20. Theory of Invariants in the Geometry of Paths. II. Equivalence Problems.

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§ 1. The method hitherto utilized in the case of the generalized rheonomic geometry of paths is, after some suitable modifications, available to geometries of paths such as ordinary, intrinsic, and rheonomic. This is due to the fact that the transformation group (I, 0.2) of the generalized rheonomic geometry is an extension, in some meaning, of the transformation group (I, i), (I, ii), (I, iii) on which the other geometries are based. However the results obtained by our method are somewhat complicated for these geometries compared with the results hitherto already known.

Concerning the intrinsic geometry of paths we give here some brief additional notes for the results of A. Kawaguchi—H. Hombu [5][1]. Under the intrinsic group (I, ii), as in the case of generalized rheonomic group G (I, 0.2), the Pfaffians $\mathfrak{d}x^{(r)i}$ $(r=0, 1, \dots, m-1)$ defined by (I, 2.2), instead of the ordinary differentials $dx^{(r)i}$ $(r=0, 1, \dots, m-1)$, are subject to the transformation law (I, 2.3). Hence we must use $\mathfrak{d}x^{(r)i}$ instead of $dx^{(r)i}$ in the fundamental theorem[2] concerning the covariant differential of the line-element. Owing to this modification the covariant derivative[3] ∇v^i of a vector field v^i of weight p turns to be our (I, 5.6). As to the results of S. Hokari [3] the same considerations are necessary.

§ 2. In this paragraph we study the equivalence problem in the geometry of paths in the case of generalized rheonomic geometry.

Let us consider two systems of paths defined by

$$(2.1)\qquad \begin{aligned} &\text{(i)} \quad x^{(m)i}+H^i(t, x, x^{(1)}, \dots, x^{(m-1)})=0\,, \\ &\text{(ii)} \quad \bar{x}^{(\overline{m})\alpha}+\overline{H}^\alpha(\bar{t}, \bar{x}, \bar{x}^{(\overline{1})}, \dots, \bar{x}^{(\overline{m-1})})=0\,. \end{aligned}$$

If there exists a transformation belonging to the generalized rheonomic group G such that the relations (I, 2.1) hold between the two sets of quantities

$$(t, x^i, x^{(1)i}, \dots, x^{(m-1)i}, H^i) \quad \text{and} \quad (\bar{t}, \bar{x}^\alpha, \bar{x}^{(\overline{1})\alpha}, \dots, \bar{x}^{(\overline{m-1})\alpha}, \overline{H}^\alpha)\,,$$

we say that the two systems of paths are equivalent to each other

1) Numbers in brackets refer to the bibliography at the end of the paper.
2) [5], P. 58, Theorem 16.
3) [5], P. 60, (3.21).

under G. Our purpose is to find a necessary and sufficient condition for the equivalence. According to our method for the generalized rheonomic geometry, we construct the parameters of connection Γ, Γ^i_j, $\Lambda^{(r)i}_{(s)j}$ and Γ^i_{jk} and the corresponding functions using the functions H^i and $\overline{H}^\alpha$ respectively. Then, if the two systems are equivalent, there exist the relations (I, 3.2), (I, 3.3), (I, 4.6) and (I, 5.2). Therefore we know that the n^2+mn+2 functions

$$\bar{t},\ \bar{x}^\alpha,\ \bar{x}^{(\bar{1})\alpha},\ \dots,\ \bar{x}^{(\overline{m-1})\alpha},\quad \left(\frac{d\bar{t}}{dt}\right)^{-1}\equiv\sigma,\quad \frac{\partial\bar{x}^\alpha}{\partial x^i}\equiv u^\alpha_i \tag{2.2}$$

of the $mn+1$ independent variables

$$t,\ x^i,\ x^{(1)i},\ \dots,\ x^{(m-1)i} \tag{2.3}$$

satisfy the following differential equations of the first order:

