13. *Contributions to Uniformly Distributed Functions. I*

*Discrepancy of Fractal Sets**),**)*

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1. Introduction. Let (M, d) denote a compact, arcwise connected metric space and μ a positive, regular, normalized Borel measure on M. A continuous function x: $[0, \infty) \to M$ is called *μ-uniformly distributed* (for short: μ-u.d.) if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x(t))dt = \int_M f(x)d\mu(x) \tag{1.1}$$

holds for all continuous, real-valued functions f on M. In the case of the K-dimensional torus $M=(R/Z)^K$ (with Lebesgue measure μ) E. Hlawka [7] introduced a quantitative measure for u.d., the so called *discrepancy*

$$D_T(x) := \sup_J \left| \frac{1}{T}\int_0^T \chi_J(x(t))dt - \mu(J) \right|, \tag{1.2}$$

where the supremum is extended over all intervals J parallel to the coordinate axes and χ_J denotes the characteristic function of J. It is an easy observation that $x(t)$ is u.d. if and only if $D_T(x)$ tends to 0 for $T \to \infty$. For a detailed survey on the theory of uniform distribution we refer to the monographs [10] and [8]. In [4] we have generalized the concept of discrepancy to compact metric spaces and obtained some lower bounds. Instead of the family of intervals (parallel to the coordinate axes) more general classes of subsets were considered: so-called *discrepancy systems*.

In the context of the present paper a discrepancy system $\mathcal{D}$ is a family of measurable subsets $E \subseteq M$ satisfying the following condition: *For every open ball $B(x, r)$ with center $x \in M$ and radius $r>0$ there exist a set $E \in \mathcal{D}$ and a ball $B(x, R)$ such that*

$$B(x, r) \subseteq E \subseteq B(x, R) \quad \text{and} \quad R/r \leq \beta \tag{1.3}$$

with an absolute constant β.

Furthermore we assume the following additional property for the measure μ: *There exists a constant $K>1$ such that*

$$\mu(B(x, r)) \leq \alpha r^K \tag{1.4}$$

for every open ball $B(x, r)$, α denoting an absolute constant.

Then for every continuous function x: $[0, \infty) \to M$, the discrepancy (with respect to $\mathcal{D}$) can be defined by

$$D_T(\mathcal{D}, x) := \sup_{E\in\mathcal{D}} \left| \frac{1}{T}\int_0^1 \chi_E(x(t))dt - \mu(E) \right|. \tag{1.5}$$

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In [4] curves $x(t)$ with arclength $s(T)$ (tending to ∞ for $T\to\infty$) were considered and the following lower bound was obtained

(1.6) $$D_T(\mathcal{D}, x)\geq C_K(\mathcal{D})(1/s(T))^{K/(K-1)} \qquad (\text{for } T\geq T_0).$$

For special spaces M (torus and sphere) and special discrepancy systems a detailed investigation of such estimates is given in [2]. The aim of the present note is to establish lower bounds for the discrepancy without assuming existence of the arclength.

Thus we obtain lower bounds for the discrepancy of fractal sets. For this purpose we assume in the following a Lipschitz-condition of the type

(1.7) $$d(x(t+h), x(t))\leq R_T|h|^{\gamma} \qquad (0\leq t\leq T)$$

with a constant $R_T\geq 1$ and an exponent $\gamma>0$. It should be mentioned here that there is a strong connection of such Lipschitz conditions and the Hausdorff-dimension of the graph $x(t)$ $(0\leq t\leq T)$; (cf. [6]).

Theorem 1. *Let $\mathcal{D}$ be a discrepancy system on the compact arcwise connected metric space (M, d) and μ a positive normalized Borel measure with property* (1.4). *Assume further that $x\colon [0,\infty)\to M$ is a continuous function satisfying a Lipschitz condition of type* (1.7) *with $\gamma>1/K$. Then there exists a constant $C_K(\mathcal{D}, \mu)$ such that*

(1.8) $$D_T(\mathcal{D}, x)\geq C_K(\mathcal{D}, \mu)(1/R_T^{1/\gamma}T)^{K/(K-1/\gamma)} \qquad (T\geq T_0).$$

The estimate (1.8) is—in some sense—a generalization of (1.6) to fractal curves. Such curves can be constructed by different methods. For instance, in the case of the two-dimensional torus we can take $x(t)=t, at+y(t) \bmod 1$, where a is an irrational and $y(t)$ a periodic fractal function satisfying a Lipschitz condition as above for $\gamma>1/2$.

