Surfaces carrying no singular functions

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(Communicated by Heisuke HIRONAKA, M.J.A., Nov. 12, 2009)

Abstract: From a finite number of Riemann surfaces W_j ($j \in J := \{1, 2, \cdots, m\}$) we form two kinds of Riemann surfaces, one of which is a united surface $\uplus_{j\in J} W_j$ and the other is simply a bunched surface $\bigcup_{j\in J} W_j$. We compare the space $H(\uplus_{j\in J} W_j)$ of harmonic functions on $\uplus_{j\in J} W_j$ and the space $H(\bigcup_{j\in J} W_j)$ of harmonic functions on $\bigcup_{j\in J} W_j$ and show that these are canonically isomorphic, i.e.

$$H\Big(\uplus_{j\in J} W_j\Big) \cong H\Big(\bigcup_{j\in J} W_j\Big)$$

in the sense that there is a bijective mapping t of the former space onto the latter space such that t is linearly isomorphic, t preserves orders, i.e. $tu \geqq 0$ if and only if $u \geqq 0$, and t fixes the real number field $\mathbf{R}$, i.e. $t\lambda = \lambda$ for every $\lambda \in \mathbf{R}$, under the standing assumption that all the W_j are hyperbolic. The result is then applied to give a sufficient condition better than our former one for an afforested surface to belong to the class $\mathcal{O}_s$ of hyperbolic Riemann surfaces carrying no nonzero singular harmonic functions when its plantation and trees on it are all in $\mathcal{O}_s$.

Key words: Afforested surface; hyperbolic; parabolic; Parreau decomposition; quasibounded; singular.

We denote by $H(R)$ the real vector space of harmonic functions on a Riemann surface R and by $HP(R)$ the vector subspace of $H(R)$ consisting of essentially positive $u \in H(R)$ in the sense that $|u|$ admits a harmonic majorant on R. Then $HP(R)$ forms a vector lattice with lattice operations of join $\vee$ and meet $\wedge$ so that $u \vee v$ ($u \wedge v$, resp.) is the least (the greatest, resp.) harmonic majorant (minorant, resp.) of u and v in $HP(R)$ on R. A $u \in HP(R)$ is said to be *quasibounded* if

(1) $$u = \lim_{s,t\in\mathbf{R}^+,\ s,t\uparrow\infty} (u \wedge s) \vee (-t)$$

locally uniformly on R and a $u \in HP(R)$ is said to be *singular* if

(2) $$(u \wedge s) \vee (-t) = 0$$

for every pair of s and t in $\mathbf{R}^+ := \{t \in \mathbf{R} : t \geqq 0\}$, where $\mathbf{R}$ is the real number field. On denoting by $HP_q(R)$ ($HP_s(R)$, resp.) the vector sublattice of $HP(R)$ consisting of quasibounded (singular, resp.) $u \in HP(R)$, we obtain the direct sum decomposition referred to as the Parreau decomposition of $HP(R)$:

(3) $$HP(R) = HP_q(R) \oplus HP_s(R).$$

We recall that $\mathcal{O}_G$ is the class of parabolic Riemann surfaces R characterized by the nonexistence of the Green function $g(\cdot, \zeta; R)$ on R with its pole ζ in R so that $R \notin \mathcal{O}_G$ means that R is hyperbolic in the sense that the Green function $g(\cdot, \zeta; R)$ on R exists for one and hence for every point ζ in R. The notation $\mathcal{O}_{HP}$ denotes the class of Riemann surfaces R with $HP(R) = \mathbf{R}$. Then we know the following important result of Sario and Tôki (cf. e.g. [8]):

(4) $$\mathcal{O}_G < \mathcal{O}_{HP} \quad \text{(the strict inclusion relation)},$$

and therefore, as far as we are concerned with the space $HP(R)$, it is natural to assume that $R \notin \mathcal{O}_G$ in advance in order to avoid the trivial case $HP(R) = \mathbf{R}$ including $HP_q(R) = \mathbf{R}$ and $HP_s(R) = \{0\}$. Even if $R \notin \mathcal{O}_G$ it can happen the case $HP_s(R) = \{0\}$. Then the main theme of the present paper is the class

(5) $$\mathcal{O}_s$$

2000 Mathematics Subject Classification. Primary 30F20; Secondary 30F15, 30F25.

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doi: 10.3792/pjaa.85.163

of Riemann surfaces $R \notin \mathcal{O}_G$ such that $HP_s(R) = \{0\}$. A typical example R in the class $\mathcal{O}_s$ is furnished by $R \in \mathcal{O}_{HP} \setminus \mathcal{O}_G$ (cf. (4) above).

