On WKB analysis of higher order Painlevé equations with a large parameter

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Abstract: We announce a generalization of the reduction theorem for 0-parameter solutions of the traditional (i.e., second order) Painlevé equations with a large parameter to those of some higher order Painlevé equations, i.e., each member of the Painlevé hierarchies (P_J) $(J = \mathrm{I}$, II-1 and II-2) discussed in [KKNT]. Thus the scope of applicability of the reduction theorem ([KT1, KT2]) has been substantially enlarged; only six equations were covered by our previous result, while the result reported here applies to infinitely many equations.

Key words: Painlevé transcendent; Painlevé hierarchy; turning point; Lax pair.

0. Introduction. The purpose of this article is to report that a 0-parameter solution of a higher order Painlevé equation $(P_J)_m$ $(J = \mathrm{I}$, II-1, II-2; $m = 1, 2, \ldots)$ can be formally reduced to a 0-parameter solution of $(P_{\mathrm{I}})_1$, i.e., the traditional Painlevé equation (P_{I}) with a large parameter, near its turning point of the first kind (in the sense of [KKNT]). This is a substantial generalization of our earlier result ([KT2]; its core part was announced in [KT1]), which is concerned with the traditional (i.e., second order) Painlevé equations; thus it covers only six equations (P_J) $(J = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI})$, while the result announced in this article applies to infinitely many equations, i.e., each member of the Painlevé hierarchy $(P_J)_m$ $(J = \mathrm{I}$, II-1, II-2; $m = 1, 2, \ldots)$ with a large parameter η. Here and in what follows we use the same notions and notations as in [KKNT]. In order to give the reader some idea of the "higher order Painlevé equations" discussed here, we recall the definition of $(P_{\mathrm{I}})_m$ together with the underlying Lax pair $(L_{\mathrm{I}})_m$, i.e., a system of linear differential equations whose compatibility condition is described by $(P_{\mathrm{I}})_m$. See [KKNT] for $(P_J)_m$ and $(L_J)_m$ $(J = \mathrm{II}$-1, II-2). See also [S], [GJP] and [GP] for the equations without the large parameter.

Definition 0.1. The m-th member of P_{I}-hierarchy with a large parameter η is the following system of non-linear differential equations:

(0.1)
$$(P_{\mathrm{I}})_m : \begin{cases} \dfrac{du_j}{dt} = 2\eta v_j \qquad (j = 1, \ldots, m) & (0.1.\mathrm{a}) \\ \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j) & (0.1.\mathrm{b}) \\ \qquad\qquad (j = 1, \ldots, m) & \\ u_{m+1} = 0, & \end{cases}$$

where w_j is a polynomial of u_k and v_l $(1 \le k,\ l \le j)$ that is determined by the following recursive relation:

(0.2)
$$w_j = \frac{1}{2}\biggl(\sum_{k=1}^{j} u_k u_{j+1-k}\biggr) + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\biggl(\sum_{k=1}^{j-1} v_k v_{j-k}\biggr) + c_j + \delta_{jm} t \quad (j = 1, \ldots, m).$$

Here c_j is a constant and $\delta_{j,m}$ stands for Kronecker's delta.

Remark 0.1. The system $(P_{\mathrm{I}})_m$ is seen to be equivalent to a single $2m$-th order differential equation. For example, $(P_{\mathrm{I}})_1$ is equivalent to

$$u_1'' = \eta^2(6u_1^2 + 4c_1 + 4t), \tag{0.3}$$

the traditional Painlevé equation (P_{I}), and $(P_{\mathrm{I}})_2$ is equivalent to the following fourth order equation:

$$u_1^{(4)} = \eta^2(20u_1u_1'' + 10(u_1')^2) + \eta^4(-40u_1^3 - 16c_1u_1 + 16c_2 + 16t). \tag{0.4}$$

The underlying Lax pair $(L_{\mathrm{I}})_m$ of $(P_{\mathrm{I}})_m$ is given by the following:

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$$(0.5)\quad (L_{\mathrm{I}})_m : \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\vec{\psi} = 0 & (0.5.\mathrm{a}) \\ \left(\dfrac{\partial}{\partial t} - \eta B\right)\vec{\psi} = 0 & (0.5.\mathrm{b}) \end{cases}$$

where $\vec{\psi} = {}^t(\psi_1, \psi_2)$,

(0.6)

$$A = \begin{pmatrix} V(x)/2 & U(x) \\ (2x^{m+1} - xU(x) + 2W(x))/4 & -V(x) \end{pmatrix},$$

and

(0.7)

$$B = \begin{pmatrix} 0 & 2 \\ u_1 + x/2 & 0 \end{pmatrix},$$

with

$$(0.8)\qquad U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j},$$

$$(0.9)\qquad V(x) = \sum_{j=1}^{m} v_j x^{m-j},$$

and

$$(0.10)\qquad W(x) = \sum_{j=1}^{m} w_j x^{m-j}.$$

See [KKNT, Proposition 1.1.1] for the proof of the fact that $(P_{\mathrm{I}})_m$ is the compatibility condition for $(L_{\mathrm{I}})_m$.

