146. *On the Structure of Fourier Hyperfunctions**)

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(Comm. by Kôsaku YOSIDA, M. J. A., Nov. 13, 1972)

We show below a complete analogue of the following structure theorem for the temperate distributions: Every element $u \in \mathcal{S}'$ can be expressed in the form $u=(1-\Delta)^N f$, where f is a temperate continuous function. Thus Corollary 1.13 in [3] is improved, and Remark 1.15 there should be cut away. We refer to [3] for the terminology employed here.

Theorem. *For every Fourier hyperfunction $u \in Q$ we can find an elliptic local operator $J(D)$ and an infinitely differentiable function $f(x)$ of infra-exponential growth satisfying $u=J(D)f$.*

By the word "infra-exponential" we mean the following type of estimate:

$$|f(x)| \leq C_\varepsilon \exp(\varepsilon |x|), \quad {}^\forall \varepsilon > 0, \quad {}^\exists C_\varepsilon > 0.$$

Note that a continuous function of infra-exponential growth is "temperate" in the sense of hyperfunction theory. Especially it can be considered as a Fourier hyperfunction in a standard way.

Now let us say that a continuous function $\psi(r) \geq 0$ of one variable $r \geq 0$ is infra-linear if it satisfies the estimate

$$\psi(r) \leq \varepsilon r + C_\varepsilon, \quad {}^\forall \varepsilon > 0, \quad {}^\exists C_\varepsilon > 0.$$

Before the proof of our theorem we prepare

Lemma. *Let $\psi_k(r)$, $k=1,2,\cdots$, be a sequence of infra-linear functions. Then we can find an infra-linear function $\psi(r)$ and a sequence of constants C_k, $k=1,2,\cdots$, satisfying*

$$(1) \qquad \psi_k(r) \leq \psi(r) + C_k.$$

Proof. Approximating the graphs of $\psi_k(r)$ by polygons from above, and smoothing the corners, we can assume that $\psi_k(r)$ are monotone increasing, concave and differentiable. Further, replacing $\psi_k(r)$ by $\sum_{j=1}^k \psi_j(r)$ if necessary, we can assume that $\psi_k(r) \leq \psi_l(r)$ and $\psi_k'(r) \leq \psi_l'(r)$ for $k \leq l$.

Now choose a_k by the following induction process:

$$(2) \qquad \psi_k'(a_k) \leq \frac{1}{k},$$

$$(3) \qquad \frac{\psi_k(a_k)-\psi_k(a_{k-1})}{a_k - a_{k-1}} \leq \frac{1}{k}.$$

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*) Partially supported by Fûjukai.

Since $\psi_k(r)$ are infra-linear, this induction really proceeds. Now define $\psi(r)$ by

$$\psi(r)=\begin{cases}\psi_1(r), & r\leq a_1,\\ \psi_k(r)+\sum_{j=1}^{k-1}(\psi_j(a_j)-\psi_{j+1}(a_j)), & a_{k-1}\leq r\leq a_k.\end{cases}$$

Then $\psi(r)$ is continuous. We see easily from (2), (3) that $\psi(r)$ is infra-linear. By the normalization of $\psi_k(r)$ made at the beginning, we also see easily that we can choose C_k satisfying (1). q.e.d.

Proof of the theorem. Let $V(\zeta)$ be a defining function of the Fourier transform $\tilde{u}(\xi)\in Q$ of $u(x)$. $V(\zeta)$ is holomorphic in $\boldsymbol{C}^n \# \boldsymbol{R}^n$ and satisfies the estimate

$$|V(\zeta)|\leq C_{k,\varepsilon}\exp(\varepsilon|\zeta|),\quad \forall\varepsilon>0,\quad \exists C_{k,\varepsilon}>0,$$
$$\text{for } 1\geq|\operatorname{Im}\zeta_j|\geq\frac{1}{k},\ j=1,\cdots,n.$$

As in the proof of Lemma 1.1 in [3], we can find monotone increasing, positive valued, continuous functions $\varphi_k(r)\nearrow\infty$ satisfying

$$|V(\zeta)|\leq C_k\exp(|\zeta|/\varphi_k(|\zeta|)),\qquad \text{for } 1\geq|\operatorname{Im}\zeta_j|\geq\frac{1}{k},\ j=1,\cdots,n.$$

