59. A Relation between the Theories of Fourier Series and Fourier Transforms.

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1. Let $f(x)$ be defined in $(-\infty, \infty)$ and belong to some class $L_p(p \geqq 1)$. If there exists a function $F(t)$ such that

$$\lim_{A\to\infty}\int_{-\infty}^{\infty}\left|F(t)-\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)e^{-itx}dx\right|^q dt=0\,,$$

then $F(t)$ is called the Fourier transform of $f(x)$ in L_q. The Titchmarsh theory states that if $f(x)\in L_p$ $(1\leqq p\leqq 2)$, then $f(x)$ has the Fourier transform $F(t)$ in $L_{p'}$ where $1/p+1/p'=1$.

Let $\varphi(x)$ be a periodic function with period $2R\,(R>0)$ and belong to $L_p(-R, R)$ and consider its Fourier series

$$\varphi(x)\sim\sum_{-\infty}^{\infty}c_n e^{-\frac{in\pi}{R}x}\,,\qquad c_n=\frac{1}{2R}\int_{-R}^{R}\varphi(x)e^{-\frac{in\pi}{R}x}dx\,.$$

It is well known that there exist close analogies between the Fourier transforms and Fourier series. The Fourier coefficient c_n corresponds to the Fourier transform. For example the convergence of $\sum|c_n|^a$ stands for the integrability of $|F(t)|^a$ in $(-\infty, \infty)$. Thus the analogy of Hausdorff-Young theorem on Fourier series is Titchmarsh theorem on Fourier transform which asserts that $\int_{-\infty}^{\infty}|F(t)|^{p'}dt<\infty$, if $1<p\leqq 2$.[1]

In this paper I shall prove theorems which make the analogies of this type clearer. The case where $F(t)$ is the Fourier-Stieltjes transform of a probability distribution was discussed recently by the author.[2]

2. *Theorem 1. Suppose that* $f(x)\in L_p(-\infty, \infty)$ $(p>1)$ *and has the Fourier transform* $F(t)$ *in* $L_q(-\infty, \infty)$ *for some* $q\,(\geqq 1)$. *We define a periodic function* $\varphi(t)$ *with period* $2R$ *which concides with* $F(t)$ *in* $(-R, R)$. *If* c_n *is the Fourier coefficient of* $\varphi(t)$, *then*

$$\sum_{n=-\infty}^{\infty}|c_n|^p\leqq\frac{A_p}{R^{p-1}}\int_{-\infty}^{\infty}|f(x)|^p dx\,, \tag{2.1}$$

where A_p *is a constant depending only on* p *and not of* $f(x)$ *and* R.

Theorem 2. Let $\varphi(t)\in L_1(-R, R)$ *and its Fourier series be*

$$\varphi(t)\sim\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi}{R}t}\,.$$

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1) E. C. Titchmarsh, A contribution to the theory of Fourier transforms, Proc. London Math. Soc., **23** (1924), 279–289.

A. Zygnumed, Trigonometrical series, Warszawa, 1935. p. 316.

2) T. Kawata, The Fourier series of the characteristic function of a probability distribution, Tohoku Math. Journ. **47** (1940).

Suppose that

$$\sum_{n=-\infty}^{\infty} |c_n|^p < \infty\,, \qquad (p>1). \tag{2.2}$$

We define a function $F(t)$ *as follows:*

$$F(t)=\varphi(t)\,, \qquad -R<t<R\,,$$
$$=0 \quad , \qquad |t| \geqq R\,,$$

and let its Fourier transform be $f(x)$ (*which clearly exists in* L_∞ *since* $F(t)\in L_1(-\infty,\infty)$). *Then we have*

$$\int_{-\infty}^{\infty} |f(x)|^p dx \leqq A_p \cdot R^{p-1} \sum_{n=-\infty}^{\infty} |c_n|^p\,, \tag{2.3}$$

where A_p *is a constant depending only on* p *and not of* $F(t)$ *and* R.

