The Riemann hypothesis and functional equations for zeta functions over $\mathbf{F}_1$

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Abstract: We prove functional equations for the absolute zeta functions. We also show that the absolute zeta functions satisfy the tensor structure in the sense that their singularities possess an additive property under the tensor product. Moreover those singularities satisfy the analog of the Riemann hypothesis.

Key words: Zeta functions; the field with one element; absolute mathematics.

1. Introduction. Let A be an $\mathbf{F}_1$-algebra which by definition is a commutative monoid. In the recent paper [1], we defined the absolute Hasse (or Weil) zeta function of A by

$$\zeta_{\mathbf{F}_1}(s, A) = \exp\left(\sum_{m=1}^{\infty} \frac{|\mathrm{Hom}(A, \boldsymbol{\mu}_m)|}{m} e^{-ms}\right), \tag{1}$$

where $\boldsymbol{\mu}_m$ is the multiplicative group of m-th roots of 1, and Hom means the group of homomorphisms for $\mathbf{F}_1$-algebras. Here we replace the previous notation T in [1] by e^{-s}, and $\zeta^W(T, A)$ by $\zeta_{\mathbf{F}_1}(s, A)$.

If A satisfies that $|\mathrm{Hom}(A, \boldsymbol{\mu}_m)| = O(e^{mc})$ for some constant $c > 0$, the sum in (1) is absolutely convergent in Re $s > c$. In particular, when $|\mathrm{Hom}(A, \boldsymbol{\mu}_m)|$ is bounded in m, it holds that (1) is valid for $\mathrm{Re}(s) > 0$.

In this paper we prove the functional equations of $\zeta_{\mathbf{F}_1}(s, A)$ for finitely generated abelian groups A. We also deduce an analog of the Riemann Hypothesis, and establish the tensor structure of singularities of such zeta functions. In the proof a determinant expression of $\zeta_{\mathbf{F}_1}(s, A)$ is crucial. In Section 2 we start with an elementary example of zeta functions having a determinant expression.

2. Determinant expression. For a bijection $\sigma \in \mathrm{Aut}(X)$ from a given set X to itself, we define its zeta function as

$$\zeta_\sigma(s) = \exp\left(\sum_{m=1}^{\infty} \frac{|\mathrm{Fix}(\sigma^m)|}{m} e^{-ms}\right).$$

When $|X| = n$, we identify σ as an element in $S_n = \mathrm{Aut}(\{1, 2, \ldots, n\})$. The following proposition belongs to a folklore. We are giving its proof, since we cannot find a suitable reference.

Proposition 1. *Let X and Y be finite sets. Put $|X| = n$.*

(i) *$\zeta_\sigma(s)$ has the determinant expression*

$$\zeta_\sigma(s) = \det(1 - M(\sigma)e^{-s})^{-1},$$

where $M(\sigma) = (\delta_{i,\sigma(j)})_{i,j=1,\ldots,n}$ is the matrix representation $M : S_n \to GL_n(\mathbf{C})$.

(ii) *$\zeta_\sigma(s)$ satisfies an analog of the Riemann hypothesis: $\zeta_\sigma(s) = \infty$ implies $\mathrm{Re}(s) = 0$.*

(iii) *$\zeta_\sigma(s)$ satisfies the functional equation*

$$\zeta_\sigma(-s) = \zeta_\sigma(s)(-1)^n \mathrm{sgn}(\sigma) e^{-ns}.$$

(iv) *$\zeta_\sigma(s)$ has the Euler product*

$$\zeta_\sigma(s) = \prod_{P \in \mathrm{Cycle}(\sigma)} (1 - N(P)^{-s})^{-1},$$

where $N(P) = e^{\mathrm{length}(P)}$.

