153. Note on Idempotent Semigroups. I

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§ 1. The purpose of this paper is to present the structure theorem of some special idempotent semigroups, called regular (see definition below), which would be considered to form a fairly wide category of idempotent semigroups.

A semigroup is called *left singular* (*right singular, rectangular*) if it satisfies an identity $ab=a$ ($ba=a$, $aba=a$). All of them are idempotent. Also a left (right) singular semigroup is rectangular. Further, a rectangular semigroup is the direct product of a left singular semigroup and a right singular semigroup. This direct product decomposition is unique up to isomorphism.

A global structure theorem on idempotent semigroup was carried out by David McLean [1]. The theorem can be stated as follows:

Let S be an idempotent semigroup. Then there exist, up to isomorphism, a unique semilattice Γ, and a disjoint family of rectangular subsemigroups of S indexed by Γ, $\{S_\gamma : \gamma \in \Gamma\}$, such that

(i) $S=\bigcup\{S_\gamma : \gamma\in\Gamma\}$,

and (ii) $S_\alpha S_\beta \subset S_{\alpha\beta}$ *for all* $\alpha, \beta \in \Gamma$.

In what follows we call Γ the *structure semilattice* of S, and S_γ the *γ-kernel*. And we denote the decomposition by the notation

$$S \sim \sum\{S_\gamma : \gamma\in\Gamma\},$$

and call it the *structure decomposition* of S.

REMARK. It is to be noted that there are, in general, non-isomorphic idempotent semigroups which have isomorphic structure semilattices and the corresponding isomorphic kernels.

§ 2. Now we shall settle here the necessary and sufficient condition for an idempotent semigroup whose every kernel is left (right) singular.

Before going into the theorem, we need the following definitions.

An idempotent semigroup S is called (1) *left regular*, (2) *right regular*, (3) *regular*, if it satisfies the following corresponding identities:

(1) $$aba=ab,$$
(2) $$aba=ba,$$
(3) $$abaca=abca.$$

REMARK. We can replace idempotency of S by the weaker condition $S^2=S$ in the definition of left (right) regularity above.

THEOREM 1. *An idempotent semigroup S is left (right) regular if and only if every kernel of S is left (right) singular.*

PROOF. Let $S\sim\sum\{S_\gamma:\gamma\in\Gamma\}$ be the structure decomposition of an idempotent semigroup S.

(i) Let S be left regular. Let a, b be any arbitrary elements of S_γ. Then $aba=a$ by the rectangularity of S_γ. On the other hand $aba=ab$ by the left regularity of S. Thus $ab=a$. This proves that S_γ is left singular.

(ii) Suppose each γ-kernel S_γ is left singular.

Let $a\in S_\alpha$, $b\in S_\beta$. Then both ab and ba belong to $S_{\alpha\beta}$. Thus we have

$$aba=ab^2a=(ab)(ba)=ab,$$

by the left singularity of $S_{\alpha\beta}$. This proves that S is left regular.

The following lemmas are straightforward from the definitions.

LEMMA 1. A left (right) regular idempotent semigroup is regular.

LEMMA 2. The direct product of (left, right) regular idempotent semigroups is also (left, right) regular.

LEMMA 3. A subsemigroup of a (left, right) regular idempotent semigroup is also (left, right) regular.

§3. Let A, B be idempotent semigroups both of which have the same structure semilattice Γ. Let $\varphi:A\to\Gamma$, $\psi:B\to\Gamma$ be their canonical homomorphisms. Then

$$P=\{(x, y): x\in A,\ y\in B,\ \varphi(x)=\psi(y)\}$$

forms a subsemigroup of an idempotent semigroup $A\times B$, the direct product of A and B. We call P the *spined product* (Kimura [2]) of A and B with respect to Γ. Note that this product depends not only on A and B and Γ but also on the canonical homomorphisms φ and ψ.

Then by the preceding lemmas we have the following

LEMMA 4. The spined product of a left regular idempotent semigroup and a right regular idempotent semigroup is regular.

The converse of this lemma also holds, namely.

LEMMA 5. Let $S\sim\sum\{S_\gamma:\gamma\in\Gamma\}$ be a regular idempotent semigroup. Then there exist a left regular idempotent semigroup $A\sim\sum\{A_\gamma:\gamma\in\Gamma\}$ and a right regular idempotent semigroup $B\sim\sum\{B_\gamma:\gamma\in\Gamma\}$, both of which have the same structure semilattice Γ, such that S is isomorphic to the spined product of A and B with respect to Γ.

