33. *Construction of Rank Two Reflexive Sheaves with Similar Properties to the Horrocks-Mumford Bundle*

By Nobuo SASAKURA,*) Yoichi ENTA,*) and Masataka KAGESAWA**)

(Communicated by Shokichi IYANAGA M. J. A., May 12, 1993)

0. In a paper of Horrocks-Mumford [2] a rank two vector bundle $\mathfrak{F}$ on P_4 with 15000 symmetries was constructed. It is to be noticed that $4 = 5 - 1$ and 5 is a prime $\equiv 1 \pmod 4$ so that -1 is a quadratic residue modulo 5. Let p be any prime $\equiv 1 \pmod 4$, so that $\left(\frac{-1}{p}\right) = 1$. We shall construct, in this note, a rank two reflexive sheaf $\mathfrak{E}_p$ on $\boldsymbol{P}_{p-1}(\boldsymbol{C})$ with similar properties to $\mathfrak{F}$. We shall consider certain graphs which have elements of $\boldsymbol{F}_p$ as vertices using quadratic residue, by means of which we shall define $\mathfrak{E}_p$. At the same time we define certain $K3$ surfaces over $\boldsymbol{F}_p$. If $p = 5$, then $\mathfrak{E}_5 \simeq \mathfrak{F}$. If $p > 5$ then $\mathfrak{E}_p > 5$ has a singularity at codimension four. It turns out that the fourth Chern class of $\mathfrak{E}_p$ is expressed in terms of rational points on these surfaces, cf. Theorem 4.2. The sheaf $\mathfrak{E}_p$ admits a group of symetries of order $p^{(p+3)/2}(p-1)$. (When $p = 5$, it has symmetries of order $p^{(p+3)/2}(p^2-1)$. But, when $p > 5$, we do not know if the group of symmetries is bigger than the above one.) Details of this note will appear elsewhere. We thank to T. Terasoma for valuable discussions on arithmetic of algebraic varieties.

1. Quadratic residue graph. First, using the quadratic residue, we define a graph for $\boldsymbol{F}_p$: The set of the vertices is $\boldsymbol{F}_p$, and a subset $\{i, j\} \subset \boldsymbol{F}_p$, $i \neq j$, is an edge if and only if $i - j = k^2$ with an element $k \in \boldsymbol{F}_p^*$.

Since $p \equiv 1 \pmod 4$, the graph is not endowed with an order. For an element $i \in \boldsymbol{F}_p$, define subsets $S_i^{\pm}$ of $\boldsymbol{F}_p$ as follows:

$$S_i^+ = \{j \in \boldsymbol{F}_p - \{i\} \mid \{i, j\} \text{ is an edge}\}$$
$$S_i^- = \{j \in \boldsymbol{F}_p - \{i\} \mid \{i, j\} \text{ is not an edge}\}.$$

More generally, take a subset $I = \{i_1, \dots, i_d\}$, $1 \leq d \leq p$, of $\boldsymbol{F}_p$. Form a symbol $\mu = (\mu_1, \dots, \mu_d)$, where μ_k denotes $+$ or $-$ $(1 \leq k \leq d)$. We set $S_I^{\mu} = \cap_{k=1}^{d} S_{i_k}^{\mu_k}$. If each $\mu_k = +$ (or $-$), then we write S_I^{μ} as S_I^+ (or S_I^-). For two disjoint subsets K, L, of $\boldsymbol{F}_p$, we write S_{KL}^{+-} for $S_K^+ \cap S_L^-$.