As in the case of the traditional Painlevé equations (cf. [KT2]), we can construct the so-called 0-parameter solution $(\hat{u}_j, \hat{v}_j)$ of $(P_{\mathrm{I}})_m$ of the following form:

$$(0.11)\qquad \hat{u}_j(t,\eta) = \hat{u}_{j,0}(t) + \eta^{-1}\hat{u}_{j,1}(t) + \cdots,$$

$$(0.12)\qquad \hat{v}_j(t,\eta) = \hat{v}_{j,0}(t) + \eta^{-1}\hat{v}_{j,1}(t) + \cdots.$$

In what follows we always substitute the 0-parameter solution into the coefficients of $(L_{\mathrm{I}})_m$. Accordingly the matrices A and B are also expanded in powers of η^{-1}; their top degree parts are respectively denoted by A_0 and B_0.

In studying the structure of 0-parameter solutions, we can readily find the structure of $\hat{v}_j$ from that of $\hat{u}_j$, thanks to (0.1.a). Hence we concentrate our attention to $\hat{u}_j$'s, or rather the solutions

(0.13)

$$b_j(t,\eta) = b_{j,0}(t) + \eta^{-1}b_{j,1}(t) + \cdots \quad (1 \le j \le m)$$

of the equation $U(b_j(t,\eta)) = 0$, that is,

$$(0.14)\qquad b_j(t,\eta)^m - \sum_{j=1}^{m} \hat{u}_j(t,\eta) b_j(t,\eta)^{m-j} = 0.$$

We note that $\{b_j\}_{j=1,\ldots,m}$ appear as a straightforward counterpart of the traditional Painlevé transcendents in the original formulation of Shimomura ([S]) of higher order Painlevé equations from the viewpoint of the Garnier system. The passage from $\{b_j\}$ to their elementary symmetric polynomials $\{u_j\}$ seems to ameliorate the global behavior of functions in question, which is not our immediate concern here (cf. [S]).

Now, our goal (Theorem 3.1 below) is to relate $b_j(t,\eta)$ with a 0-parameter solution of the traditional Painlevé-I equation through a formal transformation. In constructing the required transformation, we first rewrite $(L_J)_m$ ($J = $ I, II-1, II-2) as a pair of a Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$ (Section 1) and then analyze solutions of the Riccati equation associated with $(SL_J)_m$ near $x = b_{j,0}(t)$, the top order part of $b_j(t,\eta)$ (Section 2). Making full use of the results in Section 2, we construct an appropriate semi-global transformation that brings $(SL_J)_m$ to $(SL_{\mathrm{I}})_1$ and the constructed transformation is used to reduce b_j to a 0-parameter solution of $(P_{\mathrm{I}})_1$.

The details of this article shall be published elsewhere.

1. Derivation of a Schrödinger equation $(SL_J)_m$ and its deformation equation $(D_J)_m$. If we let ψ denote

$$(1.1)\qquad \exp\left(-\int^x \frac{U_x}{2U}dx\right)\psi_1 = \frac{1}{\sqrt{U}}\psi_1$$

for the first component ψ_1 of the unknown vector $\vec{\psi}$ of (0.5.a), we find ψ satisfies the following Schrödinger equation $(SL_{\mathrm{I}})_m$:

$$(SL_{\mathrm{I}})_m\qquad \frac{\partial^2\psi}{\partial x^2} = \eta^2 Q_{(\mathrm{I},m)}\psi$$

where

(1.2)

$$Q_{(\mathrm{I},m)} = \frac{1}{4}(2x^{m+1}U - xU^2 + 2UW) + \frac{1}{4}V^2 - \frac{\eta^{-1}VU_x}{2U} + \frac{\eta^{-1}V_x}{2} + \frac{3\eta^{-2}U_x^2}{4U^2} - \frac{\eta^{-2}U_{xx}}{2U}.$$

Making use of (0.5.b), we can find its deformation equation $(D_{\mathrm{I}})_m$, an equation compatible with

$(SL_{\mathrm{I}})_m$:

$(D_{\mathrm{I}})_m$ $$\frac{\partial\psi}{\partial t} = \mathfrak{a}_{(\mathrm{I},m)}\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial\mathfrak{a}_{(\mathrm{I},m)}}{\partial x}\psi,$$

where

$$\mathfrak{a}_{(\mathrm{I},m)} = \frac{2}{U}. \tag{1.3}$$

Now we note that $Q_{(\mathrm{I},m),0}$, the highest degree term in η of $Q_{(\mathrm{I},m)}$, has the form

$$\begin{aligned}\frac{1}{4}&(x+2\hat{u}_{1,0})U_0(x)^2 \\ &= \frac{1}{4}(x+2\hat{u}_{1,0})\left(x^m - \sum_{j=1}^{m}\hat{u}_{j,0}x^{m-j}\right)^2.\end{aligned} \tag{1.4}$$