In the special case of the torus (1.6) is due to Taschner [11]. For special manifold and special discrepancy systems the above estimate can be improved by Beck's Fourier-transform method (cf. [1], [4]). Let $\mathcal{D}_Q$ ($\mathcal{D}_B$, respectively) the discrepancy system of all cubes in arbitrary position (balls, respectively) on the K-dimensional torus $(K\geq 2)$ with Lebesgue measure and let D_C be the discrepancy system of all spherical caps on the K-dimensional sphere $(K\geq 2)$ with spherical surface measure. Generalizing [4] the following improvement of (1.8) can be established:

Theorem 2. *In the three cases $\mathcal{D}=\mathcal{D}_Q$, $\mathcal{D}=\mathcal{D}_B$ and $\mathcal{D}=\mathcal{D}_C$ we have for any curve satisfying the assumptions of Theorem* 1:

(1.9) $$D_T(\mathcal{D}, x)\geq C_K(\mathcal{D}, \mu)(1/R_T^{1/\gamma}T)^{K/2(K-1/\gamma)} \qquad (T\geq T_0).$$

A detailed proof is given in section 2. Further results on the discrepancy of fractals will be established in the subsequent paper [5]. In the final section 3 the estimates (1.8) and (1.9) are extended to functions $x(t_1, \cdots, t_n)$ in n variables. A first result of this kind is due to Taschner. Finally a continuous version of a recent result of E. Hlawka [9] concerning the so-called *Lipschitz discrepancy* is established.

2. Proofs of Theorems 1, 2.

Proof of Theorem 1. Let $\Gamma=\Gamma_T$ denote the graph of $x(t)$, $0\leq t\leq T$. Furthermore we set for $\delta>0$ $L(\delta, \Gamma) := card\,\{n : \exists$ open sets U_i $(i=1, \cdots, n)$

with $d(U_i)\leq\delta$ and $\bigcup_{i=1}^n U_i\supseteq\Gamma\}$, i.e. $L(\delta,\Gamma)$ (Lebesgue number) is the minimal number of open subsets U_i of diameter $d(U_i)\leq\delta$ such that $U_1,\cdots,U_n$ is a covering of the (compact) set Γ. For (small) $\delta>0$ we define $h=(\delta/R_T)^{1/\gamma}$ and $t_i, =ih$, $i=0,\cdots,[T/h]$. Because of the Lipschitz condition (1.8) the balls $B_i=B(x(t_i),\delta)$ form a covering of the graph Γ_T. Then we have the upper bound

$$L(\delta,\Gamma_T)\leq R_T^{1/\gamma}T/\delta^{1/\gamma}. \tag{2.1}$$

Furthermore there exists a ball B_i and an open interval $E=(a,b)\subseteq(0,T)$ such that

$$x(E)\subseteq B_i \quad\text{and}\quad b-a\geq T/L(\delta,\Gamma_T), \tag{2.2}$$

and by (2.1)

$$b-a\geq\delta^{1/\gamma}/R_T^{1/\gamma}. \tag{2.3}$$

Thus we obtain

$$\begin{aligned}\left|\frac{1}{T}\int_0^T \chi_E(x(t))\,dt-\mu(E)\right| &\geq\frac{1}{T}\int_0^T \chi_{B_i}(x(t))dt-\alpha\delta^K\\ &\geq((b-a)/T)-\alpha\delta^K\leq C_K(\mathcal{D},\mu)(1/R_T^{1/\gamma}T)^{K/(K-1/\gamma)}.\end{aligned}$$

Remark 1. The essential tool in the proof is the upper bound (2.1) for the Lebesgue number. Obviously

$$\delta^s L(\gamma,\Gamma)\geq\mathcal{H}^s(\Gamma),$$

where $\mathcal{H}^s(\Gamma)$ denotes the s-dimensional Hausdorff measure (cf. [6]). If the $L(\delta,\Gamma)$ could be estimated also from above in terms of the Hausdorff measure then a lower bound for the discrepancy in terms of the Hausdorff-measure could be established.