Theoeem 1 and 2 can be stated in the following form:

Theorem 3. If c_n *is the Fourier coefficient of an integrable perodic function* $F(t)$ *with period* $2R$, *then the series* $\sum |c_n|^p$ *converges if and only if* $F(t)$ *coincides almost everywhere in* $(-R,R)$ *with a Fourier transform of a function* $f(x)$ *in* $L_p(-\infty,\infty)$ *where* $p>1$. *Further we can choose* $f(x)$ *such that*

$$R^{p-1}\sum_{n=-\infty}^{\infty} |c_n|^p \leqq A_p \int_{-\infty}^{\infty} |f(x)|^p dx \leqq B_p R^{p-1} \sum_{n=-\infty}^{\infty} |c_n|^p\,, \tag{2.4}$$

A_p, B_p *being constants depending only on* p.

3. We can prove these theorems reducing to another theorem of Titchmarsh which is a discrete analogue of a well known theorem on conjugate function.[1]

Theorem A. (Titchmarsh) *Let* $\sum_{n=-\infty}^{\infty} |a_n|^p < \infty$, $(p>1)$ *and put*

$$b_m = \sum_{n=-\infty}^{\infty} \frac{a_n}{m+n+\frac{1}{2}}$$

which is obviously convergent by Hölder inequality. Then we have

$$\sum_{m=-\infty}^{\infty} |b_m|^p \leqq A_p \sum_{n=-\infty}^{\infty} |a_n|^p\,, \tag{3.1}$$

A_p *being a constant depending only on* p.

We first prove Theorem 1. We have

$$c_n = \frac{1}{2R}\int_{-R}^{R} \varphi(u) e^{-i\frac{n\pi}{R}u} du$$
$$= \frac{1}{2R}\int_{-R}^{R} F(u) e^{-\frac{in\pi}{R}u} du$$
$$= \frac{1}{2R}\int_{-R}^{R} e^{-\frac{in\pi}{R}u} du \ \underset{T\to\infty}{\text{l. i. m.}} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} f(t) e^{-itu} dt\,,$$

1) Titchmarsh, Reciprocal formulae for series and integrals, Math. Zeits., **25** (1926), 321–347.

where l. i. m. means the limit in the mean with index q,

$$=\frac{1}{\sqrt{2\pi}\cdot 2R}\int_{-\infty}^{\infty}f(t)\,dt\int_{-R}^{R}e^{-i\left(\frac{n\pi}{R}+t\right)u}\,du$$

$$=\frac{1}{\sqrt{2\pi}\cdot R}\int_{-\infty}^{\infty}\frac{\sin(n\pi+tR)}{n\pi/R+t}f(t)\,dt$$

$$=\frac{(-1)^n}{\sqrt{2\pi}\cdot R}\int_{-\infty}^{\infty}\frac{\sin \pi t}{n+t}\psi(t)\,dt=\frac{(-1)^n}{\sqrt{2\pi}\cdot R}d_n\,,$$

say, where $\psi(t)=f\left(\frac{\pi}{R}t\right)$.

We divide d_n in three parts:

$$d_n=\int_{-n+1}^{\infty}+\int_{-\infty}^{-n-1}+\int_{-n-1}^{-n+1}=I_{n,1}+I_{n,2}+I_{n,3}\,,$$

say. We have

$$I_{n,1}=\sum_{k=-n+1}^{\infty}\int_k^{k+1}\frac{\psi(t)\sin t\pi}{t+n}\,dt \tag{3.2}$$

$$=\sum_{k=-n+1}^{\infty}\frac{1}{k+n+\frac{1}{2}}\int_k^{k+1}\psi(t)\sin t\pi\,dt$$

$$+\sum_{k=-n+1}^{\infty}\int_k^{k+1}\frac{\psi(t)\sin t\pi\cdot\left(k+\frac{1}{2}-t\right)}{(t+n)\left(k+n+\frac{1}{2}\right)}dt$$

$$=\sum_{k=-n+1}^{\infty}\frac{a_k}{k+n+\frac{1}{2}}+I'_{n,1}\,,$$

say, where a_k denotes $\int_k^{k+1}\psi(t)\sin t\pi\,dt$. We have

$$|I'_{n,1}|\leqq\sum_{k=-n+1}^{\infty}\frac{1}{(k+n)^2}\int_k^{k+1}|\psi(t)|\,dt=\sum_{k=-n+1}^{\infty}\frac{b_k}{(k+n)^2}\,, \tag{3.3}$$