(See [KKNT, §2.1] for the details.) Hence $x = b_{j,0}$ $(1 \le j \le m)$ is a double turning point of $(SL_{\mathrm{I}})_m$. Similar observations are made also for $(SL_J)_m$ (J = II-1 and II-2). Thus, it is natural to expect that the setting of [KT2] may be also applicable to $(SL_J)_m$ (J = I, II-1, II-2), and this expectation is really validated as is discussed below. For the reference we note that the deformation equation $(D_J)_m$ (J = II-1, II-2) for $\psi = x^{1/2}T_m^{-1/2}\psi_1$ (in the case of $(L_{\text{II-1}})_m$) and $\psi = T_m^{-1/2}\psi_1$ (in the case of $(L_{\text{II-2}})_m$; for the sake of simplicity we assume $c_j = 0$ $(1 \le j \le m-1)$ in (1.3.9) of [KKNT]. To avoid some degeneracy we also assume $c \neq 0$ in (1.2.1) (resp., $\delta \neq 0$ in (1.3.1)) of [KKNT]) is given respectively with

$$\mathfrak{a}_{(\text{II-1},m)} = \frac{2gx}{T_m} \tag{1.5}$$

and

$$\mathfrak{a}_{(\text{II-2},m)} = \frac{g}{2T_m}, \tag{1.6}$$

where g is a non-zero constant and T_m is a polynomial of degree m in x whose coefficients are given in terms of (0-parameter) solutions of $(P_J)_m$.

2. Regularity of S_{odd} near $x = b_{j,0}(t)$. In this section we omit the suffix (J,m) of $Q_{(J,m)}$ and $\mathfrak{a}_{(J,m)}$. Let $S^{\pm}$ respectively denote the solution of the Riccati equation associated with $(SL_J)_m$, i.e.,

$$(S^{\pm})^2 + \frac{\partial S^{\pm}}{\partial x} = \eta^2 Q, \tag{2.1}$$

that begins with $\pm\eta\sqrt{Q}$. Then S_{odd} is, by definition,

$$S_{\mathrm{odd}} = \frac{1}{2}(S^+ - S^-). \tag{2.2}$$

We note that this definition of S_{odd} is different from that used in [KT2]; one important point is that S_{odd} thus defined may contain a term whose degree in η is even. Although we do not discuss the details here, S_{odd} thus defined is free from even degree terms for J = I, just like S_{odd} in [KT2], but not for J = II-1 or II-2. As is shown in [AKT, §2], we can verify

$$\frac{\partial S_{\mathrm{odd}}}{\partial t} = \frac{\partial}{\partial x}(\mathfrak{a}S_{\mathrm{odd}}) \tag{2.3}$$

for S_{odd} thus defined. Using (2.3), we can prove the following

Theorem 2.1. *The series S_{odd} and $\mathfrak{a}S_{\mathrm{odd}}$ are holomorphic on a neighborhood of $x = b_{j,0}(t)$ $(1 \le j \le m)$ in the sense that each of their coefficients as formal power series in η^{-1} is holomorphic on a neighborhood of $x = b_{j,0}(t)$.*

3. Reduction of $b_j(t,\eta)$ $(j = 1,\cdots,m)$ to a 0-parameter solution of $(P_{\mathrm{I}})_1$. Let $t = \tau$ be a turning point of the first kind of $(P_J)_m$ (J =I, II-1, II-2) in the sense of [KKNT]. (We note that every turning point is of the first kind if $m = 1$, i.e., for the traditional Painlevé equations.) Let us further assume that τ is simple in the sense of [AKKT] (with using a local parameter of the Riemann surface $\mathcal{R}$ of the 0-parameter solution as independent variable. Note that, as is explained in [KKNT] and [NT], the Stokes geometry of $(P_J)_m$ lies on $\mathcal{R}$ and that a turning point of the first kind is in general a square-root type branch point of $\mathcal{R}$.) Then there exist a double turning point $b_{j,0}(t)$ and a simple turning point $a(t)$ of $(SL_J)_m$ which merge at τ, and there exists an analytic function $\nu_j(t)$ for which

$$\int_{\tau}^{t}\nu_j(s)ds = 2\int_{a(t)}^{b_{j,0}(t)}\sqrt{Q_{(J,m),0}(x,t)}dx \tag{3.1}$$

holds. (See [KKNT, §2] for the proof.) Note that a Stokes curve of $(P_J)_m$ that emanates from τ is, by definition, given by

$$\mathrm{Im}\int_{\tau}^{t}\nu_j(s)ds = 0. \tag{3.2}$$

It follows from (3.1) that

$$\mathrm{Im}\int_{a(t)}^{b_{j,0}(t)}\sqrt{Q_{(J,m),0}(x,t)}dx = 0 \tag{3.3}$$

holds if t lies in the Stokes curve of $(P_J)_m$. Otherwise stated, if t lies in the Stokes curve of $(P_J)_m$, the double turning point $b_{j,0}(t)$ and a simple turning point $a(t)$ of $(SL_J)_m$ are connected by a Stokes segment γ. Using Theorem 2.1, we can prove the following Proposition 3.1 in this geometrical setting:

