# On a unit group generated by special values of Siegel modular functions. II

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**1. Introduction.** In the preceding paper [1], we constructed a group of units with full rank for the ray class field $k_6$ of $\mathbf{Q}(\exp(2\pi i/5))$ modulo 6 using special values of Siegel modular functions and circular units. Our work was based on Shimura's reciprocity law [3] which describes explicitly the Galois action on the special values of theta functions and numerical computation. In this paper, we construct certain units of the ray class field $k_{18}$ of $\mathbf{Q}(\exp(2\pi i/5))$ modulo 18.

**2. Siegel modular functions.** We argue in a situation similar to [1]. So we explain notations briefly. We denote as usual by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ by the ring of rational integers, the field of rational numbers, real numbers and complex numbers, respectively. For a positive integer $n$, let $I_n$ be the unit matrix of dimension $n$ and $\zeta_n = \exp(2\pi i/n)$. Let $\mathfrak{S}_2$ be the set of all complex symmetric matrices of degree 2 with positive definite imaginary parts. For $u \in \mathbf{C}^2$, $z \in \mathfrak{S}_2$ and $r, s \in \mathbf{R}^2$, put as usual

$$\Theta(u, z; r, s) = \sum_{x \in \mathbf{Z}^2} e\left(\frac{1}{2}{}^t(x+r)z(x+r) + {}^t(x+r)(u+s)\right),$$

where $e(\xi) = \exp(2\pi i\xi)$ for $\xi \in \mathbf{C}$. Let $N$ be a positive integer. If we define

$$\Phi(z; r, s; r_1, s_1) = \frac{2\Theta(0, z; r, s)}{\Theta(0, z; r_1, s_1)}$$

for $r, s, r_1, s_1 \in (1/N)\mathbf{Z}^2$, then $\Phi(z; r, s; r_1, s_1)$ is a Siegel modular function of level $2N^2$.

Let $\Gamma_1 = S_p(2, \mathbf{Z}) = \{\alpha \in GL_4(\mathbf{Z}) \mid {}^t\alpha J\alpha = J\}$, where

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

We let every element

$$\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

act on $\mathfrak{S}_2$ by $\alpha(z) = (Az+B)(Cz+D)^{-1}$ for $z \in \mathfrak{S}_2$.

If $\alpha$ is a matrix in $M_4(\mathbf{Z})$ such that ${}^t\alpha J\alpha = vJ$ and $\det(\alpha) = v^2$ with positive integer $v$ prime to $2N^2$, then there exists a matrix $\beta_\alpha$ in $\Gamma_1$ with

$$\alpha \equiv \begin{pmatrix} I_2 & 0 \\ 0 & vI_2 \end{pmatrix} \beta_\alpha \pmod{2N^2}.$$

We let $\alpha$ act on $\Phi(z; r, s; r_1, s_1)$ by $\Phi^\alpha(z; r, s; r_1, s_1) = \Phi(\beta_\alpha(z); r, vs; r_1, vs_1)$. Then $\Phi^\alpha$ is also a Siegel modular function of level $2N^2$.

In what follows, we fix $\zeta = \zeta_5$ and $k = \mathbf{Q}(\zeta)$. Let $\sigma$ be the element of the Galois group $G(k/\mathbf{Q})$ such that $\zeta^\sigma = \zeta^2$ and define the endomorphism $\varphi$ of $k^\times$ by $\varphi(a) = a^{1+\sigma^3}$ for $a \in k^\times$. Furthermore put

$$z_0 = \begin{pmatrix} \zeta^2+\zeta^4 & \zeta^3 \\ \zeta^4+\zeta^3 & \zeta \end{pmatrix}^{-1} \begin{pmatrix} -\zeta & \zeta^4 \\ -\zeta^2 & \zeta^3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2+\zeta-\zeta^3-2\zeta^4 & 2-\zeta+\zeta^2-2\zeta^3 \\ 2-\zeta+\zeta^2-2\zeta^3 & \zeta+2\zeta^2-2\zeta^3-\zeta^4 \end{pmatrix}.$$

We note that $z_0$ is a CM-point associated to a Fermat curve $y^2 = 1 - x^5$. For an element $\omega$ in the integer ring $\mathfrak{O}_k$ of $k$, let $R(\omega) \in M_4(\mathbf{Z})$ be the regular representation of $\omega$ with respect to the basis $\{-\zeta, \zeta^4, \zeta^2+\zeta^4, \zeta^3\}$. Then, $R(\varphi(\omega))z_0 = z_0$, ${}^tR(\varphi(\omega))JR(\varphi(\omega)) = vJ$ and $\det R(\varphi(\omega)) = v^2$, where $v = N_{k/\mathbf{Q}}(\omega)$.