say. Similarly we have

$$I_{n,2}=\sum_{k=-\infty}^{-n-2}\int_k^{k+1}\frac{\psi(t)\sin t\pi}{t+n}\,dt \tag{3.4}$$

$$=\sum_{k=-\infty}^{-n-2}\frac{a_k}{k+n+\frac{1}{2}}+I'_{n,2}\,,$$

where

$$|I'_{n,2}|\leqq\sum_{k=-\infty}^{-n-2}\frac{b_k}{(k+n)^2}\,. \tag{3.5}$$

From (3.2) (3.3), (3.4) and (3.5) we have

$$d_n=\sum_{k=-\infty}^{\infty}\frac{a_k}{k+n+\frac{1}{2}}+I'_{n,1}+I'_{n,2}+I_{n,3}+2a_{n-1}-2a_n\,,$$

$$\sum_{n=-\infty}^{\infty}|d_n|^p\leqq A_p\sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty}\frac{a_k}{k+n+\frac{1}{2}}\right|^p \tag{3.6}$$

$$+A_p\sum_{n=-\infty}^{\infty}\Big(\sum_{k=-n+1}^{\infty}\frac{b_k}{(k+n)^2}\Big)^p$$

$$+A_p\sum_{n=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{-n-1}\frac{b_k}{(k+n)^2}\Big)^p+A_p\sum_{n=-\infty}^{\infty}|I_{n,3}|^p+A_p\sum_{n=-\infty}^{\infty}|a_n|^p.$$

The first term of the right hand side of (3.6) does not exceed, by Theorem A,

$$A_p\sum_{n=-\infty}^{\infty}|a_n|^p\leqq A_p\sum_{n=-\infty}^{\infty}\left|\int_n^{n+1}\psi(t)\sin t\pi\,dt\right|^p \tag{3.7}$$

$$\leqq A_p\sum_{n=-\infty}^{\infty}\int_n^{n+1}|\psi(t)|^p\,dt=A_p\int_{-\infty}^{\infty}|\psi(t)|^p dt=A_p\cdot R\int_{-\infty}^{\infty}|f(t)|^p dt.$$

The last term is evidently

$$\leqq A_pR\int_{-\infty}^{\infty}|f(t)|^p\,dt\,. \tag{3.8}$$

Also

$$\sum_{n=-\infty}^{\infty}|I_{n,3}|^p=\sum_{n=-\infty}^{\infty}\left|\int_{-n-1}^{-n+1}\frac{\psi(t)\sin t\pi}{t+n}\,dt\right|^p \tag{3.9}$$

$$\leqq\sum_{n=-\infty}^{\infty}\Big(\int_{-n-1}^{-n+1}|\psi(t)|\,dt\Big)^p\leqq\sum_{n=-\infty}^{\infty}\int_{-n-1}^{-n+1}|\psi(t)|^p dt$$

$$\leqq 2R\int_{-\infty}^{\infty}|f(t)|^p dt\,.$$

Next we treat the second and third terms of the right hand side of (3.6). We have

$$\sum_{n=-\infty}^{\infty}\Big(\sum_{k=-n+1}^{\infty}\frac{b_k}{(k+n)^2}\Big)^p=\sum_{n=-\infty}^{\infty}\Big(\sum_{m=1}^{\infty}\frac{b_{m-n}}{m^2}\Big)^p.$$

Now $\{l_n\}$ is any sequence of positive numbers. By Hölder's inequality, we have

$$\sum_{n=-\infty}^{\infty}l_n\sum_{m=1}^{\infty}\frac{b_{m-n}}{m^2}=\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=-\infty}^{\infty}l_nb_{m-n}$$

$$\leqq\sum_{m=1}^{\infty}\frac{1}{m^2}\Big(\sum_{n=-\infty}^{\infty}l_n^q\Big)^{\frac{1}{q}}\Big(\sum_{n=-\infty}^{\infty}b_{m-n}^p\Big)^{\frac{1}{p}}$$

