Construction of Poissonian Fock space: a simple proof

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(Communicated by Kenji FUKAYA, M.J.A., Feb. 12, 2010)

Abstract: We introduce an operator which enable us to give a simple construction of the isomorphism from the so-called Fock space to the L^2-space with respect to a Poisson measure without combinatorial arguments in Schmidt's orthogonalization procedure.

Key words: Poisson measure; Fock space; random point field.

1. Introduction. The Fock space associated with a Poisson measure seems to have a long history in physics and mathematics and the isomorphism was known among specialists to be constructed by Schmidt's orthogonalization method at least in 1960's soon after the Wiener-Ito decomposition was fully established. In the present note we give a simple method of constructing the isomorphism by introducing a densely defined operator on the Fock space, denoted by T in below. It is based upon the semigroup structure of the configuration space and enables us to replace somewhat complicated combinatorial arguments in Schmidt's orthogonalization procedure by a simple operational calculus.

Let λ be a nonnegative Radon measure on a Polish space R. The Poisson measure with intensity λ is the (unique) probability Borel measure π_λ on the locally finite configuration space $Q(R)$ over R whose Laplace transform is given by

$$\int_{Q(R)} \pi_\lambda(d\xi) e^{-\langle \xi, f\rangle} = \exp \int_R (e^{-f(x)} - 1)\lambda(dx)$$

for any continuous function f with compact support where

$$\langle \xi, f\rangle = \sum_{i=1}^{\infty} f(x_i)$$

if a locally finite configuration is expressed as $\xi = \sum_{i=1}^{\infty} \delta_{x_i}$.

We denote the finite configuration space by $\hat{R}$. It can be identified with the union of the symmetrizations of product spaces R^n with $n \geq 0$. Thus, we can define a measure $\hat{\lambda}$ on $\hat{R}$ by

$$\hat{\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{\otimes n}$$

where $\lambda^{\otimes n}$ stands for the n-fold product measure on R^n. Now set $H = L^2(R, \lambda)$ and $\mathbf{H} = L^2(\hat{R}, \hat{\lambda})$. Then $\mathbf{H}$ is isomorphic to the direct Hilbert sum of the n-fold product Hilbert spaces $H^{\otimes n}$ with norms weighted by $\sqrt{1/n!}$. In other words, $\mathbf{H}$ is isomorphic to the so-called symmetric (or boson) Fock space over H.

Theorem 1. *The isomorphism* $\mathbf{I}$ *from the Fock space* $\mathbf{H}$ *onto the* $\mathcal{H} = L^2(Q(R), \pi_\lambda)$ *is given by the formula*

$$\mathbf{I}(\mathbf{f})(\xi) = \langle \xi_*, T^{-1}\mathbf{f}\rangle$$

where ξ_* *stands for the lift of* ξ *onto* $\hat{R}$. (See, Definition 2.)

Notice that the measure λ may have both atomic and continuous parts.

The first motivation of the present paper was to simplify the Fock space arguments applied to the study of the scattering length in [T1-3].

It turns out that the operator T has some unexpected applications, such as an explicit formula between correlation functions and (local) density functions in Gibbs random field theory, simplification of the arguments for fermion (or determinantal) and other processes in [SrT1-4] (also, cf. [So]), etc. These results will be published elsewhere.

2. Locally finite configuration space and finite configuration space. We denote the space of continuous functions on R with compact support by $C_c(R)$. Its dual space is the space of Radon measures on R. We denote by $Q(R)$ the set of nonnegative integer-valued Radon measures on R. An element ξ of $Q(R)$ is called a locally finite configuration over R and is expressed as a finite or

2000 Mathematics Subject Classification. Primary 60G55; Secondary 60G60, 82B05.

doi: 10.3792/pjaa.86.60
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infinite sum of δ-measures (cf. [SgT]). Also, we denote by $\hat{R}$ its subset of finite configurations and for a subset Λ of R we write

$$\hat{\Lambda} = \{X \in \hat{R} : X(\Lambda) = X(R)\}.$$

An element $X \in \hat{R}$ is expressed as a finite sum of δ-measures and its total mass is denoted by $|X|$. The subset $R_n = \{X \in \hat{R} : |X| = n\}$ will be identified with the n-fold symmetric product of the space R and $\hat{R}$ with their union.

