147. On the Property of Lebesgue in Uniform Spaces. V

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In a series of my Note [5], we studied the uniform spaces having Lebesgue property for finite covering. In this Note, we shall study the uniform space with Lebesgue property for any covering. Such a space was also studied by S. Kasahara [6, 7], one of my colleagues, and he obtained some important results.

First, we shall consider a uniform space with a unique structure.

Let S be a uniform space with a unique structure and S^* a compactification of S. This is possible by P. Samuel [8]. The unique structure of the compact space S^* induces on S a uniform structure. Therefore, the original structure on S coincides with the induced structure. Hence S is precompact. On the other hand, there is a uniform structure on S for which every continuous function is uniformly continuous (A. Weil [9], p. 16). Hence, if S is metrisable, S has the Lebesgue property, and this shows that S is compact. Further, if S has the Lebesgue property, then S is compact, since a theorem of my Note ([5], p. 441).

Therefore, we have

Theorem 1. If a uniform space with a unique structure has the Lebesgue property, then it is compact.

If E is a uniform space, there is a strongest uniform structure on E compatible with the original structure. We say such a structure the universal uniform structure. J. Dieudonné [3] proved the following

Theorem. If E is paracompact, the filter of neighbourhoods of the diagonal in $E\times E$ is the filter of surroundings of the universal structure of E.

On the other hand, P. Samuel [8] proved the following

Theorem. If every continuous mapping of S onto any uniform space is uniformly continuous, then the uniform structure of S is universal.

Theorem 2. If any covering of a uniform space S has the Lebesgue property, then the uniform structure of S is universal.

Proof. We shall prove that every continuous mapping onto a uniform space is uniformly continuous. By the theorem mentioned above, we have the conclusion of Theorem 2. Let F be a given uniform space, and $f(x)$ a continuous mapping of E onto F. For any symmetric surrounding V of F, let $O_x = f^{-1}(V(y))$, where $f(x)=y$. Then $\{O_x \mid x \in E\}$ is an open covering of E. Since E has the Lebesgue property, there is a surrounding U such that $U(x) \subset O_{x_1}$ for some x' depending on x. For $(a, b) \in U$, we have $b \in U(a) \subset O_{x'}$ and $f(b) \in f(U(a)) \subset f(O_x) = V(y)$, and $f(a) \in V(y)$. Therefore, $(f(a), y) \in V$ and $(y, f(b)) \in V$. This shows that $V \ni (a, b)$ implies $(f(a), f(b)) \in V \circ V$. Hence $f(x)$ is uniformly continuous.

If E is a paracompact uniform space for the universal structure, every continuous function is uniformly continuous, and therefore by a theorem of Kasahara brothers [7], E has the Lebesgue property. Hence E is complete (see a paper of S. Kasahara [6]). This proposition has also proved by A. Dickinson [1]. This is a positive answer for J. Dieudonné conjecture [3].

We shall give an example that every continuous function on a normal space E is uniformly continuous, but E has not the Lebesgue property for any open covering. This shows that *a finite Lebesgue property does not imply the Lebesgue property of any open covering.*

Let E be a well-ordered set consisting of 1st and 2nd class ordinal numbers. The fundamental system of neighbourhoods of a point of E is defined by $\{z \mid y < z \leq x\}$ for each y of E such that $y < x$. J. Dieudonné [2] has proved that E is not complete. Therefore, E has not the Lebesgue property. On the other hand, he has also proved that at least one of disjoint closed sets in E is compact. Hence, by a theorem of R. Doss [4], E is a uniform space with unique structure. Therefore, every continuous function is uniformly continuous.

Hence E has the Lebesgue property for any finite open covering.

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