$$\leqq A_p\Big(\sum_{n=-\infty}^{\infty}l_n^q\Big)^{\frac{1}{q}}\Big(\sum_{n=-\infty}^{\infty}b_n^p\Big)^{\frac{1}{p}},$$

where $1/p+1/q=1$. If we take $l_n=\Big(\sum_{m=1}^{\infty}\frac{b_{m-n}}{m^2}\Big)^{p-1}$, then we get

$$\sum_{n=-\infty}^{\infty}\left(\sum_{n=1}^{\infty}\frac{b_{m-n}}{m^2}\right)^p \leqq A_p\left(\sum_{n=-\infty}^{\infty}\left(\sum_{m=1}^{\infty}\frac{b_{m-n}}{m^2}\right)^p\right)^{\frac{1}{p}}\left(\sum_{n=-\infty}^{\infty}b_n^p\right)^{\frac{1}{p}}.$$

Thus we get

$$\sum_{n=-\infty}^{\infty}\left(\sum_{m=1}^{\infty}\frac{b_{m-n}}{m^2}\right)^p \leqq A_p\sum_{n=-\infty}^{\infty}b_n^p.$$

Hence the second term does not exceed

$$A_p\sum_{n=-\infty}^{\infty}b_n^p \leqq A_p\int_{-\infty}^{\infty}|\psi(t)|^p dt = A_p\cdot R\int_{-\infty}^{\infty}|f(t)|^p dt. \tag{3.10}$$

The similar inequality holds for the third term.

Above estimations and (3.6) show

$$\sum_{n=-\infty}^{\infty}|d_n|^p \leqq A_pR\int_{-\infty}^{\infty}|f(t)|^p dt, \tag{3.11}$$

which is equivalent to (2.1).

4. The proof of Theorem 2 can be done by the similar method.

$$\begin{aligned} f(-x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(t)e^{ixt}dt = \frac{1}{\sqrt{2\pi}}\int_{-R}^{R}\varphi(t)e^{ixt}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-R}^{R}e^{ixt}\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi}{R}t}dt \\ &= \sqrt{\frac{2}{\pi}}\sum_{n=-\infty}^{\infty}\frac{\sin\left(x+\frac{n\pi}{R}\right)R}{x+n\pi/R}, \end{aligned} \tag{4.1}$$

which is, by putting $x=\frac{t\pi}{R}$,

$$\frac{\sqrt{2}\,R}{\pi^{3/2}}\sum_{n=-\infty}^{\infty}c_n(-1)^n\frac{\sin t\pi}{t+n} = \frac{\sqrt{2}\,R}{\pi^{3/2}}D(t), \tag{4.2}$$

say. Now let $k<t\leqq k+1$ and write as

$$\begin{aligned} D(t) = \sum_{n=-\infty}^{\infty}(-1)^n c_n\frac{\sin t\pi}{n+t} &= \sum_{n=-k+1}^{\infty} + \sum_{n=-\infty}^{-k-2} + (-1)^{-k-1}c_{-k-1}\frac{\sin t\pi}{t-k-1} \\ &\quad + c_k(-1)^k\frac{\sin t\pi}{t-k} \\ &= I_{k,1}+I_{k,2}+I_{k,3}+I_{k,4}, \end{aligned}$$

say. We have

$$\begin{aligned} I_{k,1} &= \sin t\pi\sum_{n=-k+1}^{\infty}\frac{(-1)^n c_n}{k+n+\frac{1}{2}} + \sin t\pi\sum_{n=-k+1}^{\infty}(-1)^n c_n\frac{k-t+\frac{1}{2}}{(t+n)\left(k+n+\frac{1}{2}\right)} \\ &= S_k+S_k', \end{aligned}$$

say.

