Value distribution of the third Painlevé transcendents in sectorial domains

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Abstract: This article concerns the value distribution of the third Painlevé transcendents in sectorial domains around fixed singular points. We show that the cardinality of the zeros of a third Painlevé transcendent in a sector has an asymptotic growth of finite order, thereby giving an improvement of the known estimation.

Key words: Painlevé transcendents; value distribution; sector.

1. Introduction. In order to find new special functions, in the end of the ninteenth century, P. Painlevé studied on the second order algebraic differential equations which are free from movable singularities except for poles. And, he and G. Gambier, one of his colleagues, obtained six *Painlevé equations*. The third Painlevé equation

$$y'' = \frac{1}{y}(y')^2 - \frac{1}{x}y' + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \tag{1}$$

is one of them. Here $' = \mathrm{d}/\mathrm{d}x$ and $\alpha, \beta, \gamma, \delta \in \mathbf{C}$.

About ninety years later, it is established that the Painlevé equations has the *irreducibility*, i.e. the Painlevé equations define new special functions as solutions of them. The irreducibility of (1) is showed by Umemura and Watanabe in [5]. So, it is suitable to call solutions of the Painlevé equations *Painlevé transcendents*.

In this paper, we study the value distribution of the third Painlevé transcendents. We omit the case where $(\alpha, \gamma) = (0, 0)$ or $(\beta, \delta) = (0, 0)$ because (1) is integrable in this case([4]), and concentrate on the generic case.

Define

$$S(\phi, r, R) := \{x \,|\, |\arg x| < \phi < \pi, r < |x| < R\},$$

$$n(y, \phi, r, R) := \sharp\{x \in S(\phi, r, R) | y(x) = 0\}.$$

The aim of this article is to establish

Theorem. *If $(\alpha, \gamma) \neq (0, 0)$ and $(\beta, \delta) \neq (0, 0)$, then there exists a positive constant C, independent of $(\alpha, \beta, \gamma, \delta)$, such that for any solution $y = y(x)$ to (1), one has*

$$n(y, \phi, r, R) = O(r^{-C}) \text{ as } r \to 0,$$
$$n(y, \phi, r, R) = O(R^{C}) \text{ as } R \to \infty.$$

The theorem sharpens the estimation obtained by Shimomura in [3]. A similar result can be obtained for a-points for any non-zero complex number.

2. Proof of Theorem.

2.1. Cases and path construction. To prove the theorem, we make a change of dependent variable in each case, and then apply the following lemma obtained in [1]:

Lemma 2.1. *Let $u(s)$ be a solution of*

$$\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} = 1 + f_0(s, u) + f_1(s, u)\frac{\mathrm{d}u}{\mathrm{d}s} + f_2(s, u)\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2,$$

around $s = 0$. Suppose $f_j(s, u)(j = 0, 1, 2)$ is analytic in $D_0 = \{(s, u) \in \mathbf{C}^2 | |s| < 1, |u| < R_0\}$, $0 < R_0 < 1$, and satisfying $|f_0(s, u)| < 1/200$, $|f_1(s, u)| < K$, $|f_2(s, u)| < K$ in D_0, where K is some positive number. Put $\theta := \min\{ R_0^{1/2}/4, (200K)^{-1/2}, (200K)^{-1}\}$. If $|u(0)| \leq \theta^2/8$, then $|u(s)| \leq 15\theta^2$ in the disk $|s| < \lambda_0$ and $|u(s)| \geq \theta^2/4$ on the circle $|s| = 3\lambda_0/4$, where

$$\lambda_0 = \begin{cases} 4\theta & \text{if } |\frac{\mathrm{d}u}{\mathrm{d}s}(0)| \leq \theta, \\ \frac{4}{3}\theta^2/|\frac{\mathrm{d}u}{\mathrm{d}s}(0)| & \text{if } |\frac{\mathrm{d}u}{\mathrm{d}s}(0)| > \theta. \end{cases}$$

For the study of behavior around $x = \infty$, we devide into 4 cases; Case (i) $\delta \neq 0$, $\delta \neq -16\gamma$: let $y = \eta$, $\mu = 2$, $u = \eta - \mu$, $x = a + k^{-1/2}s$ and $k = 8\gamma + \delta/2$; Case (ii) $\delta \neq 0$, $\delta = -16\gamma$: let $y = \eta$, $\mu = 3$, $u = \eta - \mu$, $x = a + k^{-1/2}s$ and $k = 27\gamma + \delta/3$; Case (iii) $\delta = 0$, $\gamma \neq 0$: let $y = \eta$, $\mu = 2$, $u = \eta - \mu$, $x = a + k^{-1/2}s$ and $k = 8\gamma$; Case (iv) $\delta = 0$, $\gamma = 0$:

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let $y = x\eta$, $\mu = 2$, $u = \eta - \mu$, $x = a + k^{-1/2}s$ and $k = 4\alpha$. Note that the zeros of $y(x)$ coincide with the zeros of $\eta(x)$.