We denote by $\dim R$ for any Riemann surface $R \notin \mathcal{O}_G$ the *harmonic dimension* of R which is the cardinal number of the set of minimal Martin boundary points of R (cf. e.g. [1]). We have shown in [4] the following result:

$$(6) \qquad \dim \mathcal{O}_s := \{\dim R : R \in \mathcal{O}_s\} = \mathbf{N} \cup \{\aleph_0\},$$

where $\mathbf{N}$ is the set of positive integers and $\aleph_0 := \operatorname{card} \mathbf{N}$, the cardinal number of $\mathbf{N}$, as a refinement of the former result in [3] that $\dim R \leqq \aleph_0$ for $R \in \mathcal{O}_s$. In the course of proving (6) we have introduced the notion of, what we call, afforested surfaces.

An *afforested surface*

$$W := \langle P, (T_i)_{i \in \mathbf{N}_\xi}, (\sigma_i)_{i \in \mathbf{N}_\xi} \rangle$$

consists of three ingredients: a Riemann surface P called a plantation; a finite or infinite sequence $(T_i)_{i \in \mathbf{N}_\xi}$ of Riemann surfaces T_i, each of which is called a tree, where $\mathbf{N}_\xi := \{1, 2, \cdots, \xi\}$ is a finite set if $\xi \in \mathbf{N}$ and $\mathbf{N}_\xi = \mathbf{N}_{\aleph_0} := \mathbf{N}$ is an infinite set if $\xi = \aleph_0$; a sequence $(\sigma_i)_{i \in \mathbf{N}_\xi}$ of slits σ_i commonly included in P and T_i for each $i \in \mathbf{N}_\xi$, which are mutually disjoint and do not accumulate in P, and each σ_i of which is called the root of each tree T_i and at the same time the root hole in P. We paste each $T_i \setminus \sigma_i$ to $P \setminus (\bigcup_{j \in \mathbf{N}_\xi} \sigma_j)$ crosswise along each σ_i for every $i \in \mathbf{N}_\xi$ and the resulting Riemann surface is the afforested surface $W := \langle P, (T_i)_{i \in \mathbf{N}_\xi}, (\sigma_i)_{i \in \mathbf{N}_\xi} \rangle$.

Our question is whether the condition $P \in \mathcal{O}_s$ and $T_i \in \mathcal{O}_s$ $(i \in \mathbf{N}_\xi)$ assures that $W := \langle P, (T_i)_{i \in \mathbf{N}_\xi}, (\sigma_i)_{i \in \mathbf{N}_\xi} \rangle \in \mathcal{O}_s$ or not. We have seen in [5] that this is not the case in general but on the other hand we have also seen in [4] that if $\xi \in \mathbf{N}$ or if $\xi = \aleph_0$ and

$$(7) \qquad \sum_{i \in \mathbf{N}} (4M_i + 1) \frac{\sup_{P \setminus V_i} g(\,\cdot\,, \zeta_i; P)}{\inf_{\sigma_i} g(\,\cdot\,, \zeta_i; P)} < 1,$$

