5. On Order Star-Finite and Closure-Preserving Covers

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1. Introduction. R. Telgársky and Y. Yajima [6] have studied the structural properties of order star-finite covers and order locally finite covers. Moreover, in [6], they have proved the closure-preserving sum theorem for covering dimension; if a normal space X has a closure-preserving closed cover $\mathfrak{F}$ such that F is countably compact and $\dim F \leqq n$ for each $F \in \mathfrak{F}$, then $\dim X \leqq n$. This paper is a continuation of [6]. We first give a characterization of paracompact spaces in terms of order star-finiteness which is a generalization of star-finiteness. Secondly, using this result, we state a relation between order star-finite covers and order locally finite covers. Finally, we show that the closure-preserving sum theorem for large inductive dimension, as well as the above one, holds. All spaces are assumed to be Hausdorff spaces. N denotes the set of all natural numbers.

2. Order star-finite covers. A family $\{A_\lambda : \lambda \in \Lambda\}$ of subsets of a space X is said to be *order star-finite* [4] (*order locally finite* [1]), if one can introduce a well-ordering $<$ in the index set Λ such that for each $\lambda \in \Lambda$ the set A_λ meets at most finite many A_μ with $\mu < \lambda$ (the family $\{A_\mu : \mu < \lambda\}$ is locally finite at each point of A_λ). Then we may use, without loss of generality, the notation $\{A_\xi : \xi < \alpha\}$ instead of $\{A_\lambda : \lambda \in \Lambda\}$.

Proposition 1. *Every point-finite open cover of a collectionwise normal space X has an order star-finite open refinement.*

Proof. We modify the proof of E. Michael ([2], Theorem 2). Let $\mathfrak{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a point-finite open cover of X. For $k \in N$, let Λ_k be the family of all $\gamma \subset \Lambda$ such that γ has exactly k elements. We shall construct a sequence $\{\mathfrak{V}_i : i \in N\}$ of families of open sets of X, where $\mathfrak{V}_i = \{V_\gamma : \gamma \in \Lambda_i\}$, satisfying the following conditions:

(1) $\mathrm{Cl}\, V_\gamma \subset \bigcap_{\lambda \in \gamma} U_\lambda$ for each $\gamma \in \Lambda_i$.

(2) $\mathfrak{V}_i$ is discrete for each $i \in N$.

(3) $\{\delta \in \bigcup_{j=1}^{i-1} \Lambda_j : V_\delta \cap V_\gamma \neq \varnothing\}$ is finite for each $\gamma \in \Lambda_i$.

(4) If $x \in X$ is an element of at most i elements of $\mathfrak{U}$, then $x \in \bigcup_{j=1}^{i} V_j$, where $V_j = \bigcup\{V_\gamma : \gamma \in \Lambda_j\}$.

Assume that $\mathfrak{V}_i = \{V_\gamma : \gamma \in \Lambda_i\}$ $(i = 1, \cdots, k)$ have been constructed to satisfy (1)–(4) for all $i \leqq k$. For each $\gamma \in \Lambda_{k+1}$, let $F_\gamma = (X \setminus \bigcup_{i=1}^{k} V_i) \cap (X \setminus \bigcup\{U_\lambda : \lambda \notin \gamma\})$. Then it follows from [2] that $\{F_\gamma : \gamma \in \Lambda_{k+1}\}$ is a discrete family of closed subsets of X such that $F_\gamma \subset \bigcap_{\lambda\in\gamma} U_\lambda$. Let $\gamma \in \Lambda_{k+1}$ and $\delta \in \bigcup_{i=1}^k \Lambda_i$ with $\delta \not\subset \gamma$. Then it is easy to show from (1) and the definition of F_γ that F_γ cannot intersect $\mathrm{Cl}\, V_\delta$. Since $\bigcup_{i=1}^k \mathfrak{V}_i$ is locally finite in X, we have $F_\gamma \cap \mathrm{Cl}\,(\bigcup\{V_\delta : \delta \in \bigcup_{i=1}^k \Lambda_i, \delta \not\subset \gamma\}) = \varnothing$. So, by collectionwise normality of X, we can choose a discrete family $\mathfrak{V}_{k+1} = \{V_\gamma : \gamma \in \Lambda_{k+1}\}$ of open sets such that $F_\gamma \subset V_\gamma \subset \mathrm{Cl}\, V_\gamma \subset \bigcap_{\lambda\in\gamma} U_\lambda$ and $V_\gamma \cap \bigcup\{V_\delta : \delta \in \bigcup_{i=1}^k \Lambda_i, \delta \not\subset \gamma\} = \varnothing$. Then $\{\delta \in \bigcup_{i=1}^{k+1} \Lambda_i : V_\gamma \cap V_\delta \neq \varnothing\} \subset \{\delta : \delta \subset \gamma\}$ holds. Hence $\mathfrak{V}_i = \{V_\gamma : \gamma \in \Lambda_i\}$ $(i=1, \cdots, k+1)$ satisfy (1)–(3) for all $i \leqq k+1$. Further, as in [2], they satisfy (4) for all $i \leqq k+1$. Here, it is easy to verify from (1)–(4) that $\bigcup_{i=1}^\infty \mathfrak{V}_i$ is an order star-finite open refinement of $\mathfrak{U}$. The proof is completed.