In addition to the above, for the study of behavior around $x = 0$, we make the change of independent variable $x = 1/z$. Then we are able to treat it as behavior around $z = \infty$ as well. That is convenient. So, we add 4 more cases: Case (v) $\delta \neq 0$, $\gamma \neq 0$; Case (vi) $\delta \neq 0$, $\gamma = 0$; Case (vii) $\delta = 0$, $\gamma \neq 0$; Case (viii) $\delta = \gamma = 0$. For Case (v) and (vii), let $y = z^2\eta$, $\mu = 2$, $u = \eta - \mu$, $x = a + k^{-1/2}s$ and $k = 8\gamma$; and for Case (vi) and (viii), let $y = z^3\eta$, $\mu = 2$, $u = \eta - \mu$, $x = a + k^{-1/2}s$ and $k = 4\alpha$. In what follows, when we state something common in all cases, we may denote the independent variable x not only in Case (i),...,(iv) but also in Case (v),...,(viii) instead of z for brevity.

Then, in each case, we have

$$\frac{\mathrm{d}^2u}{\mathrm{d}s^2} = 1 + f_0(s,u) + f_1(s,u)\frac{\mathrm{d}u}{\mathrm{d}s} + f_2(s,u)\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2$$

and $f_0(s,u) = O(u) + O(1/x)$, that is just in the form of Lemma 2.1. Applying Lemma 2.1 to the equation of η in Case (j), we can verify

Lemma 2.2. *For each $(\alpha, \beta, \gamma, \delta) \in \mathbf{C}^4$ satisfying the condition of Case (j), there exists a quartet of positive numbers T_0, $\mu \neq 0$, Δ and A_0, each of them independent of $\eta(x)$, with the properties:*
for $x = a$ satisfying $|a| > T_0$, if $|\eta(a) - \mu| \leq \Delta$, then $|\eta(x) - \mu| \geq 2\Delta$ on the circle $|x - a| = \epsilon_a$, and $\eta(x) \neq 0$ in the disk $|x - a| \leq \epsilon_a$. Here, $\epsilon_a > 0$ satisfies

$$\epsilon_a \leq A_0, \quad \epsilon_a^{-1} \leq A_0(1 + |\eta'(a)|).$$

We construct a path which will be used in estimating integrals. In what follows, T_0, μ, Δ and A_0 denote the constants given in Lemma 2.2. Similarly to [1, Lemma 1.3], we establish

Lemma 2.3. *Put $S_\phi^{L_0} := \{x \,|\, |\arg x| < \phi, L_0 \leq |x|\}$ with $L_0 := \min\{T_0, A_0/\sin\phi\}$, $0 < \phi < \pi$. Let $\sigma \in S_\phi^{L_0}$ be an arbitrary point satisfying $\eta(\sigma) = 0$, $|\sigma| > 2L_0$. Then there exists a path $\Gamma(\sigma)$ with the properties:* (I) *$\Gamma(\sigma) \subset S_\phi^{L_0}$ starts from $a_0(\sigma) \in \partial S_\phi^{L_0} \cap \{x \,|\, |x| = L_0\}$ and ends at σ;* (II) *the length of $\Gamma(\sigma)$ does not exceed $(\pi + 1)|\sigma|$;* (III) *$|y(x) - \mu| > \Delta$ along $\Gamma(\sigma)$.*

Remark. Even if the sectorial angle ϕ is small, taking sufficiently large L_0, a circle with radius A_0 crosses at most only one of two lines; $\arg x = \pm\phi$.