$$I_{k,2}=\sin t\pi\sum_{n=-\infty}^{-k-2}\frac{(-1)^n c_n}{k+n+\frac{1}{2}}+\sin t\pi\sum_{n=-\infty}^{-k-2}(-1)^n c_n\frac{k-t+\frac{1}{2}}{(t+n)\left(k+n+\frac{1}{2}\right)}$$

$$=T_k+T_k',$$

say. Then we have

$$D(t)=\sin t\pi\sum_{n=-\infty}^{\infty}\frac{(-1)^n c_n}{k+n+\frac{1}{2}}+\sin t\pi\left\{\frac{(-1)^{-k-1}c_{-k-1}}{t-k-1}+\frac{(-1)^{-k}c_{-k}}{t-k}\right\}$$

$$+S_k'+T_k'.$$

Thus we have

$$\int_k^{k+1}|D(t)|^p dt\leqq A_p\left|\sum_{n=-\infty}^{\infty}\frac{(-1)^n c_n}{k+n+\frac{1}{2}}\right|^p+A_p(|c_{-k-1}|^p+|c_{-k}|^p)$$

$$+A_p|S_k'|^p+A_p|T_k'|^p.$$

Summing up with respect to k from $-\infty$ to ∞, we have

$$\int_{-\infty}^{\infty}|D(t)|^p dt\leqq A_p\sum_{k=-\infty}^{\infty}\left|\sum_{n=-\infty}^{\infty}\frac{(-1)^n c_n}{k+n+\frac{1}{2}}\right|^p+A_p\sum_{k=-\infty}^{\infty}|c_k|^p$$

$$+A_p\sum_{k=-\infty}^{\infty}\left\{\sum_{n=-k+1}^{\infty}\frac{|c_n|}{(k+n)^2}\right\}^p+A_p\sum_{k=-\infty}^{\infty}\left\{\sum_{n=-\infty}^{-k-2}\frac{|c_n|}{(k+n)^2}\right\}^p.$$

The first, third and fourth terms on the right do not exceed $A_p\sum_{k=-\infty}^{\infty}|c_k|^p$ which is got as in the estimations of terms in (3.6). (4.1) and (4.2) show

$$\int_{-\infty}^{\infty}|f(x)|^p dx\leqq A_p R^{p-1}\int_{-\infty}^{\infty}|D(t)|^p dt$$

$$\leqq A_p R^{p-1}\sum_{n=-\infty}^{\infty}|c_n|^p.$$

5. We show here that Titchmarsh theorem on Fourier transform is an immediate consequence of Hausdorff-Young theorem if Theorem 1 is used. The original proof of Titchmarsh is also to reduce the theorem to Fourier series theorem and his proof is more direct than ours. But our reduction is also of some interest since it clarifies the relation between two theorems.

Theorem B. Let $f(x)\in L_p\ (-\infty,\infty)$, $1<p\leqq 2$ *and its Fourier transform be* $F(t)$. *We have*

$$(5.1)\qquad \left(\int_{-\infty}^{\infty}|F(t)|^q dt\right)^{\frac{1}{q}}\leqq A_p\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{\frac{1}{p}},\qquad \left(\frac{1}{p}+\frac{1}{q}=1\right).$$

Let $\varphi(t)$ be a periodic function with period $2R$ and coincide with $F(t)$ in $(-R,R)$. Then Hausdorff-Young theorem states

$$\left(\frac{1}{2R}\int_{-R}^{R}|\varphi(t)|^q dt\right)^{\frac{1}{q}} \leqq A_p\left(\sum_{n=-\infty}^{\infty}|c_n|^p\right)^{\frac{1}{p}}, \tag{5.2}$$

where c_n represents the Fourier coefficient of $\varphi(t)$. A_p depends only on p and not of R. From this we have

$$\left(\int_{-R}^{R}|F(t)|^q dt\right)^{\frac{1}{q}} \leqq \left(\int_{-R}^{R}|\varphi(t)|^q dt\right)^{\frac{1}{q}}$$

$$\leqq A_p R^{\frac{1}{q}}\left(\sum_{n=-\infty}^{\infty}|c_n|^p\right)^{\frac{1}{p}},$$

which does not exceed by Theorem 1

$$A_p R^{\frac{1}{q}}\left(\frac{1}{R^{p-1}}\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{\frac{1}{p}} = A_p\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{\frac{1}{p}}$$

that is

$$\left(\int_{-R}^{R}|F(t)|^q dt\right)^{\frac{1}{q}} \leqq A_p\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{\frac{1}{p}}.$$

Letting $R \to \infty$, we get the result.