^n\right)\Big|U_{11}\cdot\prod_{n=1}^{\infty}(1-q^n)^{12}$$
$$= \sum_{11n\geq 5}^{\infty}\Delta_5(11n-5)q^n\prod_{n=1}^{\infty}(1-q^n)^{12}$$
$$= q\sum_{11n\geq 5}^{\infty}\Delta_5(11n-5)q^{n-1}\prod_{n=1}^{\infty}(1-q^n)^{12}$$
$$= q\sum_{n\geq 0}^{\infty}\Delta_5(11n+6)q^n\prod_{n=1}^{\infty}(1-q^n)^{12}.$$

We define another two eta-products by

$$L_1(z) := \frac{\eta^{18}(2z)\eta^2(11z)}{\eta^2(z)\eta^8(4z)},\quad L_2(z) := \frac{\eta^{16}(4z)\eta^2(11z)}{\eta^6(2z)\eta^2(z)}.$$

Setting $N=88$, it is easy to verify that both $L_1(z)$ and $L_2(z)$ satisfy the conditions in Proposition 2.1 and both are holomorphic at all the cusps of $\Gamma_0(88)$ by using Proposition 2.2, hence both $L_1(z)$ and $L_2(z)$ are in $\mathcal{M}_5(\Gamma_0(88),\chi)$, where $\chi(d)=(\frac{-1}{d})$ is a Dirichlet character modulo 88. So $L(z):=L_1(z)+2^8L_2(z)$ is in $\mathcal{M}_5(\Gamma_0(88),\chi)$. On the other hand,

$$\text{(4)}\quad L(z) = \frac{\eta^{16}(2z)}{\eta^8(4z)}\cdot\frac{\eta^2(2z)\eta^2(11z)}{\eta^2(z)}$$
$$+2^8\frac{\eta^{16}(4z)}{\eta^8(2z)}\cdot\frac{\eta^2(2z)\eta^2(11z)}{\eta^2(z)}$$
$$= E_4(2z)\cdot\frac{\eta^2(2z)\eta^2(11z)}{\eta^2(z)}$$
$$= E_4(2z)\cdot\eta^{20}(z)\eta^2(2z)\cdot\frac{\eta^2(11z)}{\eta^{22}(z)}$$
$$\equiv E_4(2z)\cdot\eta^{20}(z)\eta^2(2z) \pmod{11}$$
$$= E_4(q^2)\cdot q\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^n)^{20}.$$

We find that Theorem 1.2 is equivalent to the following congruence of modular forms by using the expressions (3) and (4):

$$L(z) \equiv 8H(z)|U_{11} \pmod{11}.$$

Using Sturm's criterion i.e Proposition 2.3, it suffices to verify the congruence above holds for the first $\frac{5}{12}\cdot[SL_2(\mathbf{Z}):\Gamma_0(88)]+1=61$ terms, which is easily completed by using Mathematica 6.0. □

## References

[ 1 ] G. E. Andrews and P. Paule, MacMahon's partition analysis. XI. Broken diamonds and modular forms, Acta Arith. **126** (2007), no. 3, 281–294.

[ 2 ] G. E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis. VIII. Plane partition diamonds, Adv. in Appl. Math. **27** (2001), no. 2–3, 231–242.

[ 3 ] S. H. Chan, Some congruences for Andrews-Paule's broken 2-diamond partitions, Discrete Math. **308** (2008), no. 23, 5735–5741.

[ 4 ] B. Gordon and K. Hughes, Ramanujan congruences for $q(n)$, in *Analytic number theory* (*Philadelphia, Pa.,* 1980), 333–359, Lecture Notes in Math., 899 Springer, Berlin.

[ 5 ] M. D. Hirschhorn and J. A. Sellers, On recent congruence results of Andrews and Paule for broken $k$-diamonds, Bull. Austral. Math. Soc. **75** (2007), no. 1, 121–126.

[ 6 ] G. Ligozat, *Courbes modulaires de genre* 1, 43, Soc. Math. France, Paris, 1975.

[ 7 ] J. Lovejoy, Ramanujan-type congruences for three colored Frobenius partitions, J. Number Theory **85** (2000), no. 2, 283–290.

[ 8 ] K. Ono, Congruences for Frobenius partitions, J. Number Theory **57** (1996), no. 1, 170–180.

[ 9 ] K. Ono, *The web of modularity*: *arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[ 10 ] P. Paule and S. Radu, Infinite families of strange partition congruences for broken 2-diamonds, Ramanujan J. **23** (2010), no. 1–3, 409–416.

[ 11 ] M. Newman, Construction and application of a class of modular functions. II, Proc. London Math. Soc. (3) **9** (1959), 373–387.

[ 12 ] J. Sturm, On the congruence of modular forms, in *Number theory* (*New York,* 1984–1985), 275–280, Lecture Notes in Math., 1240, Springer, Berlin.