2.2. Auxiliary functions. Now we present the auxiliary function $\Psi_{(j)}$ for Case (j) ($j = \mathrm{i}, \mathrm{ii}, \ldots, \mathrm{viii}$):

$$\begin{aligned}
\Psi_{(\mathrm{i})}(\mu, x) :=& \frac{x^2(\eta')^2}{\eta^2} + \frac{2x(\eta+\mu)\eta'}{\eta(\eta-\mu)} \\
&-2\alpha x\eta + \frac{2\beta x}{\eta} - \gamma x^2\eta^2 + \frac{\delta x^2}{\eta^2},\\
\Psi_{(\mathrm{ii})}(\mu, x) :=& \Psi_{(\mathrm{i})}(\mu, x),\ \Psi_{(\mathrm{iii})}(\mu, x) := \Psi_{(\mathrm{i})}(\mu, x),\\
\Psi_{(\mathrm{iv})}(\mu, x) :=& \frac{x^2(\eta')^2}{\eta^2} + \frac{4x\eta'}{\eta-\mu} + 2\frac{\eta+\mu}{\eta-\mu} \\
&-2\alpha x^2\eta + \frac{2\beta}{\eta},\\
\Psi_{(\mathrm{v})}(\mu, z) :=& \frac{z^2(\dot\eta)^2}{\eta^2} + \frac{2z(\eta-3\mu)\dot\eta}{\eta(\eta-\mu)} \\
&-2\alpha z\eta + \frac{2\beta}{z^3\eta} - \gamma z^2\eta^2 + \frac{\delta}{z^6\eta^2},\\
\Psi_{(\mathrm{vi})}(\mu, z) :=& \frac{z^2(\dot\eta)^2}{\eta^2} + \frac{4z(\eta-2\mu)\dot\eta}{\eta(\eta-\mu)} \\
&-2\alpha z^2\eta + \frac{2\beta}{z^4\eta} + \frac{\delta}{z^8\eta^2},\\
\Psi_{(\mathrm{vii})}(\mu, z) :=& \Psi_{(\mathrm{v})}(\mu, z),\ \Psi_{(\mathrm{viii})}(\mu, z) := \Psi_{(\mathrm{vi})}(\mu, z),
\end{aligned}$$

where $\mu \neq 0, \infty$, $' = \mathrm{d}/\mathrm{d}x$ and $\dot{} = \mathrm{d}/\mathrm{d}z$.

In Case (iv), for example, the points satisfying $\eta(x_0) = 0, \infty$ are singularities of P_{III}, however, the auxiliary function $\Psi_{(\mathrm{iv})}(\mu, x)$ is singular not at $\eta(x_0) = 0, \infty$, but at $\eta(x_0) = \mu$. It is similar in every other case. How to construct such functions is explained in [2].

Moreover, $\Psi_{(j)}(\mu, x)$ satisfies the first order linear differential equation

$$\text{(2)} \qquad \frac{\mathrm{d}\Psi_{(j)}(\mu, x)}{\mathrm{d}x} = P_{(j)}(x)\Psi_{(j)}(\mu, x) + Q_{(j)}(x),$$

where $P_{(j)}(x)$ and $Q_{(j)}(x) = Q_{0(j)}(x,\eta) + Q_{1(j)}(x,\eta)\eta'$ are singular only at $\eta(x_0) = \mu$ as well. If $\eta(x) \neq \mu$ on a path $\Gamma(x)$ starting from x_0 and ending at x, then (2) posesses the solution as follows:

$$\begin{aligned}
\Psi_{(j)}(\mu, x) =& \Xi_{(j)}(x) + E_{(j)}(x)\left\{\Psi_{(j)}(\mu, x_0) - \frac{\Xi_{(j)}(x_0)}{E_{(j)}(x_0)}\right\}\\
&+E_{(j)}(x)\int_{\Gamma(x)} E_{(j)}(t)^{-1}\{Q_{0(j)}(t,\eta) + P_{(j)}(t)\Xi_{(j)}(t)\}\mathrm{d}t,
\end{aligned}$$

where $E_{(j)}(x) := \exp\int_{\Gamma(x)} P_{(j)}(t)\mathrm{d}t$ and $\Xi_{(j)}(x)$ is a function satisfying $Q_{1(j)}(x,\eta)\eta' = \mathrm{d}\Xi_{(j)}(x)/\mathrm{d}x$.