(v) *The singularities of $\zeta_\sigma(s)$ satisfy an additive structure under the tensor product. Namely, a sum of a pole of $\zeta_\sigma(s)$ for $\sigma \in \mathrm{Aut}(X)$ and a pole of $\zeta_\tau(s)$ for $\tau \in \mathrm{Aut}(Y)$ is a pole of $\zeta_{\sigma\otimes\tau}(s)$, and all poles of $\zeta_{\sigma\otimes\tau}(s)$ are given by this way. Here for $\sigma \in \mathrm{Aut}(X)$ and $\tau \in \mathrm{Aut}(Y)$, we denote their tensor product by $\sigma \otimes \tau \in \mathrm{Aut}(X \times Y)$.*

(vi) *The Laurent expansion of $\zeta_\sigma(s)$ around $s = 0$ is given as follows:*

$$\zeta_\sigma(s) = s^{-m} c(\sigma)^{-1} + O(s^{-m+1}),$$

where m is the multiplicity of the eigenvalue 1 of $M(\sigma)$ and $c(\sigma) = \prod_{P \in \mathrm{Cycle}(\sigma)} \mathrm{length}(P)$.

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Proof. (i) Let

$$U^{-1}M(\sigma)U = \begin{pmatrix} \alpha_1 & * & \cdots & * \\ 0 & \alpha_2 & \ddots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

be an upper triangularization. Then

$$\begin{aligned} \det(1-M(\sigma)u) &= \det(1-U^{-1}M(\sigma)Uu) \\ &= (1-\alpha_1 u)\cdots(1-\alpha_n u). \end{aligned}$$

Hence

$$\begin{aligned} \det(1-M(\sigma)u)^{-1} &= (1-\alpha_1 u)^{-1}\cdots(1-\alpha_n u)^{-1} \\ &= \prod_{j=1}^{n} \exp\left(\sum_{m=1}^{\infty} \frac{\alpha_j^m u^m}{m}\right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{\alpha_1^m + \cdots + \alpha_n^m}{m} u^m\right). \end{aligned}$$

Then it is sufficient to show that

$$|\mathrm{Fix}(\sigma^m)| = \alpha_1^m + \cdots + \alpha_n^m.$$

We compute

$$\begin{aligned} \alpha_1^m + \cdots + \alpha_n^m &= \mathrm{tr}((U^{-1}M(\sigma)U)^m) \\ &= \mathrm{tr}(U^{-1}M(\sigma)^m U) \\ &= \mathrm{tr}(U^{-1}M(\sigma^m)U) \\ &= \mathrm{tr}(M(\sigma^m)) \\ &= \sum_{i=1}^{n} \delta_{i,\sigma^m(i)} \\ &= |\mathrm{Fix}(\sigma^m)|. \end{aligned}$$

(ii) By (i), if s is a pole of $\zeta_\sigma(s)$, we have $e^s = \alpha_j$ for some j. Then

$$e^{\mathrm{Re}(s)} = |e^s| = |\alpha_j| = 1.$$

Hence $\mathrm{Re}(s) = 0$.

(iii)

$$\begin{aligned} &\zeta_\sigma(-s) \\ &\quad = \det(1-M(\sigma)e^s)^{-1} \\ &\quad = \det((-M(\sigma)e^s)(1-M(\sigma)^{-1}e^{-s}))^{-1} \\ &\quad = (-1)^n(\det M(\sigma))^{-1}e^{-ns}\det(1-M(\sigma)^{-1}e^{-s})^{-1} \\ &\quad = (-1)^n \mathrm{sgn}(\sigma)e^{-ns}\zeta_\sigma(s). \end{aligned}$$

(iv) We put the decomposition into cyclic permutations as

$$\begin{aligned} \sigma &= \sigma_1\cdots\sigma_r \\ &= (i_1,\ldots,i_{l(1)})(i_{l(1)+1},...,i_{l(1)+l(2)}) \\ &\qquad \cdots(i_{l(1)+\cdots+l(r-1)+1},...,i_n). \end{aligned}$$

Let $\pi \in S_n$ be the permutation such that $\pi(k) = i_k$ for $k = 1,2,3,\ldots n$. Then

$$\begin{aligned} \pi^{-1}\sigma\pi &= (1\cdots l(1))(l(1)+1\cdots l(1)+l(2)) \\ &\qquad \cdots(l(1)+\cdots+l(r-1)+1\cdots n). \end{aligned}$$