The following result holds from Lemma 2 in [1] and Proposition 1.

Theorem 1. *For a regular space X, the following are equivalent.*

(a) *X is a paracompact space.*

(b) *Every open cover of X has an order locally finite open refinement.*

(c) *Every open cover of X has an order star-finite open refinement.*

Prof. R. Telgársky suggested, kindly, the following result to the author. Using our Theorem 1 and his technique in the proof of Lemma 1 in [4], we can obtain it.

Theorem 2. *Let $\{E_\xi : \xi < \alpha\}$ and $\{U_\xi : \xi < \alpha\}$ be order locally finite covers of a paracompact space X, where E_ξ is closed in X and U_ξ is an open neighborhood of E_ξ for each $\xi < \alpha$. Then there exist order star-finite covers $\{F_\eta : \eta < \beta\}$ and $\{V_\eta : \eta < \beta\}$ of X, where $\{F_\eta : \eta < \beta\}$ refines $\{E_\xi : \xi < \alpha\}$, $\{V_\eta : \eta < \beta\}$ refines $\{U_\xi : \xi < \alpha\}$, F_η is closed in X and V_η is an open neighborhood of F_η for each $\eta < \beta$.*

Proof. For each $\xi < \alpha$, we can construct order star-finite and locally finite families $\{E_{\xi,\zeta} : \zeta <_\xi \beta_\xi\}$ and $\{U_{\xi,\zeta} : \zeta <_\xi \beta_\xi\}$ such that $\{E_{\xi,\zeta} : \zeta <_\xi \beta_\xi\}$ is a closed cover of E_ξ, $U_{\xi,\zeta}$ is open in X, $E_{\xi,\zeta} \subset U_{\xi,\zeta} \subset U_\xi$ and $\{(\eta, \nu) : \eta < \xi, \nu <_\eta \beta_\eta \text{ and } U_{\eta,\nu} \cap U_{\xi,\zeta} \neq \varnothing\}$ is a finite set for each $\zeta <_\xi \beta_\xi$. Then the families $\{E_{\xi,\zeta} : \zeta <_\xi \beta_\xi, \xi < \alpha\}$ and $\{U_{\xi,\zeta} : \zeta <_\xi \beta_\xi, \xi < \alpha\}$ are order star-finite. So, we rewrite them $\{F_\eta : \eta < \beta\}$ and $\{V_\eta : \eta < \beta\}$, respectively. Then $\{F_\eta : \eta < \beta\}$ and $\{V_\eta : \eta < \beta\}$ satisfy all the conditions in Theorem 2. The proof is complete.

3. Closure-preserving covers. The following lemma is well-known (e.g., [3], Proposition 4.4.11).

Lemma 1. *Let A be a closed subset of a totally normal space X with* $\mathrm{Ind}\, A \leqq n$. *If* $\mathrm{Ind}\, F \leqq n$ *for each closed subset F of X such that $F \cap A = \varnothing$, then* $\mathrm{Ind}\, X \leqq n$.

Now we make use of the topological game $G(\boldsymbol{K}, X)$ introduced and studied by R. Telgársky [5]. Let $\mathrm{Ind}_n = \{Y : Y \text{ is normal and } \mathrm{Ind}\, Y \leqq n\}$.

Proposition 2. *Let X be a totally normal space. If a Player I* *has a winning strategy in* $G(\mathrm{Ind}_n, X)$, *then* $\mathrm{Ind}\, X \leqq n$.

Proof. The idea of the proof is essentially due to that of R. Telgársky ([5], Theorem 11.1). Let s be a winning strategy of Player I in $G(\mathrm{Ind}_n, X)$. Now let E and F be closed subsets of X with $F \subset E$. Since X is totally normal, there is a sequence $\{\mathfrak{S}_i(E,F) : i \in N\}$ of families of closed subsets of X such that

(1) $\bigcup \{H : H \in \mathfrak{S}_i(E,F), i \in N\} = E \backslash F$

and

(2) $\mathfrak{S}_i(E,F)$ is locally finite in $E \backslash F$ for each $i \in N$.

Fix $m \in N$ and $i_0, \cdots, i_m \in N$. Let $T(i_0, \cdots, i_m)$ be the set of all admissible sequences $(E_0^{i_0}, E_1^{i_0}, \cdots, E_{2m-1}^{i_{m-1}}, E_{2m}^{i_m})$ for $G(\mathrm{Ind}_n, X)$ such that

(3) $E_0^{i_0} = X$ and $E_1^{i_0} = s(X)$ for each $i_0 \in N$,

(4) $E_{2k+1}^{i_k} = s(E_0^{i_0}, E_1^{i_0}, \cdots, E_{2k-1}^{i_{k-1}}, E_{2k}^{i_k})$

and

(5) $E_{2k+2}^{i_{k+1}} \in \mathfrak{S}_{i_{k+1}}(E_{2k}^{i_k}, E_{2k+1}^{i_k})$ for $k = 0, \cdots, m-1$.