The functions are as follows:

$$P_{(\mathrm{i})}(x) = \frac{4\mu\eta}{x(\eta-\mu)^2}, \quad \Xi_{(\mathrm{i})}(x) = \frac{-12\mu}{\eta-\mu} + \frac{-8\mu^2}{(\eta-\mu)^2},$$

$$Q_{0(\mathrm{i})}(x,\eta) = \alpha\frac{-4\mu\eta(\eta+\mu)}{(\eta-\mu)^2} + \beta\frac{4(\eta+\mu)}{(\eta-\mu)^2} + \gamma\frac{-4\mu^2 x\eta^2}{(\eta-\mu)^2} + \delta\frac{4x}{(\eta-\mu)^2},$$

$$P_{(\mathrm{iv})}(x) = \frac{-4\mu\eta}{x(\eta-\mu)^2}, \quad \Xi_{(\mathrm{iv})}(x) = \frac{-8\mu}{\eta-\mu} + \frac{-8\mu^2}{(\eta-\mu)^2},$$

$$Q_{0(\mathrm{iv})}(x,\eta) = \frac{8\mu\eta(\eta+\mu)}{x(\eta-\mu)^3} + \alpha\frac{-4\mu x\eta(\eta+\mu)}{(\eta-\mu)^2} + \beta\frac{4(\eta+\mu)}{x(\eta-\mu)^2},$$

$$P_{(\mathrm{v})}(z) = \frac{4\mu\eta}{z(\eta-\mu)^2}, \quad \Xi_{(\mathrm{v})}(z) = \frac{-16\mu}{\eta-\mu} + \frac{-8\mu^2}{(\eta-\mu)^2},$$

$$Q_{0(\mathrm{v})}(z,\eta) = \alpha\frac{4\mu\eta(\eta+\mu)}{(\eta-\mu)^2} + \beta\frac{-4(\eta+\mu)}{z^4(\eta-\mu)^2} + \gamma\frac{4z\mu^2\eta^2}{(\eta-\mu)^2} + \delta\frac{-4}{z^7(\eta-\mu)^2},$$

$$P_{(\mathrm{vi})}(z) = \frac{4\mu\eta}{z(\eta-\mu)^2}, \quad \Xi_{(\mathrm{vi})}(z) = \frac{16\mu}{\eta-\mu} + \frac{-8\mu^2}{(\eta-\mu)^2},$$

$$Q_{0(\mathrm{vi})}(z,\eta) = \alpha\frac{4\mu z\eta(\eta+\mu)}{(\eta-\mu)^2} + \beta\frac{-4(\eta+\mu)}{z^5(\eta-\mu)^2} + \delta\frac{-4}{z^9(\eta-\mu)^2}.$$

If there is any points satisfying $\eta(x) = \mu, 0, \infty$ on $\{|x| = L_0\}$, retake L_0 a little bit larger so that $\eta(x) \neq \mu, 0, \infty$ on $\{|x| = L_0\}$. Similarly to [1, Lemma 1.5], we obtain an estimation of the auxiliary function in each case:

Lemma 2.4. *For a zero $x = \sigma$ satisfying $|\sigma| > 2L_0$, $\Psi_{(j)}(\mu, x)$ is estimated as $|\Psi_{(j)}(\mu, x)| \leq K_0|x|^{C_0}$ in $U(\sigma) = \{x||x-\sigma| < \rho(\sigma)\}$ with*

$$\rho(\sigma) := \sup\{\rho||\eta(x)| < |\mu|/2 \text{ for } |x-\sigma| < \rho < 1\},$$

$C_0 > 2$ independent of σ, $\eta(x)$ and the parameters of (1), K_0 independent of σ.

2.3. Proof of Theorem. For each solution of (1) satisfying the condition of Case (j), take a zero $\sigma \in S_\phi^{2L_0}$. If σ is a simple zero, put $\eta(x) =: Y(x)$. Then, as the results of the above steps show, we have $|\Psi(\mu, x)| \leq K_0|x|^{C_0}$ and $|Y(x) - \mu| \neq 0$ in $U(\sigma)$. If σ is a double zero, put $\eta(x) =: Y(x)^2$. Local analysis tells us that every zero is eihter simple or double.

Put $Y(x) =: Y_\sigma(\xi)$ with the local parameter $\xi = x - \sigma$ and define $h_\sigma^\pm(\xi)$ by

$$\frac{\mathrm{d}Y_\sigma(\xi)}{\mathrm{d}\xi} = \pm K_{(j)}(1 + h_\sigma^\pm(\xi))$$