Hence

$$M(\pi)^{-1}M(\sigma)M(\pi) = \mathrm{diag}(C_{l(1)}, C_{l(2)}, \cdots, C_{l(r)})$$

with

$${}^tC_l = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$$

of size l. Then

$$\begin{aligned} \det(1-M(\sigma)e^{-s}) &= \det(1-M(\pi)^{-1}M(\sigma)M(\pi)e^{-s}) \\ &= \prod_{j=1}^{n}(1-e^{-l(j)s}) \\ &= \prod_{P\in\mathrm{Cycle}(\sigma)}(1-N(P)^{-s}). \end{aligned}$$

(v) By (i) we have

$$\begin{aligned} \zeta_{\sigma\otimes\tau}(s) &= \det(1-M(\sigma\otimes\tau)e^{-s})^{-1} \\ &= \det(1-M(\sigma)\otimes M(\tau)e^{-s})^{-1}, \end{aligned}$$

where $\otimes$ denotes the Kronecker tensor product of matrices. We put the eigenvalues of $M(\sigma)$ and $M(\tau)$ as α_j $(j = 1,\ldots,|X|)$ and β_k $(k = 1,2,\ldots,|Y|)$, respectively, We see from (i) that the poles of $\zeta_\sigma(s)$ and $\zeta_\tau(s)$ are given by $s \equiv \log\alpha_j$ and $s \equiv \log\beta_k$ mod $2\pi i\mathbf{Z}$. Thus the set of poles of $\zeta_{\sigma\otimes\tau}(s)$ is given by

$$\{\log\alpha_j\beta_k \mod 2\pi i\mathbf{Z} \mid 1 \le j \le |X|,\ 1 \le k \le |Y|\}.$$

The result follows from

$$\log\alpha_j\beta_k \equiv \log\alpha_j + \log\beta_k \mod 2\pi i\mathbf{Z}.$$

(vi) By (i), we have

$$\begin{aligned} \zeta_\sigma(s) &= \det(1-M(\sigma)e^{-s})^{-1} \\ &= \Big((1-e^{-s})^m \prod_{\alpha\neq 1}(1-\alpha e^{-s})\Big)^{-1}. \end{aligned}$$

Hence $\zeta_\sigma(s)$ has a pole of order m at $s = 0$. The leading coefficient is calculated from (iv):

$$\begin{aligned} &\prod_{P\in\mathrm{Cycle}(\sigma)}(1-N(P)^{-s})^{-1} \\ &\quad = \prod_{P\in\mathrm{Cycle}(\sigma)}(l(P)s + O(s^2))^{-1} \\ &\quad = s^{-m}\prod_{P\in\mathrm{Cycle}(\sigma)}(l(P))^{-1} + O(s^{-m+1}). \qquad \square \end{aligned}$$

Remark 1. Under mild modifications Proposition 1 can be extended for a non-bijective map σ, since the most crucial fact

$$|\mathrm{Fix}(\sigma^m)| = \mathrm{tr}(M(\sigma)^m)$$

remains true.

We apply Proposition 1 to our absolute Hasse zeta function. For a finite abelian group

$$A \cong \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_l}$$

of order $n = \prod_{j=1}^{l} n_j$ with $n_1 | n_2 | \cdots | n_l$, we define the *absolute Frobenius operator* Φ_A *on the group*

$$A^{(2)} = \boldsymbol{\mu}_{n_1^2} \times \boldsymbol{\mu}_{n_2^2} \times \cdots \times \boldsymbol{\mu}_{n_l^2}$$

as

$$\Phi_A(\alpha_1, \ldots, \alpha_l) = (\alpha_1^{n_1+1}, \ldots, \alpha_l^{n_l+1}).$$