**3. Structure of the Galois group.** For a positive integer $N$, we denote by $k_N$ the ray class field of $k$ modulo $N$. We explain the structure of the Galois group $G(k_{18}/k)$ which is needed for our argument. For a positive integer $m$, we put $S_m = \{a \in k^\times | a \equiv 1 \pmod{m}\}$ and $\tilde{S}_m = \{(a) | a \in S_m\}$, where $(a)$ is the principal ideal of $k$ generated by $a$.

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Let $U$ be the unit group of $k$. Then we have

$$G(k_{18}/k_6) \cong \tilde{S}_6/\tilde{S}_{18} \cong S_6U/S_{18}U \\ \cong S_6/S_{18}(S_6 \cap U).$$

We put $\omega_1 = 1+6 = 1+6(-\zeta-\zeta^2-\zeta^3-\zeta^4)$, $\omega_2 = 1+6(\zeta-\zeta^4)$, $\omega_3 = 1+6(\zeta^2-\zeta^3)$, $\omega_4 = 1+6(\zeta-\zeta^2-\zeta^3+\zeta^4) = 1+2\sqrt{5}$ and $H = S_{18}(S_6 \cap U)$. Then it is easy to show that $S_6/H = \langle \omega_1 H, \omega_2 H, \omega_3 H, \omega_4 H \rangle \cong (\mathbf{Z}/3\mathbf{Z})^4$. Futhermore, if we define an endomorphism $\tilde{\varphi}$ of $S_6/H$ by $\tilde{\varphi}(aH) = \varphi(a)H$, then $\operatorname{Ker}\varphi = \omega_4 H$. Let $K$ be the intermediate field of $k_{18}/k$ corresponding to $\operatorname{Ker}\varphi$. Then $K$ is imaginary Galois extension of $\mathbf{Q}$ with $[K : \mathbf{Q}] = 1080$ and $G(K/k) \cong (\mathbf{Z}/10\mathbf{Z}) \times (Z/3\mathbf{Z})^3$. Let

$$\tau = \left(\frac{k_6/k}{(\zeta+2)}\right) \quad \text{and} \quad \eta_i = \left(\frac{k_{18}/k}{(\omega_i)}\right) \quad (i = 1, 2, 3)$$

be Artin symbols. We extend $\tau$ to $k_{18}$ and keep the notation. Then $G(K/k) = \langle \tau, \eta_1, \eta_2, \eta_3 \rangle$. We also extend $\sigma$ to $k_{18}$ and keep the notation. Then the action of $\sigma$ to $G(K/k)$ is given by

$$\sigma^{-1}\eta_1\sigma = \eta_1,\ \sigma^{-1}\eta_2\sigma = \eta_3^2,\ \sigma^{-1}\eta_3\sigma = \eta_2 \\ \text{and}\quad \sigma^{-1}\tau\sigma = \tau^7.$$

Now, let $r, s, r_1, s_1 \in (1/18)\mathbf{Z}^2$. Then it is known that $\Phi(z_0; r, s; r_1, s_1) \in k_{2\cdot 18^2}$ and

$$\Phi(z_0; r, s; r_1, s_1)^{\left(\frac{k_{2\cdot 18^2}/k}{(\omega)}\right)} = \Phi^{R(\varphi(\omega))}(z_0; r, s; r_1, s_1)$$

by Shimura's reciprocity law for $\omega \in \mathfrak{O}_k$ which is prime to 18 (cf. [3]). Moreover we know that $\Phi(z_0; r, s; 0, 0)$ is an algebraic integer and $\Phi(z_0; r, s; r_1, s_1)^{36}$ is contained in $k_{18}$ by [2]. The actions of $\tau$ and $\eta_i$ for $\Phi(z_0; r, s; r_1, s_1)$ are given by