Proof of Theorem 2. For simplicity we only consider the discrepancy system $\mathcal{D}_Q$ of all cubes (in arbitrary position) contained in the K-dimensional unit interval $[0,1]^K$. We will use J. Beck's Fourier transform method and the notation of [2]. We set

$$\hat{f}(y)=\frac{1}{(2\pi)^{K/2}}\int_{R^K} e^{-i\langle x,y\rangle}f(x)dx \tag{2.4}$$

for the Fourier transform of a function $f\in L^1(R^K)$, where $\langle x,y\rangle$ denotes the usual inner product in R^K. Let $x\colon[0,T]\to R^K/Z^K$ denote a continuous function satisfying a Lipschitz condition (1.7) and define for any measurable subset $A\subseteq R^K$

$$z(A)=\frac{1}{T}\int_0^T \chi_{A/Z^K}(x(t))dt. \tag{2.5}$$

For a rotation τ we use the notation $\chi_{r,\tau}$ for the characteristic function of the rotated cube $\tau[-r,r]^K(r\leq\frac{1}{2}K^{-1/2})$ and set

$$\mu_M(A)=\mu(A\cap[-M,M]^K) \quad\text{and}\quad z_M(A)=z(A\cap[-M,M]^K). \tag{2.6}$$

Introducing the notations

$$F_{r,\tau}=\chi_r,\ \tau*(dz_M-d\mu_M) \qquad \text{(convolution)} \tag{2.7}$$

and

$$\Phi(q)=\frac{1}{q}\int_q^{2q}\int_T\int_{R^K}|F_{r,\tau}(x)|^2dx\,d\tau\,dr \tag{2.8}$$

(T group of rotations, $d\tau$ normalized Haar measure on T), we derive by Parseval-Plancherel identity and convolution theorem

$$\Phi(q)=\int_{R^K}\varphi_q(y)|\psi(y)|^2dy, \tag{2.9}$$

where

$$\varphi_q(y)=\frac{1}{\phi}\int_q^{2q}\int_T|\hat{\chi}_{r,\tau}(y)|^2d\tau dr \quad \text{and} \quad \psi(y)=dz_M\widehat{-}d\mu_M. \tag{2.10}$$

From [1], p. 134 we immediately obtain for $0<q<p$ the lower bound

$$\varphi_{p(y)}/\varphi_{q(y)}\gg(p/q)^{K-1} \tag{2.11}$$

uniformly in all $t\in R^K$.

From (2.1) we know that $L(1/N,\Gamma_T)\le R_T^{1/r}TN^{1/r}$ balls of radius $1/N$ are sufficient to cover the graph Γ_T. Hence there also exists a covering of Γ_T consisting of $\le R_T^{1/r}TN^{1/r}$ cubes of size $2/N$. Now subdivide $[0,1]^K$ into $(N/2)^K$ cubes (take N even)

$$Q(m)=\prod_{i=1}^K\left[\frac{2m_i}{N},\frac{2(m_i+1)}{N}\right] \quad m=(m_1,\cdots,m_K),\quad 0\le m_i<\frac{N}{2}. \tag{2.12}$$

Since a cube of size $2/N$ can meet at most 2^K cubes $Q(m)$, the above covering can meet at most $2^KR_T^{1/r}TN^{1/r}$ cubes $Q(m)$. Now choose N at the minimal even number satisfying $2^KR_T^{1/r}TN^{1/r}<N^K/2$. Thus there exists a cube $Q(m)$ such that $Q(m)\cap\Gamma_T$ is void. Now consider the subcube

$$C(m)=\prod_{i=1}^K\left[\left(m_i+\frac{1}{4}\right)\frac{2}{N},\left(m_i+\frac{3}{4}\right)\frac{2}{N}\right]$$