Lemma 1. *Assume that A is a finite abelian group of order n. If*

$$A \cong \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_l}$$

with $n_1 | n_2 | \cdots | n_l$, then it holds that

$$|\mathrm{Hom}(A, \mu_m)| = \frac{|\mathrm{Fix}(\Phi_A^m)|}{|A|}.$$

Proof. First

$$|\mathrm{Hom}(A, \mu_m)| = |\mathrm{Hom}(\mu_m, A)| = |X|,$$

where

$$X = \{\alpha \in A \mid \alpha^m = 1\}.$$

Look at the canonical homomorphism

$$A^{(2)} \to A$$

defined by

$$(\alpha_1, \ldots, \alpha_l) \mapsto (\alpha_1^{n_1}, \ldots, \alpha_l^{n_l}),$$

which is a surjective $n:1$ map. Since $\mathrm{Fix}(\Phi_A^m)$ is the inverse image of X, we see that $|\mathrm{Fix}(\Phi_A^m)| = n|X|$. □

Theorem 1. *Assume that A is a finite abelian group of order n. If*

$$A \cong \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_l}$$

with $n_1 | n_2 | \cdots | n_l$, then the absolute Hasse zeta function has the following determinant expression:

$$\zeta_{\mathbf{F}_1}(s, A) = \det(1 - \Phi_A e^{-s})^{-1/|A|}. \tag{2}$$

Proof. If we identify the operator Φ_A with the square matrix of size n^2, it holds by Lemma 1 that

$$\zeta_{\mathbf{F}_1}(s, A) = \zeta_{\Phi_A}(s)^{\frac{1}{|A|}}.$$

Thus Proposition 1 leads to the result. □

3. Functional equations. For $1 \le r \in \mathbf{Z}$, we denote by $g_r(T) \in \mathbf{Z}[T]$ the Euler polynomials which are defined recursively by

$$g_1(T) = T$$

and

$$g_{r+1}(T) = \sum_{k=1}^{r} \binom{r}{k-1} (T-1)^{r-k} g_k(T). \tag{3}$$

We find inductively that $\deg g_{r+1} = r$ for $r \ge 1$. By [1, Lemma 2.2], we have

$$\sum_{\nu=1}^{\infty} \nu^{r-1} T^{\nu} = \frac{g_r(T)}{(1-T)^r}. \tag{4}$$

Example 1 (Euler polynomials)**.**

$$g_1(T) = T,$$
$$g_2(T) = T,$$
$$g_3(T) = T^2 + T,$$
$$g_4(T) = T^3 + 4T^2 + T.$$

Lemma 2 (Another expression of Euler polynomials)**.** *For $1 \le r \in \mathbf{Z}$, we define $h_r(T) \in \mathbf{Z}[T]$ recursively by*

$$h_1(T) = T$$

and

$$h_{r+1}(T) = \sum_{k=2}^{r} \binom{r}{k-1} (1-T)^{r-k} T h_k(T) + (1-T)^{r-1} T \quad (r \ge 2). \tag{5}$$

Then for $r \ge 1$,

$$g_r(T) = h_r(T).$$

Proof. We show that $h_r(T)$ satisfies

$$\sum_{\nu=1}^{\infty} \nu^{r-1} T^{\nu} = \frac{h_r(T)}{(1-T)^r}. \tag{6}$$

Then the lemma follows from (4)

Put

$$S_r = \sum_{\nu=1}^{\infty} \nu^{r-1} T^{\nu}.$$

Since

$$T S_{r+1} = \sum_{\nu=1}^{\infty} \nu^r T^{\nu+1},$$

We compute

$$(1-T)S_{r+1} = \sum_{\nu=1}^{\infty} \nu^r (T^\nu - T^{\nu+1}) \tag{7}$$
$$= \sum_{\nu=0}^{\infty} ((\nu+1)^r - \nu^r) T^{\nu+1}$$
$$= \sum_{\nu=0}^{\infty} \sum_{k=0}^{r-1} \binom{r}{k} \nu^k T^{\nu+1}$$
$$= T \sum_{k=0}^{r-1} \binom{r}{k} \sum_{\nu=0}^{\infty} \nu^k T^\nu$$
$$= T \sum_{k=1}^{r} \binom{r}{k-1} \sum_{\nu=0}^{\infty} \nu^{k-1} T^\nu$$
$$= T \left(\sum_{k=2}^{r} \binom{r}{k-1} S_k + \sum_{\nu=0}^{\infty} T^\nu \right)$$
$$= T \left(\sum_{k=2}^{r} \binom{r}{k-1} S_k + \frac{1}{1-T} \right).$$