The set S_I^{μ} is parameterized by rational points on an algebraic curve defined over $\boldsymbol{F}_p$. For example the correspondence $s \mapsto (s^2+1)^2/4s^2$ gives a surjective unramified map of degree four from $\{s \in \boldsymbol{F}_p \mid s \neq 0,\ s^4 \neq 1\}$ to S_{01}^{+-}. For an element $i \in \boldsymbol{F}_p - \{0,1\}$, set

$$E_{01i}^{+++} = \{(s, t) \in \boldsymbol{F}_p \times \boldsymbol{F}_p \mid t^2 = (s^2+1)^2 - 4is^2\}$$
$$E'^{+++}_{01i} = E_{01i}^{+++} - \{(s, t) \in E_{01i}^{+++} \mid s \cdot t = 0 \text{ or } s^4 = 1\}.$$

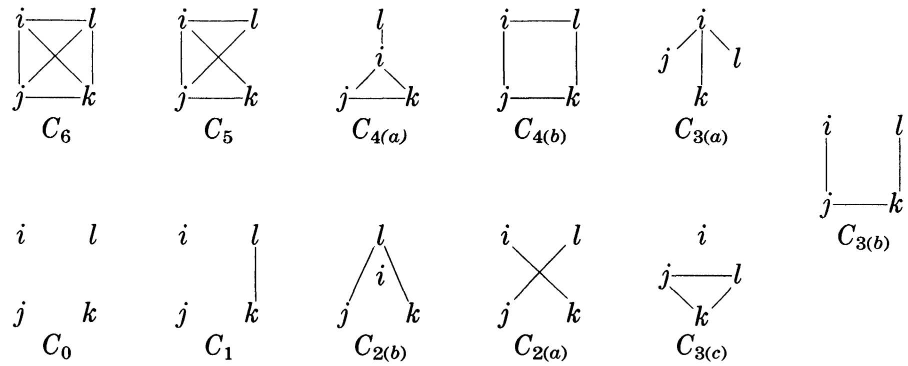
*) Department of Mathematics, Tokyo Metropolitan University.
**) Institute of Industrial Sciences, University of Tokyo.

The correspondence $(s, t) \mapsto (s^2+1)^2/4s^2$ gives a surjective unramified map of degree eight from E'^{+++}_{01i} to S^{+++}_{01i}.

Next, we say that two subsets I and J of F_p are (graph) equivalent, if there is a bijection θ from I to J such that

$(*)$ $\qquad\{\theta(i), \theta(j)\}$ is an edge $\Leftrightarrow$ $\{i, j\}$ is an edge

for each pair $\{i, j\} \subset I$. Set $\mathcal{P}_d(F_p) = \{I \subset F_p \mid \# I = d\}$, where $\# I$ is the cardinal number of I and $d \geq 2$. The (graph) equivalence divides $\mathcal{P}_d(F_p)$ into disjoint classes. We see that $\mathcal{P}_2(F_p)$ and $\mathcal{P}_3(F_p)$ are divided into two and four (graph) equivalence classes respectively. Also $\mathcal{P}_4(F_p)$ is divided into at most eleven equivalence classes. We indicate below the table of eleven possible graphs.



(Here — indicates the edge. The suffix $s = 6, \ldots, 0$ indicates the number of edges. The symbols (a), (b), (c) are used when s does not determine an equivalence class uniquely. If $p = 5$, then only $C_{3(b)}$ appears. If $p \geq 29$, then all of the graphs appear.)

Now we attach a $K3$ surface to each equivalence class $\mathcal{M}$ of $\mathcal{P}_4(F_p)$: Consider the five dimensional projective space P_5 over F_p with homogeneous coordinates $z = (z_{\alpha\beta})_{1 \leq \alpha < \beta \leq 4}$. Fix a primitive root ρ of F_p^*. Let $\mathfrak{K}_{\mathcal{M}}$ denote the subvariety of P_5 defined by the following four quadratic relations.

$$\varepsilon_{\alpha\beta} z_{\alpha\beta}^2 + \varepsilon_{\beta\gamma} z_{\beta\gamma}^2 = \varepsilon_{\alpha\gamma} z_{\alpha\gamma}^2 \qquad (1 \leq \alpha < \beta < \gamma \leq 4)$$