then $W \in \mathcal{O}_s$ can be concluded. Here $V_i := \{|z| < 1\}$ is a parametric disc about the point ζ_i which correspond to the center 0 of the slit $\sigma_i = [-s_i, s_i] \subset V_i$ in terms of the local parameter V_i for each $i \in \mathbf{N}$. Moreover it is assumed that $\overline{V}_i \cap \overline{V}_j = \emptyset$ $(i \neq j)$ and let M_i be the Harnack constant of the set $\{o\} \cup \partial V_i$ with a reference point $o \in P \setminus \bigcup_{i \in \mathbf{N}} (1/2)\overline{V}_i$ with respect to the family $H(P \setminus \bigcup_{i \in \mathbf{N}} (1/2)\overline{V}_i)^+$, where $\mathcal{F}^+$ is the class of nonnegative functions in the function space $\mathcal{F}$ (see also e.g. [4] for its precise definition). However (7) is not too good in the following two points. First, it is too restrictive in practical application; at least < 1 in the condition (7) should be desirably replaced by $< \infty$. Second the condition like (7) should be something that can take care of not only the case of $\xi = \aleph_0$ but also that of $\xi \in \mathbf{N}$. The primary purpose of this paper is to replace (7) by

$$(8) \qquad \sum_{i \in \mathbf{N}_\xi} M_i \frac{\sup_{P \setminus V_i} g(\,\cdot\,, \zeta_i; P)}{\inf_{\sigma_i} g(\,\cdot\,, \zeta_i; P)} < +\infty,$$

under which we can conclude that $P \in \mathcal{O}_s$ and $T_i \in \mathcal{O}_s$ $(i \in \mathbf{N}_\xi)$ imply $W := \langle P, (T_i)_{i \in \mathbf{N}_\xi}, (\sigma_i)_{i \in \mathbf{N}_\xi} \rangle \in \mathcal{O}_s$. Since (8) also assures that $\dim W = \xi + 1$ $(\xi \in \mathbf{N} \cup \{\aleph_0\})$ by taking P and T_i $(i \in \mathbf{N}_\xi)$ in $\mathcal{O}_{HP} \setminus \mathcal{O}_G$, we can also deduce (6). We remark that (8) is automatically satisfied for $\xi \in \mathbf{N}$ so that it is really a condition to be assumed for the case of $\xi = \aleph_0$, although the condition (7) in which $\mathbf{N}$ is replaced by $\mathbf{N}_\xi$ with $\xi \in \mathbf{N}$ may not be valid in general and the genuine (7) itself may not be true even if (8) for $\xi = \aleph_0$ holds. Anyhow we will show in the sequel that (8) assures $W \in \mathcal{O}_s$ when $P \in \mathcal{O}_s$ and $T_i \in \mathcal{O}_s$ $(i \in \mathbf{N}_\xi)$.

Let X and Y be two Riemann surfaces and γ a slit commonly contained in both of X and Y. We denote by $(X \setminus \gamma) \uplus_\gamma (Y \setminus \gamma)$ the Riemann surface obtained by pasting $X \setminus \gamma$ to $Y \setminus \gamma$ crosswise along γ. Given a finite number, say m, of open Riemann surfaces W_j $(j \in J := \{1, 2, \cdots, m\})$. Suppose a permutation $J' := \{j_1, j_2, \cdots, j_m\}$ of J is given. Let $Z_1 := (W_{j_1} \setminus \gamma_{j_1}) \uplus_{\gamma_{j_1}} (W_{j_2} \setminus \gamma_{j_1})$ for a common slit γ_{j_1} in W_{j_1} and W_{j_2}, $Z_2 := (Z_1 \setminus \gamma_{j_2}) \uplus_{\gamma_{j_2}} (W_{j_3} \setminus \gamma_{j_2})$ for a common slit γ_{j_2} in Z_1 and W_{j_3}, and finally $Z_{m-1} := (Z_{m-2} \setminus \gamma_{j_{m-1}}) \uplus_{\gamma_{j_{m-1}}} (W_{j_m} \setminus \gamma_{j_{m-1}})$ for a common slit $\gamma_{j_{m-1}}$ in Z_{m-2} and W_{j_m}. We then denote by $\biguplus_{j \in J} W_j$ the final Riemann surface Z_{m-1} neglecting how the permutation J' and the sequence of pasting slits γ_{j_i} $(i \in J)$ are chosen and call the surface $\biguplus_{j \in J} W_j$ as a *united surface* consisting of W_j $(j \in J)$. We can view $\bigcup_{j \in J} W_j$ a disconnected Riemann surface and call it as a *bunched surface* consisting of W_j $(j \in J)$. We are interested in comparing ordered vector space structures of the harmonic function spaces $H(\biguplus_{j \in J} W_j)$ and $H(\bigcup_{j \in J} W_j)$. The latter is simply given by