$$\left\{\begin{array}{lll}
\text{(i)} & \dfrac{d\bar{t}}{dt}=\dfrac{1}{\sigma},\quad \dfrac{\partial\bar{t}}{\partial x^{(s)j}}=0 & (s=0,1,\dots,m-1),\\[2ex]
\text{(ii)} & \dfrac{d\sigma}{dt}=\sigma\Gamma-\overline{\Gamma},\quad \dfrac{\partial\sigma}{\partial x^{(s)j}}=0 & (s=0,1,\dots,m-1),\\[2ex]
\text{(iii)} & \left\{\begin{array}{l}
\dfrac{\partial\bar{x}^{(\bar{r})\alpha}}{\partial x^{(r)i}}=\sigma^r u^\alpha_i \\[2ex]
\dfrac{\partial\bar{x}^{(\bar{r})\alpha}}{\partial x^{(s)j}}=0 \\[2ex]
\dfrac{\partial\bar{x}^{(\bar{r})\alpha}}{\partial x^{(s)j}}=\sigma^r u^\alpha_i\Lambda^{(r)i}_{(s)j}-\displaystyle\sum_{t=s}^{r-1}\frac{\partial\bar{x}^{(\bar{t})\beta}}{\partial x^{(s)j}}\overline{\Lambda}^{(\bar{r})\alpha}_{(\bar{t})\beta}
\end{array}\right. & \begin{array}{l}
(r=0,1,\dots,m-1),\\[2ex]
\begin{pmatrix}r=0,\ 1,\dots,m-2\\ s=r+1,\dots,m-1\end{pmatrix},\\[2ex]
\begin{pmatrix}r=1,2,\dots,m-1\\ s=0,1,\dots,\ r-1\end{pmatrix},
\end{array}\\[2ex]
\text{(iv)} & \left\{\begin{array}{l}
\dfrac{\partial\bar{x}^{(\bar{r})\alpha}}{\partial t}=\dfrac{1}{\sigma}\bar{x}^{(\overline{r+1})\alpha}-\sigma^r u^\alpha_i x^{(r+1)i}-\displaystyle\sum_{s=0}^{r-1}\frac{\partial\bar{x}^{(\bar{r})\alpha}}{\partial x^{(s)i}}x^{(s+1)i} \\[2ex]
\hspace{20em}(r=0,1,\dots,m-2),\\[2ex]
\dfrac{\partial\bar{x}^{(\overline{m-1})\alpha}}{\partial t}=-\dfrac{1}{\sigma}\overline{H}^\alpha+\sigma^{m-1}u^\alpha_i H^i-\displaystyle\sum_{s=0}^{m-2}\frac{\partial\bar{x}^{(\overline{m-1})\alpha}}{\partial x^{(s)i}}x^{(s+1)i},
\end{array}\right. & \\[2ex]
\text{(v)} & \left\{\begin{array}{l}
\dfrac{\partial u^\alpha_j}{\partial x^{(r)k}}=0 \\[2ex]
\dfrac{\partial u^\alpha_j}{\partial x^k}=u^\alpha_i\Gamma^i_{jk}-u^\beta_j u^\gamma_k\overline{\Gamma}^\alpha_{\beta\gamma}, \\[2ex]
\dfrac{\partial u^\alpha_j}{\partial t}=u^\alpha_i\Gamma^i_j-\dfrac{1}{\sigma}u^\beta_j\overline{\Gamma}^\alpha_\beta-\dfrac{\partial u^\alpha_j}{\partial x^k}x^{(1)k}.
\end{array}\right. & (r=1,2,\dots,m-1),
\end{array}\right. \tag{2.4}$$

Conversely, if (2.4) admits a solution, we may conclude after some discussions that the two systems of paths are equivalent. Thus the equivalence problem is reduced to the problem for a necessary and sufficient condition that (2.4) may have a solution. We construct for (2.4) the sequence of the sets of equations concerning the variables (2.2) and (2.3) :

$$(2.5) \qquad F^{(1)}_{\lambda_1}=0, \quad F^{(2)}_{\lambda_2}=0, \ldots, \quad F^{(N)}_{\lambda_N}=0, \ldots$$

where $F^{(1)}_{\lambda_1}=0$ is the set of equations of integrability of (2.4) and $F^{(N+1)}_{\lambda_{N+1}}=0$ for $N\geqq 1$ is the set of equations obtained by differentiating the set of equations $F^{(N)}_{\lambda_N}=0$ with respect to the variables (2.3) and eliminating the derivatives of (2.2) by means of (2.4). Then, by the well-known theorem for the existence of a solution of mixed system, we see that *a necessary and sufficient condition that the two systems of paths be equivalent is that there exists a positive integer N such that the first N sets of equations of the sequence (2.5) are algebraically consistent considered as equations for the determination of the variables (2.2) as functions of the independent variables (2.3), and that all their solutions satisfy the (N+1)-th set of equations in the sequence.*

The first set of the sequence (2.5) gives the transformation laws of the curvature and torsion tensors of the second kind, and the N-th set for $N>1$ gives the transformation laws of the covariant derivatives of the $(N-1)$-th order of them. Therefore we have the

Theorem. *The curvature tensors of the second kind, torsion tensors and their successive covariant derivatives constitute the complete system of differential invariants for the generalized rheonomic geometry of paths.*

In the same manner we may study the equivalence problem respectively in the case of rheonomic, intrinsic and ordinary geometry. In the case of rheonomic geometry, we have the same conclusion, and in the remaining two cases we must add the contravariant vectors $K^i_{(r)}$ $(r=1, 2, \ldots, m)$[4] to the curvature tensors and torsion tensors, where $K^i_{(1)}=x^{(1)i}$, and $K^i_{(r+1)}$ is the covariant derivatives along the paths of the vector $K^i_{(r)}$: $K^i_{(r+1)}=\delta_t K^i_{(r)}$ $(r=1, 2, \ldots, m-1)$.

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4) The $\mathfrak{K}^i$ defined by [5], P. 60, (3.22) is not a vector,