where $\varepsilon_{\alpha\beta}$, $\varepsilon_{\alpha\gamma}$ and $\varepsilon_{\beta\gamma}$ denote 1 or ρ and are determined by $\mathcal{M}$. This is a $K3$ surface and appears from the following consideration: Take a representative $I = \{z_1, z_2, z_3, z_4\}$ of $\mathcal{M}$. Then, according as $\{z_\alpha, z_\beta\}$ is an edge or not $z_\alpha - z_\beta = z_{\alpha\beta}^2$ or $= \rho z_{\alpha\beta}^2$ with an element $z_{\alpha\beta} \in F_p^*$. Set $\mathfrak{K}'_{\mathcal{M}} = \mathfrak{K}_{\mathcal{M}} - \{z \in \mathfrak{K}_{\mathcal{M}} \mid \Pi_{1 \leq \alpha < \beta \leq 4} z_{\alpha\beta} = 0\}$. Denote by $\widetilde{\mathfrak{K}'_{\mathcal{M}}}$ the affine cone of $\mathfrak{K}'_{\mathcal{M}}$. Setting $d_{\mathcal{M}} = \# \{\theta \mid \theta$ is a bijection: $I \to I$ satisfying$(*)\}$, the above argument leads to

Lemma 1.1. *There is a surjective unramified map from* $F_p \times \widetilde{\mathfrak{K}'_{\mathcal{M}}}$ *to* $\mathcal{M}$ *of degree* $(2^6 \cdot d_{\mathcal{M}})$.

Convention. If $\mathcal{M}$ has a representative, which is isomorphic to the one figured as C_6, $C_{4(b)}, \ldots$, we write $\mathcal{M}$ as $\mathcal{M}_6$, $\mathcal{M}_{4(b)}, \ldots$.

Lemma 1.1 was shown by Hirane (cf. [1]) when $\mathcal{M} = \mathcal{M}_6$.

Some monomials. Let $x = (x_i)_{i \in \boldsymbol{F}_p}$ be homogeneous coordinates of $P = P_{p-1}(\boldsymbol{C})$. For a subset S_I^{μ} of $\boldsymbol{F}_p$, let f_I^{μ} denote the monomial $\Pi_{\alpha \in S_I^{\mu}} x_\alpha$. (If $S_I^{\mu} = \phi$, then $f_I^{\mu} = 1$.) Take subsets $J \subset I \subset \boldsymbol{F}_p$ such that $\# J = \# I - 1$. We assume that $\# J = 1$ or that each distinct pair $\{i, j\} \subset I$ forms an edge. Define a monomial $\varphi_{J,I}$ as follows. (In the below $\{\alpha\} = I - J$.)

$$\varphi_{J,I} = \begin{cases} f_J^+ / (f_I^+ x_\alpha) & \text{if } \alpha \in S_J^+ & \cdots \text{(i)} \\ f_\alpha^+ / f_I^+ & \text{if } \alpha \in S_J^- & \cdots \text{(ii)} \\ & \text{if } \alpha \in S_{KL}^{+-} & \\ (f_L^+ / f_{L\alpha}^{++})^2 / (f_{K\alpha}^{++} / f_I^+) & \text{where } J = K \amalg L & \cdots \text{(iii)} \\ & \text{with } K \neq \phi,\ L \neq \phi. & \end{cases}$$

If J has no edge, then define a monomial $\varphi_{J,I}$ in the similar manner to the above; by changing $+$ and $-$ in (i)-(iii). (Thus, for example, $\varphi_{J,I} = f_I^- / (f_I^- x_\alpha)$ if $\alpha \in S_J^-$.) Take a subset I of $\boldsymbol{F}_p$. If $\# I = 2$ or 3, then each subset J of I with $\# J = \# I - 1$ satisfies the above assumption. Assume that $\# I = 4$. If I belongs to $\mathcal{M}_6$ or to $\mathcal{M}_0$, then define $\varphi_{J,I}$ in the above manner. If $I \notin \mathcal{M}_{4(b)}$, $\mathcal{M}_{2(a)}$, or $\mathcal{M}_{3(b)}$, then there is a subset K of I, $\# K = 3$, such that $\varphi_{K,I}$ is defined. For another subset J of I with $\# J = 3$, define $\varphi_{J,I}$ by the equation