We write $X \leq \xi$ if $X(x) \leq \xi(x)$ for each $x \in R$ and then set

$$c(\xi, X) = \prod_{x:X(x)>0} \binom{\xi(x)}{X(x)}$$

where the product over empty set is defined to be 1, as usual: $c(\xi, 0) = 1$.

Notice that if $X, Y \in \hat{R}$ and $Y \leq X$, then $c(X, Y) = c(X, X - Y)$.

Definition 2. *For $\xi \in Q(R)$ define the configuration ξ_* lifted onto $\hat{R}$ by*

$$\xi_* = \sum_{X \in \hat{R},\ X \leq \xi} c(\xi, X)\delta_X$$

and write

$$\langle \xi_*, \mathbf{f} \rangle = \sum_{X \in \hat{R},\ X \leq \xi} c(\xi, X)\mathbf{f}(X)$$

for a function $\mathbf{f}$ on $\hat{R}$ whenever the right hand side is summable.

3. Convolution on $\mathbf{F}_0$ and invariant measure $\hat{\boldsymbol{\lambda}}$. We need a space of test functions on $\hat{R}$.

Definition 3. *Let $\mathbf{F}_0$ be the set of continuous functions $\mathbf{f}$ on $\hat{R}$ which satisfy the following conditions*:

(*a*) (*compact support*) $\mathbf{f}(X) = 0$ *if* $X(\Lambda^c) > 0$ *for some compact subset Λ of R.*

(*b*) (*exponential growth*) $|\mathbf{f}(X)| \leq C^{|X|}$ *for some positive constant C.*

For a $\phi \in C_c(R)$ define $\hat{\phi} \in \mathbf{F}_0$ by

$$\hat{\phi}(X) = \prod_{i=1}^{n} \phi(x_i) \quad \text{if} \quad X = \sum_{i=1}^{n} \delta_{x_i}.$$

Such functions $\hat{\phi}$ are multiplicative on the semigroup $\hat{R}$:

$$\hat{\phi}(X + Y) = \hat{\phi}(X)\hat{\phi}(Y) \qquad X, Y \in \hat{R}.$$

Definition 4. *We define the convolution on $\mathbf{F}_0$ by*

$$\mathbf{f} * \mathbf{g}(X) = \sum_{Y \leq X} c(X, Y)\mathbf{f}(Y)\mathbf{g}(X - Y) = \mathbf{g} * \mathbf{f}(X).$$

Then, the binomial theorem shows that $\hat{\phi} * \hat{\psi} = (\phi + \psi)^\wedge$.

Lemma 5. (i) *Let $\phi \in C_c(R)$. Then,* $\int_{\hat{R}} \hat{\lambda}(dX)\hat{\phi}(X) = \alpha(\phi) := \exp \int_R \lambda(dx)\phi(x)$.

(ii) *If* $\mathbf{f}, \mathbf{g} \in \mathbf{F}_0$, *then,* $\int_{\hat{R}} \hat{\lambda}(dX)(\mathbf{f} * \mathbf{g})(X) = \Big(\int_{\hat{R}} \hat{\lambda}(dX)\mathbf{f}(X)\Big)\Big(\int_{\hat{R}} \hat{\lambda}(dX)\mathbf{g}(X)\Big)$.

Proof. (i) is immediate. (ii) follows from the following: for $\phi, \psi \in C'_c(R)$,

$$\int_{\hat{R}} \hat{\lambda}(dX)(\hat{\phi} * \hat{\psi})(X) = \int_{\hat{R}} \hat{\lambda}(dX)(\phi + \psi)^\wedge(X)$$
$$= \alpha(\phi + \psi) = \alpha(\phi)\alpha(\psi).$$

□

4. Operators $\boldsymbol{S}$ and $\boldsymbol{S^{-1}}$. For a function $\mathbf{f}$ on $\hat{R}$ define

$$S\mathbf{f}(X) = \sum_{Y \leq X} c(X, Y)\mathbf{f}(Y).$$

For instance, $S\hat{\phi}(X) = (\phi + 1)^\wedge(X)$, $S^{-1}\hat{\phi}(X) = (\phi - 1)^\wedge(X)$.

Lemma 6. *The operator S is invertible and*

$$S^{-1}\mathbf{f}(X) = \sum_{Y \leq X} (-1)^{|X|-|Y|} c(X, Y)\mathbf{f}(Y).$$

Proof. This is a version of the inclusion-exclusion formula for finite sets. □

Notice that for an $X \in \hat{R}$ and an $\mathbf{f} \in \mathbf{F}_0$, we have $\langle X_*, \mathbf{f} \rangle = S\mathbf{f}(X)$.