PROOF. Let $S\sim\sum\{S_\gamma:\gamma\in\Gamma\}$ be a regular idempotent semigroup. Since each γ-kernel S_γ is rectangular, we can assume that

$$S_\gamma = A_\gamma\times B_\gamma,$$

where A_γ is left singular and B_γ is right singular.

Define $A=\bigcup\{A_\gamma:\gamma\in\Gamma\}$, $B=\bigcup\{B_\gamma:\gamma\in\Gamma\}$.

We shall prove that A and B can be considered as idempotent semigroups.

Let (a, b), $(a, b')\in S_\alpha$, (c, d), $(c, d')\in S_\beta$.

Put $(a, b)(c, d)=(e, f)$, $(a, b')(c, d')=(e', f')$.

Then both (e, f) and (e', f') belong to $S_{\alpha\beta}$.

Since $S_{\alpha\beta}$ is rectangular, we have

$$(e, f)(e', f') = (e, f').$$

On the other hand we have

$$\begin{aligned}
(e, f)(e', f') &= (a, b)(c, d)(a, b')(c, d')\\
&= (a, b'b)(c, d'd)(a, bb')(c, d') \quad \text{(by right singularity of } B_\alpha \text{ and } B_\beta)\\
&= (a, b')(a, b)(c, d')(c, d)(a, b)(a, b')(c, d')\\
&= (a, b')(a, b)(a, b')(c, d')(a, b)(c, d)(a, b)(a, b')(c, d')\\
&\qquad \text{(by repeated use of regularity)}\\
&= (a, b'bb')(c, d')(a, b)(c, d)(a, bb')(c, d')\\
&= (a, b')(c, d')(a, b)(c, d)(a, b')(c, d')\\
&= (e', f')(e, f)(e', f') \qquad \text{(by definition)}\\
&= (e', f'). \qquad \text{(by rectangularity of } S_{\alpha\beta}).
\end{aligned}$$

Hence $(e, f')=(e', f')$ or $e=e'$.

Thus e is determined by a and c only and does not depend on x or y. Similarly, f is also determined by b and d only. Now we can define $e=p(a, c)$ and $f=q(b, d)$ by

$$(e, f)=(a, b)(c, d)=(p(a, c), q(b, d)).$$

Then $p: A\times A\rightarrow A$ and $q: B\times B\rightarrow B$ defines multiplications on A and on B. The projections

$\pi: S\rightarrow A$ defined by $\pi(a, b)=a$

and $\rho: S\rightarrow B$ defined by $\rho(a, b)=b$

give rise to homomorphisms from S onto A and B, respectively. Thus as images of homomorphisms π and ρ of an idempotent semigroup, both A and B must be idempotent semigroups.

Now it is easy to see that A_γ is left singular and B_γ is right singular for each $\gamma\in\Gamma$, so that the multiplication of A restricted on A_γ is the same to the multiplication of A_γ defined on the original semigroup. And that $A\sim\sum\{A_\gamma:\gamma\in\Gamma\}$ and $B\sim\sum\{B_\gamma:\gamma\in\Gamma\}$ are also straightforward.

Thus S can be considered as a subsemigroup of $A\times B$ identifying the γ-kernel S_γ of S with the subset $A_\gamma\times B_\gamma$ of $A\times B$. This proves that S is isomorphic to the spined product of A and B with respect to Γ.

Thus obtained A and B are called the left and the right part of S, respectively.

Combining Lemmas 4 and 5, we have the following

THEOREM 2. *An idempotent semigroup is regular if and only if it is the spined product of a left regular idempotent semigroup* *and a right regular idempotent semigroup.*

Furthermore this spined product decomposition is unique up to isomorphism. Thus we have the following

COROLLARY. *Regular idempotent semigroups are isomorphic if and only if their left parts are isomorphic and their right parts are isomorphic by isomorphisms which keep the structure semilattice invariant.*

References

[1] McLean, David: Idempotent semigroups, Amer. Math. Monthly, **61**, 110–113 (1954).
[2] Kimura, Naoki: On Semigroups, Dissertation, Tulane University (1957).