$$H\Big(\bigcup_{j \in J} W_j\Big) = \bigoplus_{j \in J} H(W_j) \quad \text{(direct sum)},$$

where we understand that $u|W_i \equiv 0$ $(i \in J \setminus \{j\})$ for $u \in H(W_j)$. In general, let $\mathcal{F}$ be an ordered vector space consisting of some real valued functions with respect to function sums and function orders containing the constant function subspace $\mathbf{R}$. We say that two such spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ are *canonically isomorphic*, $\mathcal{F}_1 \cong \mathcal{F}_2$ in notation, if there is a bijective mapping t of $\mathcal{F}_1$ onto $\mathcal{F}_2$ satisfying the following 3 conditions: t is a vector space isomorphism of $\mathcal{F}_1$ onto $\mathcal{F}_2$; t preserves order in the sense that $tf \geqq 0$ if and only if $f \geqq 0$ for $f \in \mathcal{F}_1$; $t\lambda = \lambda$ for every $\lambda \in \mathbf{R}$. We maintain the following assertion: if all the Riemann surfaces W_j are hyperbolic (i.e. $W_j \notin \mathcal{O}_G$) for all $j \in J$, then

$$H\Big(\bowtie_{j\in J} W_j\Big) \cong H\Big(\bigcup_{j\in J} W_j\Big) \tag{9}$$
(canonically isomorphic).

In passing we remark that the hyperbolicity of all the W_j ($j \in J$) is essential for the validity of (9). For, let

$$W_1 \in \mathcal{O}_{HP} \setminus \mathcal{O}_G$$

and

$$W_2 := \hat{\mathbf{C}} \setminus \{0, \infty\} \in \mathcal{O}_G,$$

where $\hat{\mathbf{C}}$ is the Riemann sphere. If $H(W_1 \bowtie W_2) \cong H(W_1 \cup W_2) = H(W_1) \oplus H(W_2)$, then a canonical isomorphism t here preserves HP and hence $HP(W_1 \bowtie W_2) \cong HP(W_1 \cup W_2) = HP(W_1) \oplus HP(W_2)$ so that $\dim HP(W_1 \bowtie W_2) = \dim HP(W_1 \cup W_2)$ must be true. Here e.g. $\dim HP(W_1 \bowtie W_2)$ is the usual vector space dimension of the vector space $HP(W_1 \bowtie W_2)$. However, $\dim HP(W_1 \bowtie W_2) = 3$ and $\dim HP(W_1 \cup W_2) = 2$. Therefore we see that

$$H(W_1 \bowtie W_2) \ncong H(W_1 \cup W_2),$$

i.e. (9) may not be true when there is a $W_j \in \mathcal{O}_G$.

For the proof of (9), by using the induction, we can assume that $J = \{1, 2\}$. Let X be a hyperbolic Riemann surface and ∞_X the ideal boundary of X in the sense of Alexandroff. Any complement A of a compact subset of X is said to be an ideal boundary neighborhood of ∞_X and any two harmonic functions u and v on A are said to coincide with each other at ∞_X, $u \doteq v$ at ∞_X in notation, if $|u - v|$ is dominated by a potential (cf. e.g. [2]) on X on an ideal boundary neighborhood of ∞_X. A function $s \in H(A)$ for an ideal boundary neighborhood A of ∞_X is said to be a *singularity* at ∞_X and any $p \in H(X)$ with $p \doteq s$ at ∞_X is said to be a (Dirichlet) *principal function* of s on X (cf. e.g. Rodin-Sario [7]). We have then the following useful result (cf. e.g. [6]):

Principal Function Theorem. *There exists a unique principal function p on a hyperbolic Riemann surface X of any given singularity s at the ideal boundary ∞_X of X.*

