

117. Equivariant Annulus Theorem for 3-Manifolds

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1. **Introduction.** Let $f: (F, \partial F) \rightarrow (M, \partial M)$ be a proper map from a bounded surface F into a 3-manifold M . The map f is called *boundary incompressible* if it is not properly homotopic to a map $g: (F, \partial F) \rightarrow (M, \partial M)$ such that $g(F) \subset \partial M$. Let F' be a surface properly embedded in M . F' is called *essential* if it is incompressible and $\text{inc}: (F', \partial F') \rightarrow (M, \partial M)$ is boundary incompressible.

In this paper we will prove an equivariant essential annulus theorem for the Haken manifolds whose boundary components are all tori.

Theorem 1.^{*)} *Let M be a bounded, Haken manifold whose boundary components are all tori and which is not homeomorphic to $T^2 \times I$ where T^2 denotes the 2-dimensional torus and I denotes the unit interval $[0, 1]$. Suppose that there is an essential annulus A' in M . If G is a finite subgroup of $\text{Diff}(M)$ then there exists an essential annulus A^* in M such that either $g(A^*) = A^*$ or $g(A^*) \cap A^* = \emptyset$ for each element g of G .*

Note that for $T^2 \times I$ this theorem does not hold. See the remark of section 2 below.

The examples of 3-manifold admitting no nontrivial, finite group actions are constructed by Raymond-Tollefson [7] and Siebenmann [9]. As an application of Theorem 1 we will give a simple construction of such 3-manifolds by using the knot theory (Theorem 2).

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2. **Proof of Theorem 1.** Throughout this paper we will work in the C^∞ -category. For the definitions of standard terms in the three dimensional topology we refer to [3] and [4].

The proof of Theorem 1 depends on the following result which is due to Nakauchi [6]. The author was informed that J. Hass had proved a similar result in his Ph. D. thesis.

Theorem A (Nakauchi). *Let M be a compact, orientable, 3-dimensional, Riemannian manifold with convex incompressible boundary and let A be a smooth annulus. Suppose that there is an essential smooth map $f: (A, \partial A) \rightarrow (M, \partial M)$. Then*

(1) *there exists an essential smooth immersion $f^*: (A, \partial A)$*

^{*)} T. Soma independently obtained the similar result.

$\rightarrow(M, \partial M)$ which has least area among all such essential smooth maps,

(2) any such least area immersion is either an embedding or a double covering map onto an embedded Möbius band,

(3) the image of any two such least area maps are disjoint or are equal or intersect each other only at one essential arc. Moreover, all the distinct images of the double covering maps are mutually disjoint.

Lemma 2.1. *Let M be a bounded Haken manifold whose boundary components are all tori and which is homeomorphic to neither $T^2 \times I$ nor the twisted I -bundle over the Klein bottle and let A_1, A_2 be essential annuli in M such that a component S_1 of ∂A_1 and a component S_2 of ∂A_2 are contained in the same component T_1 of ∂M . Then S_1 is isotopic to S_2 in T_1 .*

Proof. Assume that S_1 is not isotopic to S_2 in T_1 . By [4], [5] there is a characteristic Seifert pair Σ in M such that the components of $\text{Fr}_M \Sigma$ are all tori where $\text{Fr}_M \Sigma$ denotes the frontier of Σ in M . Let σ be a component of Σ such that $T_1 \subset \partial \sigma$. By the homotopy annulus theorem (Theorem VIII. 10 of [4]) we may suppose that $A_1 \subset \sigma, A_2 \subset \sigma$. Then σ admits such Seifert fibration that S_i ($i=1, 2$) is a regular fiber (see Theorem VIII. 34 of [4]). So σ admits two fibrations which are not isotopic. Hence by Theorem VI. 18 of [4] σ is homeomorphic to the solid torus, $T^2 \times I$ or the twisted I -bundle over the Klein bottle.

Since σ contains an essential annulus, σ is not the solid torus. If σ is the twisted I -bundle over the Klein bottle then so is M and this is a contradiction. Suppose σ is $T^2 \times I$. Since A_1 is essential in M , A_1 intersects both components of $\partial \sigma$. Hence M is $T^2 \times I$ and this is a contradiction.

This completes the proof of Lemma 2.1.

Proof of Theorem 1. It is elementary to construct an invariant metric on M with convex boundary. There is an incompressible and boundary incompressible map $f: (A, \partial A) \rightarrow (M, \partial M)$ which satisfies the conclusions of Theorem A.

If f is an embedding then set $A^* = f(A)$. By Theorem A either $g(A^*) = A^*$, $g(A^*) \cap A^* = \emptyset$ or $g(A^*) \cap A^*$ is an essential arc of A^* for each element g of G . If M is not the twisted I -bundle over the Klein bottle then by Lemma 2.1 the third case cannot occur. So we have the conclusion of Theorem 1. If M is the twisted I -bundle over the Klein bottle then there are only two proper homotopy classes in all essential annuli in M and one of them represents such an annulus that cuts M into two solid tori, the other cuts M into one solid torus (see Example VI. 5 (d) of [4]). Hence A^* and $g(A^*)$ are isotopic in M and we again cannot have the third case.