$$R(\varphi(\zeta+2)) = \begin{pmatrix} 3 & 0 & -1 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & -2 & 2 & -1 \\ -2 & 1 & 1 & 4 \end{pmatrix}$$

$$\equiv \begin{pmatrix} I_2 & 0 \\ 0 & 11I_2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 & 1 \\ 2 & 2 & 0 & -1 \\ 15 & 44 & -44 & -59 \\ 44 & 59 & 59 & 74 \end{pmatrix} \pmod{2 \cdot 9^2},$$

$$R(\varphi(\omega_2)) = \begin{pmatrix} -29 & 72 & -12 & 72 \\ 84 & 31 & -72 & -12 \\ 24 & -12 & -41 & 60 \\ -12 & 12 & 72 & 43 \end{pmatrix},$$

$$R(\varphi(\omega_3)) = \begin{pmatrix} 43 & -60 & -12 & -84 \\ -72 & -29 & 60 & -12 \\ 0 & 12 & 31 & -72 \\ 12 & 12 & -84 & -41 \end{pmatrix},$$

$$R(\varphi(\omega_1)) = 7^2 I_4 \quad \text{and} \quad R(\varphi(\omega_4)) = -179 I_4,$$

which implies that $\Phi(z_0; r, s; 0, 0)^{36}$ is an algebraic integer of $K$.

**4. Norm computation.** We explain how to compute $N_{K/\mathbf{Q}}\Phi(z_0; r, s; 0, 0)^{36}$ for $r, s \in (1/18)\mathbf{Z}^2$. Let $\omega_0 = \zeta + 2$ and $\Omega = \{\omega_0^{e_0}\omega_1^{e_1}\omega_2^{e_2}\omega_3^{e_3} \mid 0 \le e_0 \le 9, 0 \le e_1, e_2, e_3 \le 2\}$. We first note that

$$N_{K/k}\Phi(z_0; r, s; 0, 0)^{36} = \prod_{\rho \in G(K/k)} \Phi(z_0; r, s; 0, 0)^{36\rho} \\ = \prod_{\omega \in \Omega} \Phi^{R(\varphi(\omega))}(z_0; r, s; 0, 0)^{36}$$

and hence

$$N_{K/\mathbf{Q}(\sqrt{5})}\Phi(z_0; r, s; 0, 0)^{36} \\ = \prod_{\omega \in \Omega} \left| \Phi^{R(\varphi(\omega))}(z_0; r, s; 0, 0)^{36} \right|^2.$$

Now, we can write

$$\left| \Phi^{R(\varphi(\omega))}(z_0; r, s; 0, 0) \right| = \left| \Phi(z_0; r', s'; r_1', s_1') \right|$$

explicitly with $r', s' \in (1/18)\mathbf{Z}^2$ and $r_1', s_1' \in (1/2)\mathbf{Z}^2$ by transformation formula for theta series. Since $\Phi(z_0; r, s; 0, 0)^{36}$ is an algebraic integer of $k_{18}$ and since the absolute value of a conjugate of $\Phi(z_0; r, s; 0, 0)^{36}$ over $\mathbf{Q}$ is a form of $|\Phi(z_0; r', s'; r'', s'')^{36}|$ for some $r', s' \in (1/18)\mathbf{Z}^2$ and $r'', s'' \in (1/2)\mathbf{Z}^2$, we can determine $N_{K/\mathbf{Q}}\Phi(z_0; r, s; 0, 0)^{36}$ with some luck.

**Example 4.1.** Let

$$r = \begin{pmatrix} 4/18 \\ 5/18 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 3/18 \\ 1/18 \end{pmatrix}.$$

We note that

$$\prod_{\omega \in \Omega} \Phi^{R(\varphi(\omega))}(z_0; r, s; 0, 0)^2$$

is contained in $k$. So we computed the approximate values of

$$\text{(1)} \qquad \prod_{\omega \in \Omega} \left| \Phi^{R(\varphi(\omega))}(z_0; r, s; 0, 0) \right|^2 \\ \times \prod_{\omega \in \Omega} \left| \Phi^{R(\varphi(\omega))}(z_0; r', s'; r'', s'') \right|^2$$

for all $r', s' \in (1/18)\mathbf{Z}^2$ and $r'', s'' \in (1/2)\mathbf{Z}^2$ and found that $(2^{432}3^{14})^2$ is the only possible integral value for (1). Furthermore, there are 270 pairs of

