6. *Nonmonotoneity of Picard Principle for Schrödinger Operators**),**)

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Consider the time independent Schrödinger equation

$$(1) \qquad L_P u(z) \equiv (-\Delta + P(z))u(z) = 0 \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad z = x + yi\right)$$

on the punctured open unit disk $\Omega = \{0<|z|<1\}$, where the potential P in L_P is assumed to be nonnegative and of class C^∞ on $0<|z|\leqq 1$. Such a function P will be referred to as a *density* on Ω in this note. We say that the *Picard principle* is valid for a density P at the origin $z=0$ if the set $F_P(\Omega)$ of nonnegative solutions of (1) on Ω with vanishing boundary values on the unit circle $\Gamma: |z|=1$ is generated by a single element u in $F_P(\Omega)$: $F_P(\Omega) = \{cu : c \geqq 0\}$. In other words the Picard principle is valid for P at the origin if and only if the Martin ideal boundary of Ω over the origin with respect to (1) consists of one point.

Let P be a density on Ω for which the Picard principle is valid and Q be a density on Ω with $Q \leqq P$ on Ω. It is a natural question whether the Picard principle is valid for Q along with P. The most decisive positive result in this direction is that the Picard principle is valid for Q if P and Q are *rotation free*, i.e. $P(z)=P(|z|)$ and $Q(z)=Q(|z|)$ on Ω ([1], [3]). However, in general, the Picard principle is invalid for Q ([4], [5]). Moreover the Picard principle is generally invalid for Q even if Q is supposed to be rotation free ([5]). In view of these we have been interested in the question whether the Picard principle is valid for Q in the case when P is assumed to be rotation free. The purpose of this note is to resolve the question in the negative by showing the following:

Theorem. *There exist a rotation free density P on Ω and a density Q on Ω with $Q \leqq P$ on Ω such that the Picard principle is valid for P and nevertheless the Picard principle is invalid for Q.*

As an important consequence of the proof of the above result we will show the Picard principle is invalid for an almost rotation free density P even if it is valid for the density given by $P(|z|)$. We will also show a similar result to the above theorem for densities having singularities in a nondegenerate boundary component.

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1. Proof of the theorem. We only have to exhibit P and Q with properties mentioned in the theorem. The desired densities P and Q on Ω are given by

$$P(z)=\frac{5}{4|z|^2}\left[\log\frac{1}{|z|}\right]^2+\frac{3}{2|z|^2},$$

$$Q(re^{i\theta})=\frac{1}{r^2}\left[\log\frac{1}{r}\right]^2\left[\frac{1}{4}+\cos^2\theta\right]+\frac{1}{r^2}\left[\frac{1}{2}+\sin^2\theta\right], \tag{2}$$

where $z=re^{i\theta}$. By the test of Picard principle for rotation free densities ([3, Theorem on pp. 426–427]) the Picard principle is valid for P. Observe that the functions

$$E_j(re^{i\theta})=\exp\left\{\frac{1}{4}\left[\log\frac{1}{r}\right]^2+(-1)^j\left[\log\frac{1}{r}\right]\sin\theta\right\}\qquad(j=1,2)$$

are positive unbounded solutions of $L_Qu(z)=0$ on Ω and denote by e_Q the Q-unit on Ω, i.e. the unique bounded solution of $L_Qu(z)=0$ on Ω with boundary values 1 on Γ. Then E_1-e_Q and E_2-e_Q belong to $F_Q(\Omega)$. Since, as easily seen, E_1-e_Q and E_2-e_Q are nonproportional, the Picard principle is invalid for Q.

2. Almost rotation free densities. We say that a density P on Ω is *almost rotation free* if there exists a constant $c\in[1,\infty)$ such that

$$c^{-1}P(|z|)\leqq P(z)\leqq cP(|z|) \tag{3}$$

for every z in Ω ([2]). It has rather been expected that the Picard principle is valid for an almost rotation free density P if the Picard principle is valid for $P(|z|)$. The density Q on Ω given by (2) is almost rotation free and the Picard principle is valid for

$$Q(|z|)=\frac{5}{4|z|^2}\left[\log\frac{1}{|z|}\right]^2+\frac{1}{2|z|^2}$$

([3]). Hence the above expectation is dashed by the density Q given by (2). Moreover the Picard principle may not in general be valid for an almost rotation free density P even if the constant c in (3) can be chosen enough close to 1. In fact the functions