If f double covers a Möbius band F in M then by Theorem A (3) either $g(F)=F$ or $g(F)\cap F=\phi$. Then by [1] there exists a regular neighborhood R of F such that either $g(R)=R$ or $g(R)\cap R=\phi$. Then we set $A^*=\text{Fr}_M R$. A^* is an essential annulus and satisfies the conclusion of Theorem 1.

This completes the proof of Theorem 1.

Remark. We show that for $T^2 \times I$ Theorem 1 does not hold. Let $\phi: T^2 \rightarrow T^2$ be a periodic map such that $\phi_*: \pi_1(T^2) \rightarrow \pi_1(T^2)$ is represented by the 2×2 matrix $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ for the fixed basis of $\pi_1(T^2)$ and let $\Phi: T^2 \times I \rightarrow T^2 \times I$ be $\phi \times \text{id}_I$. Let A' be an essential annulus in $T^2 \times I$ and S_1 be a component of $\partial A'$ contained in $T^2 \times \{1\}$. We can identify the universal cover of $T^2 \times \{1\}$ to \mathbb{R}^2 and $\Phi|_{T^2 \times \{1\}}$ is conjugate to the map $f: T^2 \times \{1\} \rightarrow T^2 \times \{1\}$ such that f lifts to an affine map $A: \underline{x} \rightarrow B\underline{x} + \underline{c}$ of \mathbb{R}^2 where $\underline{x}, \underline{c}$ denote \mathbb{R}^2 vectors. Then S_1 lifts to a line l in \mathbb{R}^2 . Since the eigen value of B is not ± 1 , l and $A(l)$ are not coincide and intersect. Hence we have $f(S_1) \neq S_1$ and $f(S_1) \cap S_1 \neq \phi$ and so there are no Φ -equivariant essential annuli in $T^2 \times I$.

3. Theorem 2. In this section we will state Theorem 2 and prove it.

A knot K is a simple closed curve in the 3-sphere S^3 . The exterior $Q(K)$ of K is the closure of the complement of the regular neighborhood of K . K is *strongly negative amphicheiral* (*strongly invertible* resp.) if there is an orientation reversing (preserving resp.) involution g of S^3 which satisfies (i) $g(K)=K$ and (ii) $g|_K$ reverses the orientation of K . The *meridian* $m(K)$ of K is a nontrivial simple closed curve in $\partial Q(K)$ which bounds a disk in $S^3 - \text{Int } Q(K)$.

For the definitions of other standard terms in the knot theory we refer to [8].

Theorem 2. *Let K_1 be a non strongly negative amphicheiral, prime knot and K_2 be a non strongly invertible, prime knot such that $Q(K_1)$ is not homeomorphic to $Q(K_2)$. Then $M=Q(K_1 \# K_2)$ admits no nontrivial, finite group actions where $\#$ denotes the composition of knots.*

Proof. Assume that there is a periodic map f of M which is not id_M . We may suppose that the period of f is a prime integer. There is an essential annulus A in M which cuts M into M_1 and M_2 where M_i ($i=1, 2$) is homeomorphic to $Q(K_i)$. By Lemma 2.1 and the uniqueness of the prime decomposition of knots we see that the essential annulus in M is unique up to isotopy. By Theorem 1 we may suppose that $f(A)=A$ or $f(A)\cap A=\phi$.

We claim that $f(A)=A$ hence $f(M_i)=M_i$. Assume that $f(A)\cap A=\phi$. Then $f(A)$ is parallel to A and so we have $f(M_i)\subseteq M_i$ ($i=1$ or 2)

for M_2 is not homeomorphic to M_1 . Hence we have $M_i \supseteq f(M_i) \supseteq f^2(M_i) \supseteq \dots$, which contradicts the periodicity of f and the claim is established.

So we have the periodic map $f_i = f|_{M_i}: (M_i, A) \rightarrow (M_i, A)$. If f is orientation reversing involution then f_1 extends to an orientation reversing involution g of S^3 such that $g(K_1) = K_1$ and $g|_{K_1}$ reverses the orientation of K_1 , which is a contradiction. If f is orientation preserving and f preserves each component of ∂A then f extends to an orientation preserving periodic map g of S^3 such that $\text{Fix}(g) = K_1 \# K_2$ where $\text{Fix}(g)$ denotes the fixed point set of g . This contradicts the Smith conjecture [10]. If f is orientation preserving and f exchanges the components of ∂A then the order of f is two and f_2 extends to an orientation preserving involution g of S^3 such that $g(K_2) = K_2$ and $g|_{K_2}$ reverses the orientation of K_2 . This is a contradiction.

Hence M admits no nontrivial finite group actions and this completes the proof of Theorem 2.

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