Put $\mathfrak{E}(i_0, \cdots, i_m) = \{E_{2m}^{i_m} : (E_0^{i_0}, E_1^{i_0}, \cdots, E_{2m-1}^{i_{m-1}}, E_{2m}^{i_m}) \in T(i_0, \cdots, i_m)\}$ and $s\mathfrak{E}(i_0, \cdots, i_m) = \{s(E_0^{i_0}, E_1^{i_0}, \cdots, E_{2m-1}^{i_{m-1}}, E_{2m}^{i_m}) : (E_0^{i_0}, E_1^{i_0}, \cdots, E_{2m-1}^{i_{m-1}}, E_{2m}^{i_m}) \in T(i_0, \cdots, i_m)\}$. Moreover, let $X(i_0, \cdots, i_m)$ be the union of all elements of $s\mathfrak{E}(i_0, \cdots, i_m)$. Here, we define $X_k = \bigcup \{X(i_0, \cdots, i_m) : m \in N, i_0 + \cdots + i_m \leqq k\}$. First, we can see from (2) and (5) that for each $i_0, \cdots, i_m \in N$,

(6) $\mathfrak{E}(i_0, \cdots, i_m)$ is locally finite in $X \backslash \bigcup_{j=1}^{m-1} X(i_0, \cdots, i_j)$.

Now we shall show the following three facts;

(7) X_k is closed in X,

(8) $\mathrm{Ind}\, X_k \leqq n$ for $k = 0, 1, \cdots$

and

(9) $\bigcup_{k=0}^{\infty} X_k = X$.

X_0 is clearly closed in X and assume that X_k is closed in X. Let $x \notin X_{k+1}$. Take any $i_0, \cdots, i_m \in N$ with $i_0 + \cdots + i_m = k+1$. Then we obtain $x \in X \backslash \bigcup_{j=1}^{m-1} X(i_0, \cdots, i_j)$. By (6), $s\mathfrak{E}(i_0, \cdots, i_m)$ is locally finite at x. Hence we have $x \notin \mathrm{Cl}\, X(i_0, \cdots, i_m)$. From the inductive assumption, $x \notin \mathrm{Cl}\, X_{k+1}$ holds. Thus, (7) is true. Clearly, $\mathrm{Ind}\, X_0 \leqq n$, and assume that $\mathrm{Ind}\, X_k \leqq n$ holds. Let H be a closed subset in X_{k+1} with $H \cap X_k = \emptyset$. Then, by (6), $\{H \cap E : E \in s\mathfrak{E}(i_0, \cdots, i_m), i_0 + \cdots + i_m = k+1$ and $m \in N\}$ is a locally finite closed cover of H. By the locally finite sum theorem for Ind, we have $\mathrm{Ind}\, H \leqq n$. From (7), the inductive assumption and Lemma 1, $\mathrm{Ind}\, X_{k+1} \leqq n$ holds. Thus, (8) is true. Let $x \notin \bigcup_{k=0}^{\infty} X_k$. Then, there is a sequence $\{i_0, i_1, \cdots\}$ of N such that we can choose some $E_{2m}^{i_m} \in \mathfrak{E}(i_0, \cdots, i_m)$ with $x \in E_{2m}^{i_m}$ for $m = 0, 1, \cdots$. Moreover, the countable many admissible sequences $\{(E_0^{i_0}, E_1^{i_0}, \cdots, E_{2m-1}^{i_{m-1}}, E_{2m}^{i_m}) : m \in N\}$ yield a play $(E_0^{i_0}, E_1^{i_0}, E_2^{i_1}, E_3^{i_1}, \cdots)$ in $G(\mathrm{Ind}_n, X)$, where each $E_{2k+1}^{i_k}$ is chosen to satisfy (4). Since s is a winning strategy of Player I in $G(\mathrm{Ind}_n, X)$, we have $\bigcap_{m=0}^{\infty} E_{2m}^{i_m}=\varnothing$. This is a contradiction. Thus, (9) is true. From (7)–(9) and the countable sum theorem for Ind, $\mathrm{Ind}\, X \leqq n$ holds. The proof is completed.

DK denotes the class of all spaces Y which have a discrete closed cover $\{Y_\lambda : \lambda \in \Lambda\}$ with $\{Y_\lambda : \lambda \in \Lambda\} \subset \boldsymbol{K}$ (cf. [5]).

Lemma 2 (Telgársky and Yajima [6]). *If a space X has a closure-preserving closed cover $\mathfrak{F}$ such that F is countably compact and $F \in \boldsymbol{K}$ for each $F \in \mathfrak{F}$, then Player I has a winning strategy in $G(D\boldsymbol{K}, X)$.*

The following theorem holds from Proposition 2 and Lemma 2.

Theorem 3. *If a totally normal space X has a (σ-)closure-preserving closed cover $\mathfrak{F}$ such that F is countably compact and* $\mathrm{Ind}\, F \leqq n$ *for each $F \in \mathfrak{F}$, then* $\mathrm{Ind}\, X \leqq n$.

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