If we put

$$S_r = \frac{f_r(T)}{(1-T)^r}$$

and substitute it to (7), we find that $f_r(T)$ is a polynomial in T with $f_r(T) = h_r(T)$, because $f_r(T)$ also satisfies (5). □

Lemma 3 (Transformation formula for Euler polynomials). *For $r \geq 2$ the Euler polynomials $g_r(T)$ satisfy*

$$g_r(T^{-1}) = T^{-r} g_r(T). \tag{8}$$

Proof. We prove by induction on r. When $r = 2$, the result follows from $T^{-1} = T^{-2} \cdot T$. Assume that (8) is true for any k with $2 \leq k \leq r$. Then

$$g_{r+1}(T^{-1}) = \sum_{k=1}^{r} \binom{r}{k-1} (T^{-1}-1)^{r-k} g_k(T^{-1})$$
$$= \sum_{k=2}^{r} \binom{r}{k-1} (T^{-1}-1)^{r-k} T^{-k} g_k(T) + (T^{-1}-1)^{r-1} T^{-1}$$
$$= T^{-r-1} \left(\sum_{k=2}^{r} \binom{r}{k-1} (1-T)^{r-k} T g_k(T) + (1-T)^{r-1} T \right)$$
$$= T^{-r-1} g_{r+1}(T),$$

where the last identity follows from Lemma 2. □

For proving the functional equations, we recall the following fact.

Proposition 2 ([1], Proposition 2.1). *Assume A is a finitely generated free abelian group of rank r. Put*

$$A \cong \mathbf{Z}^r \times \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_k}$$

with $n_1 | n_2 | \cdots | n_k$. Then

$$\zeta_{\mathbf{F}_1}(s, A) = \begin{cases} \displaystyle\prod_{\underline{d}|\underline{n}} \left(1 - e^{-|\underline{d}|s}\right)^{-\varphi(\underline{d}) d_1^{k-2} \cdots d_k^{-1}} & \text{for} \quad r = 0, \\ \displaystyle\prod_{\underline{d}|\underline{n}} \exp\left(\frac{g_r(e^{-|\underline{d}|s}) \varphi(\underline{d}) d_1^{r+k-1} \cdots d_k^{r-1}}{(1 - e^{-|\underline{d}|s})^r} \right), & \text{for} \quad r \geq 1, \end{cases}$$

where the notation $\underline{d}|\underline{n}$ means that the product is over all tuples

$$\underline{d} = (d_1, \ldots, d_k) \in \mathbf{N}^k$$

such that $d_1 | n_1$, $d_2 | \frac{n_2}{d_1}, \ldots, d_k | \frac{n_k}{d_1 \cdots d_{k-1}}$. Further, we put

$$|\underline{d}| = d_1 \cdots d_k$$

and

$$\varphi(\underline{d}) = \varphi(d_1) \cdots \varphi(d_k).$$

Theorem 2 (Functional equations).

(i) *If A is a finite abelian group, the following functional equation holds*:

$$\zeta_{\mathbf{F}_1}(-s, A) = w(A) e^{-|A|s} \zeta_{\mathbf{F}_1}(s, A),$$

where $w(A)$ is a complex number of modulus 1 satisfying

$$w(A) = (-1)^{|A|} \det(\Phi_A)^{\frac{1}{|A|}}.$$

(ii) *Assume A is a finitely generated free abelian group of rank 1. Put*

$$A \cong \mathbf{Z} \times \boldsymbol{\mu}_{n_1} \times \boldsymbol{\mu}_{n_2} \times \cdots \times \boldsymbol{\mu}_{n_k}$$

with $n_1 | n_2 | \cdots | n_k$. Then the following functional equation holds:

$$\zeta_{\mathbf{F}_1}(-s, A) = \zeta_{\mathbf{F}_1}(s, A)^{-1} \prod_{\underline{d}|\underline{n}} e^{-\varphi(\underline{d}) d_1^{k-1} \cdots d_k}.$$

