7. A Note on Class Number One Problem for Real Quadratic Fields

By Hideo YOKOI

Graduate School of Human Informatics, Nagoya University

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In our previous paper[2], for the fundamental unit ε_p of the real quadratic field $\boldsymbol{Q}(\sqrt{p})$ of prime discriminant, we showed that there exist exactly 30 real quadratic fields $\boldsymbol{Q}(\sqrt{p})$ of class number one satisfying $\varepsilon_p < 2p$ with one more possible exception of prime discriminant p.

On the other hand, in the paper [3], for a positive square-free integer D we defined new D-invariants m_D, n_D, and using them we provided some estimate formulas of the class number and the fundamental unit of the real qnadratic field $\boldsymbol{Q}(\sqrt{D})$.

In this paper, using one of those estimate formulas of the class number we shall provide a kind of improvement of Theorem 2 in [2], which relates to class number one problem for real quadratic fields.[1)]

For any positive square-free integer D, we denote by h_D and by

$$\varepsilon_D = (t_D + u_D\sqrt{D})/2 \ (>1)$$

the class number and the fundamental unit of the real quadratic field $\boldsymbol{Q}(\sqrt{D})$ respectively, and put

$$\boldsymbol{D}_- = \{D\text{: positive square-free integer with } N\varepsilon_D = -1\}.$$

Our main purpose of this paper is to prove the following theorem:

Theorem. For arbitrarily chosen and fixed natural number h_0 and real number c greater than 2, there exists only a finite number of real quadratic fields $\boldsymbol{Q}(\sqrt{D})$ $(D \in \boldsymbol{D}_-)$ such that

(1) $$\varepsilon_D < D(e^{D^{\frac{1}{c}}} - 1) \quad \text{or} \quad t_D < D(e^{D^{\frac{1}{c}}} - 1),$$

and

(2) $$h_D \leqq h_0.$$

To prove this theorem, we need several lemmas.

Lemma 1. For any $D > 5$ in $\boldsymbol{D}_-$,

$$[t_D/D] = [\varepsilon_D/D] = [u_D^2/t_D]$$

holds, where $[x]$ means the greatest integer less than or equal to x.

For the proof, see Theorems 2.1, 2.3 and their proofs in [3].

Here, if we put

$$m_D = [t_D/D] \ (= [\varepsilon_D/D])$$

the same as in [3], then we have easily the following lemma:

Lemma 2. If $s \geqq 11.2$ and $D \geqq e^s$ for D in $\boldsymbol{D}_-$, then

$$h_D > 0.3275 \cdot s^{-1} \cdot D^{(s-2)/(2s)} / \{\log D(m_D + 1)\}$$

holds with one possible exception of D.

For the proof, see Theorem 2.3 in [3].

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1) Cf. H. Yokoi [1].

Proof of Theorem. If we assume $\varepsilon_D < D(e^{D^{\frac{1}{c}}} - 1)$ for any $c > 2$, then we get from Lemma 1

$$m_D = [\varepsilon_D / D] \leqq \varepsilon_D / D < e^{D^{\frac{1}{c}}} - 1,$$

and hence

$$m_D + 1 < e^{D^{\frac{1}{c}}}.$$

Similarly, $t_D < D(e^{D^{\frac{1}{c}}} - 1)$ implies

$$m_D = [t_D / D] \leqq t_D / D < \mathrm{e}^{D^{\frac{1}{c}}} - 1,$$

and hence also

$$m_D + 1 < e^{D^{\frac{1}{c}}}.$$

From these facts it follows that

$$\log\{D(m_D + 1)\} < D^{1/c} + \log D.$$

Here, if we put

$$\alpha = (s-2)/(2s),\ \beta = 1/c,$$

then

$$\alpha > \beta \quad \text{if and only if} \quad s > 2c/(c-2).$$

On the other hand, if we put moreover

$$f_s(D) = D^{\alpha} / (D^{\beta} + \log D),$$

then from Lemma 2 we get

$$h_D > (0.3275/s) \cdot f_s(D)$$

for any $s > 2c/(c-2)$, and under the assumption $\alpha > \beta > 0$ $f_s(D)$ tends to infinity as D tends to infinity.

Therefore, if we choose any s satisfying

$$s > \mathrm{Max}\{11.2,\ 2c/(c-2)\},$$

then there exist a positive number D_0 such that

$$h_D > h_0 \text{ holds for any } D \text{ in } \boldsymbol{D}_- \text{ with } D > D_0,$$

namely

$$h_D \leqq h_0 \quad \text{implies} \quad D \leqq D_0.$$

Hence, there exists only a finite number of $Q(\sqrt{D})$ $(D \in \boldsymbol{D}_-)$ satisfying $h_D \leqq h_0$.

Thus, our theorem was completely proved.

From this theorem we can obtain immediately the folowing two corollaries:

Corollary 1. For any fixed number c greater than 2, there exists only a finite number of real quadratic fields $Q(\sqrt{D})$ $(D \in \boldsymbol{D}_-)$ of class number one satisfying

$$\varepsilon_D < D(e^{D^{\frac{1}{c}}} - 1) \quad (\text{or } t_D < D(e^{D^{\frac{1}{c}}} - 1)).$$

Corollary 2. There exist infinitely many real quadratic fields of class number one with prime discriminant if and only if there exist infinitely many real quadratic fields of class number one with prime discriminant p satisfying

$$\varepsilon_p > p(e^{p^{\frac{1}{c}}} - 1) \quad (\text{or } t_p > p(e^{p^{\frac{1}{c}}} - 1))$$

for any fixed number c greater than 2.