of such cube $Q(m)$, $r<(2NK^{1/2})^{-1}$ and $x\in C(m)$. Then

$$F_{r,\tau}(x)=-(2r)^K \quad \text{and} \quad \Phi(q)\gg M^Kq^{2K} \tag{2.13}$$

for $q<(4NK^{1/2})^{-1}$. Since $N\gg\ll(R_T^{1/r}T)^{1/(K-1/r)}$ we have shown the following lower bound

$$\Phi\left(\frac{a_k}{(R_T^{1/r}T)^{1/(K-1/r)}}\right)\gg(R_T^{1/r}T)^{-2K/(K-1/r)}M^K, \tag{2.14}$$

where a_K is a constant depending only on the dimension K. Setting

$$q=\frac{a_k}{(R_T^{1/r}T)^{1/(K-1/r)}} \quad \text{and} \quad p=(5K^{1/2})^{-1} \tag{2.15}$$

we obtain by (2.10) and (2.11)

$$\Phi(p)=\int_{R^K}\varphi_p(y)|\psi(y)|^2dy\gg(R_T^{1/r}T)^{(K-1)/(K-1/r)}\Phi(q)\gg(R_T^{1/r}T)^{-(K+1)/(K-1/r)}M^K. \tag{2.16}$$

Since $F_{r,\tau}(x)=0$ for $x\notin[-M-1,M+1]^K$ we get from (2.16)

$$\frac{1}{p}\int_p^{2p}\int_T\int_{[-M+1,M-1]^K}|F_{r,\tau}(x)|^2dxd\tau dr\gg(R_T^{1/r}T)^{-(K+1)/(K-1/r)}M^K. \tag{2.17}$$

Thus there exists an $r\in[p,2p]$, $a\tau\in T$ and an $x\in[-M+1,M-1]^K$ with

$$|F_{r,\tau}(x)|=\left|\frac{1}{T}\int_0^T\chi_{\tau[-r,r]^K+x/ZK}(x(t))dt-\mu(\tau[-r,r]^K+x)\right|\gg(R_T^{1/r}T)^{-(K+1)/2(K-1/r)},$$

and the proof of Theorem 2 (in the case of the discrepancy system $\mathscr{D}_Q$) is complete. The other cases can be settled in a similar way (cf. [2]).

3. Generalizations. We use the notations $\boldsymbol{t}=(t_1,\cdots,t_n)$, $\boldsymbol{T}=(T_1,\cdots,T_n)$, $\boldsymbol{h}=(h_1,\cdots,h_n)$ and $\|\boldsymbol{h}\|=\max|h_j|$. For a given discrepancy system $\mathcal{D}$ and a normalized Borel measure μ on M the discrepancy of a continuous function $\boldsymbol{x}\colon [0,\infty)^n\to M$ is defined by

$$(3.1)\qquad D_{\boldsymbol{T}}(\mathcal{D},\boldsymbol{x}):=\sup_{E\in\mathcal{D}}\left|\frac{1}{T_1\cdots T_n}\int_0^1\cdots\int_0^1\chi^E(\boldsymbol{x}(\boldsymbol{t}))dt_1\cdots dt_n-\mu(E)\right|.$$

Theorem 3. *Let $\mathcal{D}$ be a discrepancy system on the compact arcwise connected metric space (M,d) and μ a positive normalized Borel measure with property* (1.4). *Assume further that $x\colon [0,\infty)^n\to M$ is a continuous function satisfying the Lipschitz condition*

$$(3.2)\qquad d(x(t+\boldsymbol{h}),x(t))\le R_T\|\boldsymbol{h}\|^{\gamma}\qquad(0\le t_i\le T_i,\ i=1,\cdots,n),$$

with exponent $\gamma>n/K$ and $R_T\ge 1$. Then there exists a constant $C_K(\mathcal{D},\mu)$ such that

$$(3.3)\qquad D_T(\mathcal{D},x)\ge C_K(\mathcal{D},\mu)(1/R_TT_1\cdots T_n)^{K/(K-n/\gamma)}\qquad(T_i\ge T_{i0},\ i=1,\cdots,n).$$