Furthermore, if we introduce an involution operator J (i.e., $J^2 = I$) by

$$J\mathbf{f}(X) = (-1)^{|X|}\mathbf{f}(X) \qquad X \in \hat{R}$$

then, $S^{-1} = JSJ$.

5. Operators $\boldsymbol{T}$ and $\boldsymbol{T^{-1}}$. From now on we fix a nonnegative Radon measure λ on R and denote

$$H = L^2(R, \lambda), \mathbf{H} = L^2(\hat{R}, \hat{\lambda}) \text{ and } \mathcal{H} = \mathrm{L}^2(\mathrm{Q(R)}, \pi_\lambda).$$

Notice that the space $\mathbf{F}_0$ is dense in $\mathbf{H}$ and that the space $\mathbf{H}$ is an L^2-realization of the symmetric (or boson) Fock space over H.

Lemma 7. *For a function $\mathbf{f} \in \mathbf{F}_0$ define*

$$T\mathbf{f}(X) = \int_{\hat{R}} \mathbf{f}(X + Y)\hat{\lambda}(dY).$$

Then, T defines an invertible operator from $\mathbf{F}_0$ to itself and

$$T^{-1}\mathbf{f}(X) = \int_{\hat{R}} (-1)^{|Y|}\mathbf{f}(X+Y)\hat{\lambda}(dY).$$

Moreover, T and T^{-1} are formally adjoint operators of S and S^{-1} in $\mathbf{H}$, respectively.

Proof. The point of the proof is the following observation: for $\phi \in C_c(R)$

$$T\hat{\phi}(X) = \alpha(\phi)\hat{\phi}(X) \quad \text{and} \quad \alpha(\phi) = \mathrm{T}\hat{\phi}(0).$$

It then follows that for $\phi, \psi \in C_c(R)$

$$\begin{aligned}\langle T\hat{\phi}, \hat{\psi}\rangle_{\mathbf{H}} &= \alpha(\phi)\alpha(\phi\psi) = \alpha(\phi(\psi+1)) \\ &= \langle \hat{\phi}, S\hat{\psi}\rangle_{\mathbf{H}}.\end{aligned}$$

Thus, formally, $T = S^*$ in $\mathbf{H}$. It then follows that T is invertible and $T^{-1} = JTJ$ takes the desired form. □

6. Exponential function $\mathbf{e}_f$. For a given $f \in C_c(R)$ denote

$$\mathbf{e}_f = \hat{\phi} \quad \text{with} \quad \phi(x) = e^{-f(x)} - 1.$$

We call such functions exponential.

Lemma 8. (i) *There holds the identity $\langle \xi_*, \mathbf{e}_f\rangle = e^{-\langle \xi, f\rangle}$ for any $f \in C_c(R)$.*

(ii) *For $\phi, \psi \in C_c(R)$, $\langle \xi_*, \hat{\phi}\rangle\langle \xi_*, \hat{\psi}\rangle = \langle \xi_*, (\phi\psi + \phi + \psi)\wedge\rangle$.*

Proof. The assertion (i) is nothing but the expansion formula for the product $\prod_{i=1}^n a_i = \prod_{i=1}^n (1 + (a_i - 1))$. Then (ii) follows for $\phi = e^{-f} - 1$ and $\psi = e^{-g} - 1$ since $e_{f+g} = (\phi\psi + \phi + \psi)^\wedge$. The rest is a routine work. □

7. Isomorphism I and proof of Main Theorem. Let us rewrite the definition of the Poisson measure π_λ by using our notations.

Lemma 9. *The Poisson measure with intensity λ is the (unique) probability Borel measure π_λ on the locally finite configuration space $Q(R)$ such that $\int_{Q(R)} \pi_\lambda(d\xi)\langle \xi_*, \mathbf{f}\rangle = \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{f}(X)$ for any $\mathbf{f} \in \mathbf{F}_0$.*

Proof. Obvious. □

Now we define the isomorphism $\mathbf{I}$ on $\mathbf{F}_0$.