with suitably taken branch, where $K_{(j)} = \kappa_{(j)}\sigma^{n_{(j)}}$ is a constant as follows: $\kappa_{(\mathrm{i})} = \kappa_{(\mathrm{ii})} = \kappa_{(\mathrm{v})} = \kappa_{(\mathrm{vi})} = \sqrt{-\delta}$, $\kappa_{(\mathrm{iii})} = \kappa_{(\mathrm{iv})} = \kappa_{(\mathrm{vii})} = \kappa_{(\mathrm{viii})} = \sqrt{-\beta/2}$; $n_{(\mathrm{i})} = n_{(\mathrm{ii})} = 0$, $n_{(\mathrm{iii})} = 1/2$, $n_{(\mathrm{iv})} = 1$, $n_{(\mathrm{v})} = 4$, $n_{(\mathrm{vi})} = 5$, $n_{(\mathrm{vii})} = 5/2$, $n_{(\mathrm{viii})} = 3$. And also define $\rho_0 := \sup\{\rho||Y_\sigma(\xi)| \leq b|x|^{-P}$ is valid for $|\xi| < \rho < 1/3\}$. Note that $0 < \rho_0 \leq 1/3$ because $\eta(\sigma) = 0$ implies $Y_\sigma(0) = 0$. Then, similarly to [1, Lemma 1.7], we can establish the following lemma:

Key lemma. *Suppose $|Y_\sigma(\xi)| \leq b|x|^{-P}$ for $|\xi| < \rho_0$ with b a sufficiently small positive number independent of σ, putting $P := n_{(j)} + C_0/2$. Then $|h_\sigma^\pm(\xi)| < 1/2$ for $|\xi| < \rho_0$.*

Proof of Theorem. For ξ satisfying $|\xi| < \rho_0$, which indicates $|Y_\sigma(\xi)| \leq b|x|^{-P}$, Key lemma gives us

$$\frac{\mathrm{d}Y_\sigma(\xi)}{\mathrm{d}\xi} = \pm K_{(j)}\{1 + h_\sigma^\pm(\xi)\}, \quad |h_\sigma^\pm(\xi)| < \frac{1}{2}.$$

Integrating the above, we have the following estimation:

$$|Y_\sigma(\xi) \mp K_{(j)}\xi| \leq |K_{(j)}| \int_0^\xi |h_\sigma^\pm(\xi)||\mathrm{d}\xi| \leq \frac{1}{2}|K_{(j)}||\xi|,$$

which indicates

$$\frac{1}{4}|K_{(j)}||\xi| \leq |Y_\sigma(\xi)| \leq \frac{7}{4}|K_{(j)}||\xi|.$$

Provided that $|\sigma| > M := 2L_0 + (3b/2|K_{(j)}|)^{1/P}$ with a sufficiently large number L_0, we have $\rho_0 \geq \kappa(\sigma) := b|\sigma|^{-P}/2|K_{(j)}|$, $\kappa(\sigma) < 1/3$. Because, for sufficiently large $|\sigma|$, $|\sigma| > 2L_0 + (3b/2|K_{(j)}|)^{1/P}$ implies $|\sigma| > 2L_0$ and $1/3 > b|\sigma|^{-P}/2|K_{(j)}|$. Suppose $\rho_0 < b|\sigma|^{-P}/2|K_{(j)}|$, then $|Y_\sigma(\xi)| \leq 7|K_{(j)}||\xi|/4 \leq 7|K_{(j)}|\rho_0/4 \leq 7b|\sigma|^{-P}/8$ for $|\xi| < \rho_0 < 1/3$, which contradicts the definition of ρ_0. So, we have $\rho_0 \geq b|\sigma|^{-P}/2|K_{(j)}| = \kappa(\sigma)$.

Hence, for $|\sigma| > M$ and $|\xi| < \kappa(\sigma)$, we have

$$\frac{1}{4}|K_{(j)}||\xi| \leq |Y_\sigma(\xi)| \leq \frac{7}{4}|K_{(j)}||\xi|,$$

which implies $|\eta(\sigma)| > 0$ for $0 < |x - \sigma| < \kappa(\sigma)$.

Therefore, the number of zeros is estimated as follows:

$$\begin{aligned}
&\sharp\{\sigma \in S(\phi, 2L_0, R)|\eta(\sigma) = 0\} \\
&\leq \frac{\mathrm{Area}(S(\phi, 2L_0, R))}{\min_{\sigma \in S(\phi, 2L_0, R)} \pi(\kappa(\sigma)/2)^2} \leq \frac{\mathrm{Area}(S(\phi, 0, R))}{\pi(\kappa(R)/2)^2} \\
&= \frac{\phi R^2}{\frac{\pi b^2}{16|\kappa_{(j)}|^2} R^{2(n_{(j)} - P)}} = O(R^{2+C_0}) \ //\text{qed.}
\end{aligned}$$

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