(iii) *If A is a finitely generated abelian group of rank $r \geq 2$, the following functional equation holds*:

$$\zeta_{\mathbf{F}_1}(-s, A) = \zeta_{\mathbf{F}_1}(s, A)^{(-1)^r}.$$

Proof. (i): Let $|A| = n$. By Proposition 1(iii), identifying Φ_A with a square matrix of size n^2, we compute

$$\begin{aligned}\zeta_{\mathbf{F}_1}(-s,A) &= \zeta_{\Phi_A}(-s)^{\frac{1}{|A|}} \\ &= \left(\zeta_{\Phi_A}(s)(-1)^{n^2}\operatorname{sgn}(\Phi_A)e^{-n^2 s}\right)^{\frac{1}{|A|}} \\ &= \zeta_{\mathbf{F}_1}(s,A)(-1)^n \det(\Phi_A)^{\frac{1}{|A|}} e^{-ns} \\ &= w(A)e^{-ns}\zeta_{\mathbf{F}_1}(s,A).\end{aligned}$$

(ii): We have $g_1(T) = T$. Thus by Proposition 2,

$$\begin{aligned}\zeta_{\mathbf{F}_1}(-s,A) &= \prod_{\underline{d}|\underline{n}} \exp\left(\frac{e^{|\underline{d}|s}\varphi(\underline{d})d_1^{k-1}\cdots d_k}{1-e^{|\underline{d}|s}}\right) \\ &= \prod_{\underline{d}|\underline{n}} \exp\left(-\frac{\varphi(\underline{d})d_1^{k-1}\cdots d_k}{1-e^{-|\underline{d}|s}}\right).\end{aligned}$$

This gives

$$\zeta_{\mathbf{F}_1}(s,A)\zeta_{\mathbf{F}_1}(-s,A) = \prod_{\underline{d}|\underline{n}} e^{-\varphi(\underline{d})d_1^{k-1}\cdots d_k}.$$

(iii): By Lemma 3, it follows for $r \geq 2$ that

$$\begin{aligned}\zeta_{\mathbf{F}_1}(-s,A) &= \prod_{\underline{d}|\underline{n}} \exp\left(\frac{g_r(e^{|\underline{d}|s})\varphi(\underline{d})d_1^{r+k-1}\cdots d_k^{r-1}}{(1-e^{|\underline{d}|s})^r}\right) \\ &= \prod_{\underline{d}|\underline{n}} \exp\left(\frac{g_r(e^{-|\underline{d}|s})\varphi(\underline{d})d_1^{r+k-1}\cdots d_k^{r-1}}{e^{-r|\underline{d}|s}(1-e^{|\underline{d}|s})^r}\right) \\ &= \prod_{\underline{d}|\underline{n}} \exp\left(\frac{g_r(e^{-|\underline{d}|s})\varphi(\underline{d})d_1^{r+k-1}\cdots d_k^{r-1}}{(-1)^r(1-e^{-|\underline{d}|s})^r}\right) \\ &= \zeta_{\mathbf{F}_1}(s,A)^{(-1)^r}.\end{aligned}$$

□

4. The Riemann Hypothesis and tensor structure. According to the functional equations obtained in the previous section, the Riemann Hypothesis for $\zeta_{\mathbf{F}_1}(s,A)$ asserts that

$$\mathrm{Re}(\rho) = 0$$

for any singularity ρ, which is a zero or a pole of $\log\zeta_{\mathbf{F}_1}(s,A)$.

Theorem 3 (The Riemann Hypothesis and tensor structure)**.**

(i) *When A is a finitely generated abelian group, any singularity ρ of $\zeta_{\mathbf{F}_1}(s,A)$ satisfies that* $\mathrm{Re}(\rho) = 0$.