$$E_{\varepsilon j}(re^{i\theta})=\exp\left\{\frac{1}{4}\left[\log\frac{1}{r}\right]^2+(-1)^j\varepsilon\left[\log\frac{1}{r}\right]\sin\theta\right\}\qquad(j=1,2)$$

are positive unbounded solutions of $L_{Q_\varepsilon}u(z)=0$ on Ω with the density

$$Q_\varepsilon(re^{i\theta})=\frac{1}{r^2}\left[\log\frac{1}{r}\right]^2\left[\frac{1}{4}+\varepsilon^2\cos^2\theta\right]+\frac{1}{r^2}\left[\frac{1}{2}+\varepsilon^2\sin^2\theta\right]$$

which satisfies (3) for $c=1+4\varepsilon^2$, where ε is an arbitrary positive constant.

We also see that the Picard principle may not in general be valid for a density P on Ω with

$$P(z)\leqq\left[\frac{1}{4}+\varepsilon\right]\frac{1}{|z|^2}\left[\log\frac{1}{|z|}\right]^2$$

in a neighbourhood of the origin $z=0$ for any positive constant ε.

3. Nonmonotoneity at a singularity in a nondegenerate boundary component. We denote by Ω^+ the upper half unit disk $\{\mathrm{Im}\, z>0\}\cap\Omega$. By a *density* on Ω^+ we mean a nonnegative C^∞ function on $\bar{\Omega}^+-\{0\}$. We say that the *Picard principle* is valid for a density P on Ω^+ at the origin $z=0$ if the set $F_P(\Omega^+)$ of nonnegative solutions of (1) on Ω^+ with vanishing boundary values on $\partial\Omega^+-\{0\}$ is generated by a single element in $F_P(\Omega^+)$. Let P be a density on Ω^+ for which the Picard principle is valid and Q be a density on Ω^+ with $Q\leqq P$ on Ω^+. The Picard principle is valid for Q if P and Q are rotation free ([3], [6]). However the Picard principle may not be valid for Q. Moreover, by the same fashion as in [5] applied to Ω, we can construct a density P on Ω^+ for which the Picard principle is valid and a rotation free density Q on Ω^+ with $Q\leqq P$ on Ω^+ for which the Picard principle is invalid.

What happens to the case when P is rotation free. The result is still in the negative as in the case of Ω. Namely the Picard principle is invalid for the density

$$Q(re^{i\theta})=\frac{4}{r^2}\left[\log\frac{1}{r}\right]^2(1+\cos^2 2\theta)+\frac{1}{r^2}(2+\sin^2 2\theta)$$

on Ω^+ although the Picard principle is valid for the rotation free density

$$P(z)=\frac{8}{|z|^2}\left[\log\frac{1}{|z|}\right]^2+\frac{3}{|z|^2}$$

on Ω^+ which dominates Q on Ω^+ ([3], [6]). In fact, the functions

$$E_j(re^{i\theta})=\exp\left\{\left[\log\frac{1}{r}\right]^2+(-1)^j\left[\log\frac{1}{r}\right]\sin 2\theta\right\}\qquad(j=1,2)$$

are positive unbounded solutions of $L_Q u(z)=0$ on Ω^+. Let u_n be the solution of $L_Q u(z)=0$ on $\Omega_n^+=\{|z|>1/n\}\cap\Omega^+$ with boundary values E_1 $(=E_2)$ on $\partial\Omega_n^+\cap\partial\Omega^+$ and 0 on $\partial\Omega_n^+\cap\Omega^+$. Then $u_n\leqq u_{n+1}$ and $u_n\leqq\min(E_1,E_2)$ on Ω_n^+ $(n=1,2,\cdots)$ so that $\{u_n\}$ converges to a positive solution E_0 of $L_Q u(z)=0$ on Ω^+ with the same boundary values as that of E_1 $(=E_2)$ on $\partial\Omega_n^+-\{0\}$. Since Q and the boundary values of E_0 are symmetric with respect to the imaginary axis, E_0 is also symmetric. Therefore the functions E_1-E_0 and E_2-E_0 are nonproportional and hence the Picard principle is invalid for Q.

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