$(r', s', r'', s'')$ such that (1) is close to $(2^{432}3^{14})^2$ and all the pairs are derived from one by the action of $G(K/k)$. Hence we can conclude that

$$N_{K/\mathbf{Q}}\Phi(z_0; r, s; 0, 0)^{36} = (2^{432}3^{14})^{36}$$

and

$$(2)\quad \left|\Phi(z_0; r, s; 0, 0)^{36\sigma}\right| = \left|\Phi\left(z_0; \begin{pmatrix}16/18\\1/18\end{pmatrix}, \begin{pmatrix}1/18\\1/18\end{pmatrix}; \begin{pmatrix}1/2\\1/2\end{pmatrix}, \begin{pmatrix}1/2\\1/2\end{pmatrix}\right)^{36}\right|$$

for some extension of $\sigma$ to $K$. Similarly we have

$$N_{K/\mathbf{Q}}\Phi\left(z_0; \begin{pmatrix}8/18\\1/18\end{pmatrix}, \begin{pmatrix}1/18\\1/18\end{pmatrix}; \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)^{36}$$
$$= N_{K/\mathbf{Q}(\sqrt{5})}\Phi\left(z_0; \begin{pmatrix}8/18\\1/18\end{pmatrix}, \begin{pmatrix}1/18\\1/18\end{pmatrix}; \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)^{36}$$
$$N_{K/\mathbf{Q}(\sqrt{5})}\Phi\left(z_0; \begin{pmatrix}0/18\\3/18\end{pmatrix}, \begin{pmatrix}1/18\\1/18\end{pmatrix}; \begin{pmatrix}1/2\\1/2\end{pmatrix}, \begin{pmatrix}1/2\\1/2\end{pmatrix}\right)^{36}$$
$$= (2^{432}3^{14})^{36}.$$

**5. Construction of units.** We denote by $E_K$ the unit group of $K$. It is easy to show that 2 and 3 inert in $k/\mathbf{Q}$, 3 is totally ramified in $K/k$ and the decomposition group of 2 for $K/k$ is the cyclic group generated by $\tau^2$. Hence we see that

$$\Phi\left(z_0; \begin{pmatrix}4/18\\5/18\end{pmatrix}, \begin{pmatrix}3/18\\1/18\end{pmatrix}; \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)^{36(1-\tau^2)}$$

is contained in $E_K$ and hence

$$\varepsilon_1 = \Phi\left(z_0; \begin{pmatrix}4/18\\5/18\end{pmatrix}, \begin{pmatrix}3/18\\1/18\end{pmatrix}; \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)^{1-\tau^2}$$

is also contained in $E_K$ because $\zeta_{36}^{\tau^2} = \zeta_{36}$. Similarly we have one more unit

$$\varepsilon_2 = \Phi\left(z_0; \begin{pmatrix}8/18\\1/18\end{pmatrix}, \begin{pmatrix}1/18\\1/18\end{pmatrix}; \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right)^{1-\tau^2}$$

of $K$. We are interested in the subgroups of $E_K$ generated by $\varepsilon_i$ with Galois actions. Let

$$H = \left\{ \tau^{e_0}\eta_1^{e_1}\eta_2^{e_2}\eta_3^{e_3} \;\middle|\; \begin{array}{l} 0 \le e_0 \le 7, \\ 0 \le e_1, e_2, e_3 \le 2, \\ e_1 + e_2 \le 3 \end{array} \right\}$$

be the subset of $G(K/k)$ and put $E_i = \langle \varepsilon_i^\rho \mid \rho \in H\rangle$ $(i = 1, 2)$. The cardinality of $H$ is 192. We determine the rank of $E_i$.