$$\varphi_{L,J}\, \varphi_{J,I} = \varphi_{L,K}\, \varphi_{K,I}$$

with $L = J \cap K$. (Note that $\# L = 2$). If $I \in \mathcal{M}_{4(b)}$, we define $\varphi_{J,I} = f_{\beta\,\delta}^{-+} f_{\alpha\,\beta\,\gamma\,\delta}^{---+}$. Here $J = \{\alpha, \beta, \gamma\}$ and $I - J = \{\delta\}$ and $\{\beta, \delta\}$ is not an edge. If $I \in \mathcal{M}_{2(a)}$, then we define $\varphi_{J,I}$ by changing $+$ and $-$ in the definition. If $I \in \mathcal{M}_{3(b)}$, then define two systems of monomials:

$$\varphi_{ilj,I} = (f_{l\,k}^{+-} / f_{i\,l\,j\,k}^{++--}) f_{i\,l\,j\,k}^{+++-}, \quad \varphi_{ilk,I} = (f_{i\,j}^{+-} / f_{i\,l\,j\,k}^{++--}) f_{i\,l\,j\,k}^{++-+},$$
$$\varphi'_{ilj,I} = (f_{k\,i}^{-+} / f_{i\,k\,l\,i}^{--++}) f_{j\,k\,l\,i}^{---+}, \quad \varphi'_{jki,I} = (f_{j\,l}^{-+} / f_{j\,k\,l\,i}^{--++}) f_{j\,k\,l\,i}^{--+-}.$$

We set $\varphi_{ijk,I} = \varphi_{ljk,I} = 0$ and $\varphi'_{ilj,I} = \varphi'_{ilk,I} = 0$. (Here the graph of $I = \{i, j, k, l\}$ is normalized as in the table of graphs, and $ijk, \ldots,$ is an abbreviation of $\{i, j, k, \ldots\}$.)

We give a main property of the monomials $\varphi_{J,I}$ defined above. Take a subset I of $\boldsymbol{F}_p$ with $\# I = 3$ or 4. According as $\# I = 3$ or 4, we have the following cocycle relations: If $\# I = 3$, then for any $\alpha \in I$,

(1) $$\varphi_{\alpha,J}\, \varphi_{J,I} = \varphi_{\alpha,K}\, \varphi_{K,I}$$

where $\{\alpha\} \subsetneq J \subsetneq I$, $\{\alpha\} \subsetneq K \subsetneq I$. If $\# I = 4$, then for any $L \subset I$ with $\# L = 2$,

(2) $$\varphi_{L,J}\, \varphi_{J,I} = \varphi_{L,K}\, \varphi_{K,I}$$

where $L \subsetneq J \subsetneq I$, $L \subsetneq K \subsetneq I$. According as $\# I = 3$ or 4, define an integer n_I as follows.

$$n_I = \begin{cases} \dfrac{p+3}{4} - d(\varphi_{J,I}) & \text{if } \# I = 3;\ \# J = 2 \text{ and } J \subset I \\ \dfrac{p+3}{4} - \{d(\varphi_{K,J}) + d(\varphi_{J,I})\} & \text{if } \# I = 4;\ \# J = 3,\ \# K = 2 \\ & \quad \text{and } K \subset J \subset I, \end{cases}$$

where $d(\varphi_{J,I})$, denotes the degree of $\varphi_{J,I}$. By (1) and (2), n_I is well defined. The integers n_I are expressed in terms of the number of rational points of algebraic curves.

2. Construction. Our sheaf $\mathfrak{E}_p$ is contructed as follows (cf. [4]).

Notation. For a subset I of $\boldsymbol{F}_p$, let X_I^d, $d = \# I$, denote the linear subspace of P defined by $x_i = 0$, $i \in I$. We set $X^d = \cup_{\# I = d} X_I^d$. The structure sheaves of $P = P_{p-1}$ and X_I^d are denoted by $\mathfrak{D}$ by $\mathfrak{D}_I$; $\mathfrak{D}(m)$ and $\mathfrak{D}_I(m)$, $m \in \boldsymbol{Z}$, denote the m times twist of $\mathfrak{D}$ and $\mathfrak{D}_I$ by the hyperplane bundle. We simply denote $X_{\{i\}}^1$ and $f_{\{i\}}^{\pm}$ as X_i^1 and $f_i^{\pm}$, $i \in \boldsymbol{F}_p$.