Finally, we provide two tables of primes less than 10^4. One of them (Table I) is the table of primes p congruent to one mod 4 satisfying $h_p = 1$ and $(e^{\sqrt{p}} - 1) < \varepsilon_p / p$. Another one (Table II) is the table of primes p

Table I. The table of primes $p \equiv 1 \pmod 4$ satisfying $h_p = 1$ and $(e^{\sqrt{p}} - 1) < \varepsilon_p / p$.
$(10^{n-1} < e^{\sqrt{p}} < 10^n,\quad 10^{m-1} < \varepsilon_p / p < 10^m)$

p	n	m
601	11	12
769	13	14
1,033	14	15
1,201	16	21
1,249	16	20
1,321	16	20
1,609	18	24
1,801	19	27
1,873	19	24
2,017	20	24
2,137	21	25
2,161	21	29
2,281	21	29
2,377	22	26
2,473	22	30
2,521	22	36
2,689	23	36
3,001	24	34
3,049	24	38
3,169	25	36
3,217	25	32
3,313	25	30
3,361	26	40
3,433	26	29
3,457	26	27
3,529	26	43
3,697	27	35
3,769	27	38
3,793	27	30
4,057	28	37
4,129	28	42
4,153	28	33
4,201	29	46
4,273	29	32
4,513	30	30
4,561	30	47
4,657	30	42
4,801	31	47
4,969	31	44
4,993	31	39
5,113	32	36
5,209	32	53
5,233	32	36
5,449	33	47
5,569	33	57
5,641	33	52
5,689	33	44
5,737	33	43
5,857	34	37
5,881	34	55
6,073	34	46
6,121	34	52
6,217	35	42
6,337	35	47
6,361	35	55
6,449	35	36
6,529	36	52
6,553	36	48
6,577	36	46
6,673	36	49
6,689	36	36
6,841	36	63
6,961	37	57
7,121	37	39
7,177	37	52
7,297	38	46
7,321	38	51
7,369	38	53
7,393	38	54
7,417	38	50
7,489	38	61
7,529	38	40
7,561	38	69
7,681	39	69
8,009	39	42
8,089	40	74
8,161	40	58
8,209	40	62
8,233	40	57
8,329	40	63
8,353	40	54
8,369	40	42
8,521	41	70
8,641	41	59
8,737	41	58
8,849	41	45
8,929	42	70
9,001	42	64
9,241	42	78
9,337	42	49
9,433	43	54
9,601	43	60
9,649	43	65
9,689	43	48
9,697	43	56
9,769	43	78
9,817	44	45

h_p : the class number of $Q(\sqrt{p})$
ε_p : the fundamental unit of $Q(\sqrt{p})$

Table II. The table of primes $p \equiv 1 \pmod 4$ satisfying $h_p = 1$ and $e^{\sqrt[3]{p}} < \varepsilon_p/p < e^{\sqrt{p}}$

97	2,113	4,049	6,301	8,269
193	2,129	4,073	6,329	8,273
241	2,221	4,177	6,353	8,293
281	2,269	4,217	6,373	8,297
313	2,273	4,241	6,421	8,317
337	2,293	4,261	6,469	8,377
409	2,297	4,289	6,473	8,389
433	2,341	4,297	6,521	8,429
449	2,389	4,337	6,569	8,461
457	2,393	4,349	6,581	8,513
521	2,417	4,457	6,661	8,537
617	2,437	4,481	6,709	8,609
641	2,441	4,549	6,733	8,629
673	2,593	4,621	6,737	8,677
709	2,609	4,673	6,761	8,681
809	2,617	4,721	6,781	8,689
829	2,633	4,729	6,793	8,713
857	2,657	4,789	6,829	8,753
881	2,729	4,793	6,833	8,821
929	2,749	4,813	6,857	8,893
937	2,753	4,817	6,869	8,941
953	2,797	4,861	6,977	8,961
977	2,801	4,909	7,001	9,013
1,021	2,833	4,937	7,069	9,041
1,049	2,857	4,957	7,109	9,109
1,069	2,897	5,009	7,193	9,137
1,097	2,909	5,021	7,237	9,157
1,153	2,953	5,077	7,309	9,161
1,213	2,969	5,101	7,333	9,209
1,217	3,041	5,153	7,433	9,221
1,289	3,061	5,197	7,457	9,257
1,361	3,089	5,281	7,477	9,277
1,381	3,109	5,393	7,537	9,349
1,409	3,209	5,413	7,541	9,377
1,433	3,257	5,437	7,549	9,397
1,453	3,301	5,441	7,577	9,421
1,481	3,329	5,581	7,589	9,461
1,549	3,389	5,653	7,621	9,473
1,553	3,449	5,657	7,649	9,497
1,621	3,469	5,701	7,669	9,521
1,657	3,541	5,749	7,741	9,613
1,669	3,593	5,801	7,753	9,629
1,697	3,613	5,849	7,789	9,661
1,721	3,617	5,861	7,793	9,721
1,741	3,637	5,869	7,829	9,733
1,753	3,673	5,897	7,841	9,781
1,777	3,709	5,953	7,901	9,857
1,789	3,733	6,029	7,933	9,901
1,861	3,761	6,037	7,937	9,929
1,889	3,833	6,089	7,993	9,941
1,913	3,881	6,101	8,017	9,949
1,933	3,889	6,229	8,053	
1,993	3,929	6,257	8,081	
2,053	4,021	6,269	8,221	

congruent to one mod 4 satisfying $h_p = 1$ and $e^{\sqrt[3]{p}} < \varepsilon_p / p < e^{\sqrt{p}}$. From these tables we may find the fact that there are only 97 primes in the Table I while there exist 521 primes less than 10^4 congruent to one mod 4 with $h_p = 1$. On the other hand, there are 267 primes in the Table II.

References

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