To prove this let A be an ideal boundary neighborhood of ∞_X such that $s \in H(\bar{A})$ and A is the complement of the closure $X \setminus A$ of a regular subregion of X and B is a regular subregion of X with $B \supset X \setminus A$. For any $f \in C(\partial A)$ ($C(\partial B)$, resp.) H_f^A (H_f^B, resp.) is the PWB (i.e. Perron-Wiener-Brelot) solution of Dirichlet problem on A (B, resp.) with the boundary data f on $\alpha := \partial A$ ($\beta := \partial B$, resp.) (cf. e.g. [1]) so that moreover the additional condition $H_f^A \doteq 0$ at ∞_X is imposed upon H_f^A. Let $T\varphi := H_f^A|\beta$ with $f = H_\varphi^B|\alpha$ for $\varphi \in C(\beta)$. Since the sup-norm of H_1^A on β is strictly less than 1, i.e. $\|H_1^A; \beta\|_\infty =: k < 1$, by virtue of $X \notin \mathcal{O}_G$, $T : C(\beta) \to C(\beta)$ is a bounded linear operator with the operator norm $\|T\| \leqq k < 1$, and the abstract integral equation

$$(I - T)\varphi = s_0, \quad s_0 := s - H_s^A$$

has a unique solution $\varphi \in C(\beta)$ in the C. Neumann series

$$\varphi := (I - T)^{-1} s_0 = \sum_{n=0}^{\infty} T^n s_0$$

so that by setting $f := H_\varphi^B$ we obtain

$$f|\alpha = H_\varphi^B|\alpha, \quad H_{f-s}^A|\beta = (f - s)|\beta. \tag{10}$$

We define a $p \in H(X)$ by $p|A = H_f^A + s_0$ and $p|B = H_\varphi^B$. We need to ascertain that these two functions are identical on $A \cap B$. In fact, by using (10), we have

$$(p|A)|\alpha = f|\alpha + s_0|\alpha = f|\alpha,$$
$$(p|B)|\alpha = H_\varphi^B|\alpha = f|\alpha$$

so that $p|A = p|B$ on $\alpha = \partial A$, and similarly

$$(p|A)|\beta = H_f^A|\beta + s_0|\beta = T\varphi + s_0|\beta$$
$$= T\varphi + (I - T)\varphi = \varphi,$$
$$(p|B)|\beta = H_\varphi^B|\beta = \varphi,$$

so that $p|A = p|B$ on $\beta = \partial B$. Since $p|A = p|B$ on $\partial(A \cap B) = \partial A \cup \partial B = \alpha \cup \beta$, we can conclude that $p|A = p|B$ on $A \cap B$. Thus p is well defined on X and $p \in H(X)$. Then $p = H_f^A + s_0$ on A shows that $p - s = H_{p-s}^A$ and $p - s \doteq H_{p-s}^A \doteq 0$ at ∞_X and therefore p is a principal function for the singularity s. The unicity of principal function for s is trivial since, if there are two principal functions p_1 and p_2, then $p_1 - p_2 \doteq s - s = 0$ at ∞_X and $p_1 \equiv p_2$ on X. □

We return to the proof of (9) in the form $H(W_1 \bowtie W_2) \cong H(W_1 \cup W_2)$. Fix an arbitrary ideal boundary neighborhood A_j of ∞_{W_j} for $j = 1$ and 2 such that $A_j \subset W_1 \bowtie W_2$ $(j = 1, 2)$ and $A := A_1 \cup A_2 \subset W_1 \bowtie W_2$ is an ideal boundary neighborhood of $\infty_{W_1 \bowtie W_2}$ so that A is also an ideal boundary neighborhood of $\infty_{W_1 \cup W_2}$. For any $u \in H(W_1 \bowtie W_2)$, let $tu = (t_1 \oplus t_2)u := t_1 u + t_2 u \in H(W_1) \oplus H(W_2) = H(W_1 \cup W_2)$ with $t_i u | W_j \equiv 0$ $(i \neq j)$ and $t_i u \doteq u | A_i$ at ∞_{W_i}. The bijectiveness of $t : H(W_1 \bowtie W_2) \to H(W_1 \cup W_2)$ can be easily seen by the principal function theorem and it is also easily checked that t is a canonical isomorphism. The proof of (9) is herewith complete.