Put $\psi_k(r)=r/\varphi_k(r)$. Then $\psi_k(r)$ are infra-linear. Applying the above lemma we can find an infra-linear function $\psi(r)$ and constants C'_k so that

$$(4)\qquad |V(\zeta)|\leq C'_k\exp(\psi(|\zeta|)),\qquad \text{for } 1\geq|\operatorname{Im}\zeta_j|\geq\frac{1}{k},\ j=1,\cdots,n,$$

holds. In the same way as at the beginning of the proof of the lemma, we can assume that $\psi(r)$ is positive valued, concave and differentiable. Hence we can assume that $r/\psi(r)$ is monotone increasing to infinity. In fact, we have

$$(r/\psi(r))'=(\psi(r)-r\psi'(r))/\psi(r)^2$$

and

$$\psi(r)-r\psi'(r)|_{r=0}=\psi(0)\geq 0,$$
$$(\psi(r)-r\psi'(r))'=-r\psi''(r)\geq 0.$$

Put $\varphi(r)=\min(r/\psi(r),\sqrt{r})$. Then we have obtained

$$(5)\qquad |V(\zeta)|\leq C'_k\exp(|\zeta|/\varphi(|\zeta|)),\qquad \text{for } 1\geq|\operatorname{Im}\zeta_j|\geq\frac{1}{k},\ j=1,\cdots,n.$$

Now, by Lemma 1.2 in [3] we can choose an elliptic local operator $J(D)$ whose Fourier transform $J(\zeta)$ satisfies

$$(6)\qquad |J(\zeta)|\geq\exp(|\zeta|/\varphi(|\zeta|)),\qquad \text{for } |\operatorname{Im}\zeta_j|\leq 1,\ j=1,\cdots,n.$$

Put $G(\zeta)=V(\zeta)/J(\zeta)^2$. $G(\zeta)$ is holomorphic in $\{\zeta\in\boldsymbol{C}^n;\, 0<\operatorname{Im}\zeta_j\leq 1,\ j=1,\cdots,n\}$ and defines a Fourier hyperfunction $\tilde{f}(\xi)$. From (5), (6) we have

$$(7)\qquad |G(\zeta)|\leq C'_k\exp(-\sqrt{|\zeta|}),\qquad \text{for } 1\geq|\operatorname{Im}\zeta_j|\geq\frac{1}{k},\ j=1,\cdots,n.$$

Let $f(x)$ be the inverse Fourier transform of $\tilde{f}(\xi)$. Then we have

$$u = J(D)^2 f.$$

We will show that f is infinitely differentiable and infra-exponential. In fact, we can calculate the defining function $F(x+iy)$ of f from that of $\tilde{f}$, along the path $\{(\xi_1 \pm i/k, \cdots, \xi_n \pm i/k)\,;\, \xi_j \in \boldsymbol{R}\}$, in the following way

$$\begin{aligned} F(x+iy) = \operatorname{sgn} y_1 \cdots \operatorname{sgn} y_n \sum_{\sigma} \sigma_1 \cdots \sigma_n \int_{-\infty \operatorname{sgn} y_1}^{0} \cdots \int_{-\infty \operatorname{sgn} y_n}^{0} \\ \cdot \exp\left\{-i\left(x\cdot\xi - \frac{1}{k} y\cdot\sigma\right) + \frac{1}{k} x\cdot\sigma + y\cdot\xi\right\} \\ \cdot G\left(\xi + i\frac{1}{k}\sigma\right) d\xi_1 \cdots d\xi_n, \end{aligned} \tag{8}$$

where $\sigma = (\sigma_1, \cdots, \sigma_n)$, $\sigma_j = \pm 1$ presents the sign. The estimate (7) shows that every derivative of F (of finite order) converges locally uniformly when we let $y_j \to 0$. Thus the boundary value $f(x)$ defined by $F(z)$ is infinitely differentiable (in fact even in some Gevrey class). Further, estimating the integral (8), we have

$$|f(x)| \le C_k'' \exp\left(\frac{1}{k}|x|\right).$$

Since k is arbitrary, we have proved our theorem.

References*)

[1] Kaneko, A.: On the structure of hyperfunctions with compact supports. Proc. Japan Acad., **47**, 956–959 (1971) (Supplement II).

[2] ——: A new characterization of real analytic functions. Proc. Japan Acad., **47**, 774–775 (1971).

[3] ——: Representation of hyperfunctions by measures and some of its applications. J. Fac. Sci. Univ. Tokyo, Sec. 1A, **19** (3) (1972) (to appear).

*) The above numbering of the references corresponds to the original order of my papers concerning these subjects. The forthcoming paper [3] contains all the results of [1], [2] and refines some of them. This paper follows [3].