**Theorem 5.1.** We have $\mathrm{rank}_{\mathbf{Z}}\, E_1 = \mathrm{rank}_{\mathbf{Z}}\, E_2 = 192$.

*Proof*. The actions of $\tau$ and $\eta_i$ for $\varepsilon_1$ are determined explicitly by Shimura's reciprocity law and that of $\sigma$ is given by (2) for suitable extension of $\sigma$. So we can construct the $192 \times 540$ matrix

$$(3)\qquad (\log |u^g|),$$

where $u$ runs over $\varepsilon_1^\rho$ $(\rho \in H)$ and $g$ is of the form $\sigma^i\rho$ $(0 \le i \le 1, \rho \in G(K/k))$. We verified numerically that the rank of (3) is 192. So we have $\mathrm{rank}_{\mathbf{Z}}\, E_1 = 192$. The same argument is applicable to $E_2$. □

Next we construct a subgroup of $E_K$ of larger rank by composing $E_1$ and $E_2$. Let

$$H' = \left\{ \tau^{e_0}\eta_1^{e_1}\eta_2^{e_2}\eta_3^{e_3} \;\middle|\; \begin{array}{l} 0 \le e_0 \le 7, \\ 0 \le e_1 \le 1, \\ 0 \le e_2, e_3 \le 2 \end{array} \right\}$$

be the subset of $H$ and put $E_{12} = \langle \varepsilon_1^{\rho_1}, \varepsilon_2^{\rho_2} \mid \rho_1, \rho_2 \in H'\rangle$. We note that the cardinality of $H'$ is 144. Furthermore we define a group of cyclotomic units $E_3 = \langle 1 - \zeta_{45}^i \mid i = 1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19\rangle$. The following is the main result in this paper.

**Theorem 5.2.** We have $\mathrm{rank}_{\mathbf{Z}}\, E_{12}E_3 = 299$.

*Proof*. The determination of $\mathrm{rank}_{\mathbf{Z}}\, E_{12}E_3$ is slightly difficult. We can consider

$$\left|\varepsilon_1^\sigma\right| = \left|\Phi\left(z_0; \begin{pmatrix}16/18\\1/18\end{pmatrix}, \begin{pmatrix}1/18\\1/18\end{pmatrix}; \begin{pmatrix}1/2\\1/2\end{pmatrix}, \begin{pmatrix}1/2\\1/2\end{pmatrix}\right)^{1-\tau^2}\right|$$

for suitable extension of $\sigma$. But we can only assert that

$$\left|\varepsilon_2^\sigma\right| = \left|\Phi\left(z_0; \begin{pmatrix}0/18\\3/18\end{pmatrix}, \begin{pmatrix}1/18\\1/18\end{pmatrix}; \begin{pmatrix}1/2\\1/2\end{pmatrix}, \begin{pmatrix}1/2\\1/2\end{pmatrix}\right)^{(1-\tau^2)\rho}\right|$$

for some $\rho \in G(K/k)$ and $\zeta_{45}^\sigma = \zeta_{45}^2, \zeta_{45}^7, \zeta_{45}^{17}, \zeta_{45}^{22}, \zeta_{45}^{32}$ or $\zeta_{45}^{37}$. We need to calculate the rank of $299 \times 540$ matrix similar to (3). Let

$$H'' = \left\{ \begin{array}{l} \tau^8, \sigma\tau^8, \tau^9, \sigma\tau^9, \eta_1^2, \sigma\eta_1^2, \tau^8\eta_1^2, \\ \sigma\tau^8\eta_1^2, \tau^9\eta_1^2, \sigma\tau^9\eta_1^2, \sigma\tau^7\eta_1^2\eta_2^2\eta_3^2 \end{array} \right\}.$$

We calculated numerically the determinants of minor matrices of dimension 299 consisting of columns associated to $H' \cup \sigma H' \cup H''$ for all possible values of $\varepsilon_2^\sigma$ and $\zeta_{45}^\sigma$ and verified that the determinants are non-zero for all cases. Hence we fortunately conclude that $\mathrm{rank}_{\mathbf{Z}}\, E_{12}E_3 = 299$. □

The computations were executed on a 64-bit work station DEC Alpha 500/333. A custom program by C and assembly language was written for calculating approximate values of theta functions with high precision. Specifying independent units and computing ranks of matrices were handled by

TC which is an interpreter of multi-precision C language developed by one of the authors.

## References


[ 1 ] T. Fukuda and K. Komatsu: On a unit group generated by special values of Siegel modular functions (to appear in Math. Comp.).

[ 2 ] K. Komatsu: Construction of normal basis by special values of Siegel modular functions (to appear in Proc. Amer. Math. Soc.).

[ 3 ] G. Shimura: Theta functions with complex multiplication. Duke Math. J., **43**, 673–696 (1976).