It is seen that the order preserving and the linear structure preserving map $t = \oplus_{j \in J} t_j$ giving a canonical isomorphism in (9) clearly preserves HP, HP_q, and HP_s:

$$HY\left(\biguplus_{j \in J} W_j\right) \cong HY\left(\bigcup_{j \in J} W_j\right) = \bigoplus_{j \in J} HY(W_j) \tag{11}$$

$$(Y = P,\ P_q,\ P_s).$$

As a consequence of this we can deduce the following

Assertion 12. *The united surface $\biguplus_{i \in J} W_j$ of hyperbolic Riemann surfaces W_j $(j \in J$ with $J = \{1, 2, \cdots, m\}$; $m \in \mathbf{N})$ belongs to the class $\mathcal{O}_s$ if and only if every $W_j \in \mathcal{O}_s$ $(j \in J)$.*

Since an afforested surface W given by $\langle P, (T_j)_{j \in \mathbf{N}_\xi}, (\sigma_j)_{j \in \mathbf{N}_\xi} \rangle$ for a $\xi \in \mathbf{N}$ is a kind of united surface $P \bowtie (\biguplus_{j \in \mathbf{N}_\xi} W_j)$, the assertion 12 assures that $W \in \mathcal{O}_s$ if and only if $P \in \mathcal{O}_s$ and every $T_j \in \mathcal{O}_s$ $(j \in \mathbf{N}_\xi)$. Hence, in particular, if $P \in \mathcal{O}_s$ and $T_j \in \mathcal{O}_s$ $(j \in \mathbf{N}_\xi)$, then $W \in \mathcal{O}_s$. Next, let $W := \langle P, (T_j)_{j \in \mathbf{N}}, (\sigma_j)_{j \in \mathbf{N}} \rangle$ and assume that $P \in \mathcal{O}_s$ and $T_j \in \mathcal{O}_s$ $(j \in \mathbf{N})$. Clearly $W_m := \langle P, (T_j)_{j \geqq m+1}, (\sigma_j)_{j \geqq m+1} \rangle \notin \mathcal{O}_G$ and therefore, again by Assertion 12, $W = W_m \bowtie (\biguplus_{1 \leqq j \leqq m} T_j) \in \mathcal{O}_s$ if and only if $W_m \in \mathcal{O}_s$. Hence, in particular, we state the following

Assertion 13. *The membership of an afforested surface $W := \langle P, (T_j)_{j \in \mathbf{N}}, (\sigma_j)_{j \in \mathbf{N}} \rangle$ with $P \in \mathcal{O}_s$ and $T_j \in \mathcal{O}_s$ $(j \in \mathbf{N})$ in $\mathcal{O}_s$ is not affected by adding or deleting of a finite number of trees to or from the sequence $(T_j)_{j \in \mathbf{N}}$.*

Suppose (8) with $\xi = \aleph_0$ is valid. Then we can find an $m \in \mathbf{N}$ such that

$$\sum_{j > m} (4M_i + 1) \frac{\sup_{P \setminus V_i} g(\,\cdot\,, \zeta_i; P)}{\inf_{\sigma_i} g(\,\cdot\,, \zeta_i; P)} \leqq 5 \sum_{j > m} M_i \frac{\sup_{P \setminus V_i} g(\,\cdot\,, \zeta_i; P)}{\inf_{\sigma_i} g(\,\cdot\,, \zeta_i; P)} < 1.$$

Then we have (7) for the afforested surface $W_m := \langle P, (T_j)_{j > m}, (\sigma_j)_{j > m} \rangle$ so that we can conclude $W_m \in \mathcal{O}_s$ by our former result (cf. [4]). Adding m trees $T_1, \cdots, T_m$ to W_m we obtain $W := \langle P, (T_j)_{j \in \mathbf{N}}, (\sigma_j)_{j \in \mathbf{N}} \rangle$ and $W \in \mathcal{O}_s$ along with $W_m \in \mathcal{O}_s$ by assertion 13.

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