Definition 10. *For an $\mathbf{f} \in \mathbf{F}_0$ define a function $\mathbf{I}(\mathbf{f})(\xi)$ on $Q(R)$ by*

$$\mathbf{I}(\mathbf{f})(\xi) = \langle \xi_*, T^{-1}\mathbf{f}\rangle.$$

Lemma 11. *Let $\phi, \psi \in C_c(R)$. Then,*

$$\langle \mathbf{I}(\hat{\phi}), \mathbf{I}(\hat{\psi})\rangle_{\mathcal{H}} = \alpha(\phi\psi) = \exp\langle \phi, \psi\rangle_H = \langle \hat{\phi}, \hat{\psi}\rangle_{\mathbf{H}}.$$

Proof. It follows from the preceeding lemmas that

$$\langle \mathbf{I}(\hat{\phi}), \mathbf{I}(\hat{\psi})\rangle_{\mathcal{H}} = \alpha(\phi\psi) = \exp\langle \phi, \psi\rangle_H = \langle \hat{\phi}, \hat{\psi}\rangle_{\mathbf{H}},$$

noting $1/\alpha(\phi) = \alpha(-\phi)$. □

Hence we obtain the following since the functions $\hat{\phi}, \phi \in C_c(R)$ are dense in $\mathbf{F}_0$ with respect to the norm $\|\cdot\|_{\mathbf{H}}$.

Lemma 12. (i) *If $\mathbf{f} \in \mathbf{F}_0$, then, $\mathbf{I}(\mathbf{f}) \in \mathcal{H}$.*
(ii) *If $\mathbf{f}, \mathbf{g} \in \mathbf{F}_0$, then,*

$$\langle \mathbf{I}(\mathbf{f}), \mathbf{I}(\mathbf{g})\rangle_{\mathcal{H}} = \langle \mathbf{f}, \mathbf{g}\rangle_{\mathbf{H}}.$$

In particular,

$$\|\mathbf{I}(\mathbf{f})\|_{\mathcal{H}} = \|\mathbf{f}\|_{\mathbf{H}}.$$

Now we can prove the main theorem.

Theorem 13. *The operator $\mathbf{I}$ is extended to the unitary operator from $\mathbf{H}$ onto $\mathcal{H}$.*

Proof. From the preceeding lemmas we can extend $\mathbf{I}$ to the norm preserving operator from $\mathbf{H} = L^2(\hat{R}, \hat{\lambda})$ to $\mathcal{H} = L^2(Q(R), \pi_\lambda)$. It remains to prove that $\mathbf{I}$ is onto. But it is obvious because the functions

$$\alpha(e^{-f} - 1)\mathbf{I}(\mathbf{e}_f) = e^{-\langle \xi, f\rangle}, \quad f \in C_c(R)$$

span the Hilbert space $\mathcal{H} = L^2(Q(R), \pi_\lambda)$. □

8. Remark: a characterization of the operator *T*. Let $C_b(R)$ be the Banach space of bounded continuous functions on R with supremum norm.

Definition 14. *For an $\mathbf{f} \in \mathbf{F}_0$ define the functional $\Phi_{\mathbf{f}}$ on $C_b(R)$ by*

$$\Phi_{\mathbf{f}}(\phi) = \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{f}(X)\hat{\phi}(X), \quad \phi \in C_b(R).$$

Lemma 15. *$\Phi_{\mathbf{f}}(\phi)$ is an analytic functional in $\phi \in C_b(R)$ and its Taylor expansion at $\phi = 1$ is given by*

$$\begin{aligned}\Phi_{\mathbf{f}}(1+\psi) &= \int_{\hat{R}} \hat{\lambda}(dX)T\mathbf{f}(X)\hat{\psi}(X) \\ &= \Phi_{\mathbf{f}}(1) + \sum_{n=1}^{\infty} \frac{1}{n!}\int_{R_n} \lambda^{\otimes n}(dX)T\mathbf{f}(X)\hat{\psi}(X), \\ &\qquad \psi \in C_b(R).\end{aligned}$$

Proof. The analyticity is obvious since $\mathbf{f} \in \mathbf{F}_0$. Now

$$\begin{aligned}\Phi_{\mathbf{f}}(1+\psi) &= \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{f}(X)(\psi+1)^\wedge(X) \\ &= \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{f}(X)S\hat{\psi}(X) \\ &= \int_{\hat{R}} \hat{\lambda}(dX)T\mathbf{f}(X)\hat{\psi}(X).\end{aligned}$$

□

The above lemma shows that the functional $\Phi_{\mathbf{f}}$ have the derivative of order n (which is a symmetric multilinear functional) for each n and it admits the density $T\mathbf{f}$ with respect to the product measure $\lambda(dx_1)\dots\lambda(dx_n)$ on R^n.

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