Proposition 3.1. *Let τ be a simple turning point of the first kind of $(P_J)_m$ ($J=$ I, II-1, II-2), and let σ ($\neq \tau$) be a point that is sufficiently close to τ and that lies in a Stokes curve of $(P_J)_m$ which emanates from τ. Then there exist a neighborhood Ω of the above mentioned Stokes segment γ, a neighborhood ω of σ and holomorphic functions $\tilde{x}_j(x,t)$ ($j=0,1,2,\cdots$) on $\Omega\times\omega$ and $\tilde{t}_j(t)$ ($j=0,1,2,\cdots$) on ω so that the following relations may hold*:

(i) *The function $\tilde{t}_0(t)$ satisfies*

$$\int_\tau^t \nu_j(s)ds = \int_0^{\tilde{t}} \sqrt{12\lambda_0(\tilde{s})}d\tilde{s}\,\Big|_{\tilde{t}=t_0(t)}, \tag{3.4}$$

where $\lambda_0=\sqrt{-\tilde{s}/6}$, and, in particular, $d\tilde{t}_0/dt\neq 0$ holds on ω, if ω is chosen sufficiently small.

(ii) $\tilde{x}_0(b_{j,0}(t),t)=\lambda_0(\tilde{t}_0(t))$ *and* $\tilde{x}_0(a(t),t)=-2\lambda_0(\tilde{t}_0(t))$.

(iii) $\partial\tilde{x}_0/\partial x\neq 0$ *on* $\Omega\times\omega$.

(iv) *Letting $\tilde{x}(x,t,\eta)$ and $\tilde{t}(t,\eta)$ respectively denote $\sum_{j\geq 0}\tilde{x}_j(x,t)\eta^{-j}$ and $\sum_{j\geq 0}\tilde{t}_j(t)\eta^{-j}$, we find the following relation*:

$$\begin{aligned} Q_{(J,m)}(x,t,\eta) &= \left(\frac{\partial\tilde{x}}{\partial x}\right)^2 \tilde{Q}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta) \\ &\quad -\frac{1}{2}\eta^{-2}\{\tilde{x}(x,t,\eta);x\}, \end{aligned} \tag{3.5}$$

where $\{\tilde{x};x\}$ denotes the Schwarzian derivative and $\tilde{Q}(\tilde{x},\tilde{t})$ is the potential of the Schrödinger equation (SL_{I}) in [KT2], *i.e.*,

$$\begin{aligned} \tilde{Q}(\tilde{x},\tilde{t}) &= 4\tilde{x}^3+2\tilde{t}\tilde{x}+\nu_{\mathrm{I}}^2-4\lambda_{\mathrm{I}}^3-2\tilde{t}\lambda_{\mathrm{I}} \\ &\quad -\eta^{-1}\frac{\nu_{\mathrm{I}}}{\tilde{x}-\lambda_{\mathrm{I}}}+\eta^{-2}\frac{3}{4(\tilde{x}-\lambda_{\mathrm{I}})^2}, \end{aligned} \tag{3.6}$$

with

(3.7)
$\lambda_{\mathrm{I}}(\tilde{t},\eta)$ *being a* 0*-parameter solution of* (P_{I}),
i.e., $\lambda_{\mathrm{I}}''=\eta^2(6\lambda_{\mathrm{I}}^2+\tilde{t})$, *and* ν_{I} *being* $\eta^{-1}d\lambda_{\mathrm{I}}/d\tilde{t}$.

Using the transformations $\tilde{x}(x,t,\eta)$ and $\tilde{t}(t,\eta)$ constructed above, we can show

$$S_{(J,m),\mathrm{odd}}(x,t)=\left(\frac{\partial\tilde{x}}{\partial x}\right)S_{\mathrm{I,odd}}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta). \tag{3.8}$$

This relation and Theorem 2.1 entail the following

Theorem 3.1. *In the situation of Proposition* 3.1, *we have*

$$\tilde{x}(x,t,\eta)\,|_{x=b_j(t,\eta)}=\lambda_{\mathrm{I}}(\tilde{t}(t,\eta),\eta). \tag{3.9}$$

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