(ii) *The singularities of $\zeta_{\mathbf{F}_1}(s,A)$ are equipped with a tensor structure, which means the following: For finitely generated abelian groups A_j $(j = 1,2)$ and singularities ρ_j for $\zeta_{\mathbf{F}_1}(s,A_j)$, the sum $\rho_1+\rho_2$ is a singularity of $\zeta_{\mathbf{F}_1}(s,A_1\times A_2)$. Conversely, all singularities of $\zeta_{\mathbf{F}_1}(s,A_1\times A_2)$ have such an expression.*

Proof. (i): Proposition 2 tells that any singularity ρ of $\zeta_{\mathbf{F}_1}(s,A)$ satisfy $e^{-|\underline{d}|\rho} = 1$ for some $\underline{d}$. Thus

$$\rho \in 2\pi i|\underline{d}|^{-1}\mathbf{Z}.$$

(ii): Since

$$|\mathrm{Hom}(A,\boldsymbol{\mu}_m)| = m^r|\mathrm{Hom}(A_{\mathrm{tor}},\boldsymbol{\mu}_m)|,$$

we have for $\mathrm{Re}(s) > 0$ and a finitely generated group A that

$$\begin{aligned}(9)\quad \zeta_{\mathbf{F}_1}(s,A) &= \exp\left(\frac{1}{n}\sum_{m=1}^{\infty} m^{r-1}\mathrm{tr}(\Phi_{A_{\mathrm{tor}}})e^{-ms}\right) \\ &= \exp\left(\frac{1}{n}\sum_{m=1}^{\infty}\sum_{\lambda} m^{r-1}\lambda^m e^{-ms}\right) \\ &= \exp\left(\frac{1}{n}\sum_{\lambda}\frac{g_r(\lambda e^{-s})}{(1-\lambda e^{-s})^r}\right),\end{aligned}$$

where the sum is taken over n^2 eigenvalues of the matrix $\Phi_{A_{\mathrm{tor}}}$ with $n = |A_{\mathrm{tor}}|$.

If we put $n = |(A_1)_{\mathrm{tor}}|$ and $k = |(A_2)_{\mathrm{tor}}|$, then $|(A_1\times A_2)_{\mathrm{tor}}| = nk$ and

$$(10)\qquad \Phi_{(A_1\times A_2)_{\mathrm{tor}}} = \Phi_{(A_1)_{\mathrm{tor}}} \otimes \Phi_{(A_2)_{\mathrm{tor}}},$$

where $\otimes$ denotes the tensor product of matrices. By the expression (9) any singularities ρ_j $(j=1,2)$ of $\zeta_{\mathbf{F}_1}(s,A_j)$ satisfy $e^{\rho_j} = \lambda_j$ with λ_j an eigenvalue of $\Phi_{(A_j)_{\mathrm{tor}}}$. From (10), the eigenvalues of $\Phi_{(A_1\times A_2)_{\mathrm{tor}}}$ are given by the form $\lambda_1\lambda_2$. Thus for any singularity ρ of $\zeta_{\mathbf{F}_1}(s,A_1\times A_2)$, there exist λ_1 and λ_2 such that

$$e^{\rho} = \lambda_1\lambda_2 = e^{\rho_1}e^{\rho_2} = e^{\rho_1+\rho_2}.$$

This shows the tensor structure of singularities. □

Remark 2. The direct product group is considered to be equal to the tensor product over $\mathbf{F}_1$. Namely,

$$A_1 \times A_2 = A_1 \underset{\mathbf{F}_1}{\otimes} A_2.$$

This is an analogous situation to the absolute tensor product (or Kurokawa tensor product) in [4–6]. In other words, $\zeta_{\mathbf{F}_1}(s,A_1\times A_2)$ is analogous to

$$\zeta_{\mathbf{F}_1}(s,A_1)\otimes\zeta_{\mathbf{F}_1}(s,A_2)$$

under the notation in [2] and [3].

Remark 3. Concerning the Euler product expression of the absolute zeta functions, we refer to [5], where weighted Euler products of the form

$$\prod_P (1-N(P)^{-s})^{-w(P)}$$

is crucial.

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