Now, for each $i \in \boldsymbol{F}_p$, form a vector $\mathfrak{f}_i = {}^t[f_i^-, f_i^+]$. To X_i^1 we attach a (2×2)-matrix $\begin{bmatrix} 1 & f_i^-/f_i^+ \\ 0 & x_i/x_{i+1} \end{bmatrix}$. Then we have canonically a locally free sheaf $\dot{\mathfrak{E}}_p$ of rank two on $P - X^2$. The sheaf $\mathfrak{E}_p$ is the direct image $\iota_* \dot{\mathfrak{E}}_p$ of $\dot{\mathfrak{E}}_p$ with the injection $\iota : P - X^2 \subset P$. We see that $\mathfrak{E}_p$ is an $\mathfrak{D}$-submodule of $\mathfrak{D}(p)^{\oplus 2}$ and, as the submodule, $\mathfrak{E}_p$ is chracterized as follows: An element $\zeta \in \mathfrak{D}(p)_x^{\oplus 2}$, $x \in P$, is in $\mathfrak{E}_{p,x}$ if and only if there is an element $\sigma_i \in \mathfrak{D}_i((p+1)/2)$ such that for each $i \in \boldsymbol{F}_p$

$$\zeta \,|\, _{X_i^1} = \sigma_i \, \mathfrak{f}_i \,|\, _{X_i^1}. \tag{3}$$

Take an element $(a, b) \in \boldsymbol{F}_p \times \boldsymbol{F}_p^*$. Also take an element $c = (c_s)_{0 \le s \le (p-1)/2} \in \boldsymbol{F}_p^{\oplus (p+1)/2}$. Define a matrix $g_{a,b,c} \in GL(p, \boldsymbol{C})$ in a similar manner to p. 66 in [2]; the (i, j)-component of $g_{a,b,c}$ is given by $\varepsilon_{c,i}\delta_{i,bj+a}$, where $\varepsilon_{c,i} = \varepsilon^{t_{c,i}}$ with $\varepsilon = \exp(2\pi\sqrt{-1}/p)$ and $t_{c,i} = \sum_{s=0}^{(p-1)/2} c_s i^s$. The matrices $g_{a,b,c}$ form a subgroup of $GL(p, \boldsymbol{C})$ with order $p^{(p+3)/2}(p-1)$. When $p = 5$, it coincides with the subgroup of the Horrocks-Mumford group whose elements make the divisor X^1 invariant.

Theorem 2.1. The above group acts on $\mathfrak{E}_p$.

This is proved by checking the action of an element $g_{a,b,c}$ on the transition matrix introduced above.

3. Filtration. The direct image sheaf $\mathfrak{E}_p$ is studied inductively on X^1, $X^2, \ldots$.

Lemma 3.1. *There is a filtration* $\{\mathfrak{E}^i\}_{0 \le i \le 4}$ *of* $\mathfrak{D}$*-submodules of* $\mathfrak{E}_p$: $\mathfrak{E}^0 \subset \mathfrak{E}^1 \subset \mathfrak{E}^2 \subset \mathfrak{E}^3 \subset \mathfrak{E}^4 \subset \mathfrak{E}_p$, *such that* $\mathfrak{E}^i = \mathfrak{E}_p$ *on* $P - X^{i+1}$, $0 \le i \le 4$. *Moreover* $\mathfrak{E}^0 \simeq \mathfrak{D}^{\oplus 2}$, $\mathfrak{E}^1/\mathfrak{B}^0 \simeq \oplus_{i \in \boldsymbol{F}_p} \mathfrak{D}_i((3-p)/2)$, $\mathfrak{E}^2/\mathfrak{E}^1 \simeq \bigoplus_{ij \in \mathcal{P}_2(\boldsymbol{F}_p)} \mathfrak{D}_{ij}((11-3p)/4)$ *and*

$$\mathfrak{E}^3/\mathfrak{E}^2 \simeq \bigoplus_{J \in \mathcal{P}_3(\boldsymbol{F}_p)} \mathfrak{D}_J(3 - p + n_J)$$