Proof. The proof runs along the same lines as the proof of Theorem 1. Instead of (2.1) we get upper bound

$$(3.4)\qquad L(\delta,\Gamma_T)\le R_T^{n/\gamma}T_1\cdots T_n/\delta^{1/\gamma}$$

for the Lebesgue number of the set $\{x(t),\ 0\le t_i\le T_i,\ i=1,\cdots,n\}$. There exists a ball B_i of the δ-cover of Γ_T and an open set $E\subseteq\prod_{i=1}^n[0,T_i]$ such that

$$(3.5)\qquad x(E)\subseteq B_i\quad\text{and}\quad\lambda(E)\ge\delta^{n/\gamma}/R_T^{n/\gamma}$$

where λ denotes the n-dimensional Lebesgue measure. The same arguments as in the proof of Theorem 1 yield the desired result.

Theorem 4. *In the three cases $\mathcal{D}=\mathcal{D}_Q$, $\mathcal{D}=D_B$ and $D=\mathcal{D}_C$ we have for any function $x(t)$ satisfying the assumption of Theorem* 3:

$$(3.6)\qquad D_T(\mathcal{D},x)\ge C_K(\mathcal{D},\mu)(1/R_T^{n/\gamma}T_1\cdots T_n)^{K/2(K-n/\gamma)}.$$

As in section 2 we only consider the discrepancy system $\mathcal{D}_Q$.

Proof. The proof runs along the same lines as the proof of Theorem 2. We use the upper bound (3.4)

$$L(1/N,\Gamma_T)\le R_T^{n/\gamma}T_1\cdots T_nN^{n/\gamma}$$

for the minimal number of balls of radius N^{-1} which is necessary to cover the graph Γ_T. Thus we obtain

$$(3.7)\qquad \Phi\left(\frac{a_K}{(R_T^{n/\gamma}T_1\cdots T_n)^{1/(K-n/\gamma)}}\right)\gg(R_T^{1/\gamma}T_1\cdots T_n)^{-2K/(K-n/\gamma)}M^K$$

instead of (2.14) in the proof of Theorem 2. From this onwards the desired result follows by the same arguments as in section 2.

In the following we consider the so-called Lipschitz discrepancy of a function $x\colon [0,\infty)\to M$ defined by

$$(3.8)\qquad D_T(\gamma,x)=\sup_{f\in K_\gamma}TR\left|\frac{1}{T}\int_0^T f(x(t))dt-\int_M f(x)d\mu(x)\right|,$$

where the supremum extended over all functions f satisfying a Lipschitz condition $|f(y)-f(z)|\le Rd(y,z)^{\gamma}$ with fixed exponent $\gamma>0$ and fixed constant R. In the case of sequences on special manifolds M and $\gamma=1$ this concept of discrepancy was extensivey studied by E. Hlawka (cf. [9]; also connections to other notions of discrepancy were established there).

Let G be a compact group (with metric θ) operating continuously and transitively on M and let μ be the Haar measure on G. We assume further that the measure μ on M is G-invariant and

$$d(gy, g'y) \leq c\theta(g, g')$$

for arbitrary $g, g' \in G$. Then for every $y \in M$ the function $g(t)y$ is μ-u.d. on M. This can be proved by the same arguments as the analogous statement in [9]. Now let $\Phi \in K^{\gamma}$ be a Lipschitz function with exponent $\gamma > 0$, i.e.

$$|\Phi(y) - \Phi(z)| \leq Rd(y, z)^{\gamma}.$$

Hence we have for the function $F_0(g) = \Phi(gy)$ (for fixed y)

$$|F_0(g) - F_0(g')| \leq Rd(gy, g'y)^{\gamma} \leq Rc^{\gamma}\theta(g, g')^{\gamma}. \tag{3.9}$$

Thus we derive the inequality

$$D_T(\gamma, g(t)y) \leq c^{\gamma} D_T(\gamma, g(t)). \tag{3.10}$$

A number of other estimates for the Lipschitz discrepancy of functions can be worked out. For more details we refer to [9]. Finally we mention here that a lot of other concepts of uniform distribution can be introduced also in the case of functions: well distribution, weak well distribution and complete uniform distribution. The first two concepts were studied in earlier papers, completely uniformly distributed functions will be studied in the subsequent article [3].

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