74. A Geometrical Proof of a Theorem on the Secular Equation.

By Masaji ITÔ.

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The well-known fact that the secular equation

$$|A-\lambda E|=|a_{ik}-\lambda\delta_{ik}|=0, \qquad (a_{ik}=a_{ki})$$

has real roots only, may geometrically be interpreted as follows.

A plane determined by n points

$$\begin{array}{llllll} P_1: & (a_{11}-\lambda, & a_{12}, & \ldots\ldots, & a_{1n} &), \\ P_2: & (a_{21}, & a_{22}-\lambda, & \ldots\ldots, & a_{2n} &), \\ \multicolumn{6}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ P_n: & (a_{n1}, & a_{n2}, & \ldots\ldots, & a_{nn}-\lambda &), \end{array}$$

passes through the origin n times, when the parameter λ varies from $-\infty$ to $+\infty$ continuously.

We will prove this theorem by simple geometrical consideration.

Let l_i be the straight line, parallel to the coordinate axis x_i, passing through $(a_{i1}, a_{i2}, \ldots\ldots, a_{in})$, along which the point P_i moves from $+\infty$ to $-\infty$, when λ varies from $-\infty$ to $+\infty$.

First consider the case, where $l_1, l_2, \ldots\ldots, l_n$ meet in a point. We transform then $l_1, l_2, \ldots\ldots$ to the coordinates axes and the origin to a point P, lying in the region, where all coordinates are of the same sign.

For the sake of simplicity, we take $n=3$.

Let Q be the orthogonal projection of P on the x_1x_2 plane, and S be the intersection of the x_3 axis with the join of P, R, where R denotes the intersection of P_1P_2 and OQ.

When the plane $P_1P_2P_3$ passes through P, S will coincide with P_3.

When λ is negative and $|\lambda|$ is sufficiently large, P_3 lies on the positive x_3 axis far from the origin, while S lies very near to O.

When λ increases gradually, P_3 moves towards O, while S towards $+\infty$.

Therefore there comes a moment, where S coincides with P_3. After that moment, S moves further and comes on the negative side on the x_3 axis, passing through infinity, when R passes through Q, that is, P_1P_2 passes through Q. Q lies in the region on the x_1x_2 plane, where the coordinates are of the same sign. Therefore, if we can prove that P_1P_2 passes through Q twice, then S passes through infinity twice, so that S coincides with P_3 three times in all.

Thus the problem is reduced to the case $n=2$.

It is evident that P_1P_2 passes through Q once, when each of P_1, P_2 moves from $+\infty$ towards O, if Q lies in the first quadrant. After this moment, P_1P_2 will coincide with x_2 axis and then with x_1 axis, or in the inverse order, according as P_1 passes through the origin before or after P_2. Therefore P_1P_2 passes through Q once more.

Thus the theorem is proved.

The general case will be proved by mathematical induction, quite similarly to the above reasoning.

We will next turn to the case, where $l_1, l_2, \ldots\ldots, l_n$ do not meet in one point.

Without any loss of generality we can assume $a_{12}, a_{13}, \ldots\ldots, a_{1n} > 0$. Let l_i be the straight line, along which P_i moves from $+\infty$ to $-\infty$. When $n=2m$, the plane passing through $l_2, l_3, \ldots\ldots, l_m$ and O, and further a point T (corresponding to $\lambda=\lambda_0$) on l_{m+1} will meet the line l_1 at a point A, corresponding to the value of λ, which satisfies

$$\begin{vmatrix} a_{11}-\lambda & a_{1,m+1} & \ldots\ldots, & a_{1n} \\ a_{21} & a_{1,m+1} & \ldots\ldots, & a_{2n} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{m+1,1} & a_{m+1,m+1}-\lambda_0 & \ldots\ldots, & a_{m+1,n} \end{vmatrix} = 0. \qquad (1)$$

Again, the plane passing through $l_{m+2}, \ldots\ldots, l_n$ and O, T will meet the line l_1 at a point B, corresponding to the value of λ satisfying

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots\ldots, & a_{1,m+1} \\ a_{m+1,1} & a_{m+1,2} & \ldots\ldots, & a_{m+1,m+1}-\lambda_0 \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{n1} & a_{n2} & \ldots\ldots, & a_{n,m+1} \end{vmatrix} = 0. \qquad (2)$$

Since $a_{ik}=a_{ki}$, (1) and (2) are the same, so that A coincides with B. If we determine λ_0 such that T lies on the plane passing through l_1 and O, then A will be uniquely determined.

When $n=2m+1$, it is easily verified, that two planes passing through the origin and $l_2, l_3, \ldots\ldots, l_{m+1}$; $l_{m+2}, \ldots\ldots, l_n$ respectively will meet l_1 at the same point. Let this point be A.

Then draw l_i' parallel to l_i, passing through A, and denote by P_i' the intersection of l_i' with OP_i. Then $P_1, P_2', \ldots\ldots, P_n'$ move along $l_1, l_2', \ldots\ldots, l_n'$, which meet in the point A. And two planes $P_1P_2\ldots\ldots P_n$, $P_1P_2'\ldots\ldots P_n'$ pass through O at the same time.

If the x_1 coordinate of A be negative, then P_2, P_3, ... will move on l_2', l_3', ... in the negative sense. Therefore, inverting the direction of l_1, A lies in the region where all coordinates are of the same sign. Therefore the plane $P_1P_2'P_3'......P_n'$, consequently $P_1P_2......P_n$, passes through O exactly n times.

Thus the reality of roots of the secular equation is established.

The Sylvester's theorem, which asserts the reality of roots of the equation

$$|A-\lambda B|=0,$$

where A, B are symmetric, and A or B is definite, can also be proved geometrically; we will publish the proof in another occasion.