$$\mathfrak{E}^4/\mathfrak{E}^3 \simeq \bigoplus_{\substack{I \in \mathcal{P}_4(\boldsymbol{F}_p) \\ I \notin \mathcal{M}_3(b)}} \mathfrak{D}_I(4 - p + n_I) \oplus \bigoplus_{\substack{L \in \mathcal{P}_4(\boldsymbol{F}_p) \\ L \in \mathcal{M}_3(b)}} \{\mathfrak{D}_L(4 - p + n_L) \oplus \mathfrak{D}_L(4 - p + n_L')\}.$$

Here the integers n_J, n_I *are as in the end of* §1.

The proof is given by applying general cohomological arguments in [3] and [5] to the present situation, where every necessary datum is written explicitly. The key point is that the integers n_J and n_I are determined by the arithmetic of the algebraic curves, cf. §§1 and 2.

4. Main results. First we have

Theorem 4.1. *The* $\mathfrak{D}$*-module* $\mathfrak{E}_p$ *is locally free over* $P - X^4$*. Take a subset* I *of* $\boldsymbol{F}_p$ *with* $\# I = 4$*. If* $I \notin \mathcal{M}_{4(b)}$ *or* $\mathcal{M}_{2(a)}$*, then* $\mathfrak{E}_p$ *is locally free at each point* $x \in X_I^4 - X^5$*. If* $I \in \mathcal{M}_{4(b)}$ *or* $\mathcal{M}_{2(a)}$*, then* $\mathfrak{E}_p$ *is not locally free at each* $x \in X_I^4$*.*

Thus $\mathfrak{E}_p$ is *of low rank* in the sense of [4]. If $p = 5$, then every $I \in \mathcal{P}_4(\boldsymbol{F}_p)$ belongs to $\mathcal{M}_{3(b)}$. If $p > 5$, however, then there is a subset $I \in \mathcal{P}_4(\boldsymbol{F}_p)$ belonging to $\mathcal{M}_{4(b)}$.

Theorem 4.2. *The Chern classes* c_i *of* $\mathfrak{E}_p$, $1 \leq i \leq 4$, *are as follows*: $c_1 = p$, $c_2 = \binom{p}{2}$, $c_3 = 0$, *and*

$$c_4 = -\binom{p}{2}(3k^2 - 8k + 2) - 6 \sum_{J \in \mathcal{P}_3(\boldsymbol{F}_p)} n_J - 6 \# \mathcal{M}_{3(b)}$$

where $k = (p-1)/4$.

This follows from Lemma 3.1 by using standard formula for Chern classes. Using graph theoretical arguments, we have

$$c_4 = -40 \# \mathcal{M}_{4(b)} = -40p(p-1) \# \mathrm{rat}(\mathfrak{K}'_{\mathcal{M}_{4(b)}}) / (2^6 \cdot 8)$$

where rat $(\mathfrak{K}'_{\mathcal{M}_{4(b)}})$ denotes the set of $\boldsymbol{F}_p$-rational points of the affine $K3$ surface $\mathfrak{K}'_{\mathcal{M}_{4(b)}}$, cf. §1.

References

[1] T. Hirane: Quadratic residue and cocycle $K3$ surface. Master thesis, Tokyo Metropolitan Univ. (1993)(in Japanese).

[2] G. Horrocks and D. Mumford: A rank 2 vector bundle on P_4 with 15000 symmetries. Topology, **12**, 63–81 (1973).

[3] N. Sasakura: Configuration of divisors and reflexive sheaves. Report note, R. I. M. S. Kyoto Univ., **634**, 407–513 (1987).

[4] ——: Configuration of divisors and reflecxive sheaves. Proc. Japan Acad., **65A**, 27–30 (1989).

[5] N. Sasakura, Y. Enta and M. Kagesawa: Rank two reflexive sheaves which are constructed from the prime field $\boldsymbol{F}_p$. Report note, R.I.M.S. Kyoto Univ., **807**, 163–188 (1992).