

# Lagrangian submanifolds in para-complex Euclidean space\*

Henri Anciaux      Maikel Antonio Samuays

## Abstract

We address the study of some curvature equations for distinguished submanifolds in para-Kähler geometry. We first observe that a para-complex submanifold of a para-Kähler manifold is minimal. Next we describe the extrinsic geometry of Lagrangian submanifolds in the para-complex Euclidean space  $\mathbb{D}^n$  and discuss a number of examples, such as graphs and normal bundles. We also characterize those Lagrangian surfaces of  $\mathbb{D}^2$  which are minimal and have indefinite metric. Finally we describe those Lagrangian self-similar solutions of the Mean Curvature Flow (with respect to the neutral metric of  $\mathbb{D}^n$ ) which are  $SO(n)$ -equivariant.

## Introduction

Symplectic manifolds enjoy a distinguished class of submanifolds, namely *Lagrangian submanifolds*. They are defined as those submanifolds of maximal dimension (half the dimension of the ambient space) such that the symplectic form vanishes on it. In the Kähler case, it is interesting to study the metric properties of Lagrangian submanifolds. In particular, Lagrangian submanifolds which are in addition *minimal*, i.e., critical points of the volume functional attached to the metric, enjoy interesting properties. For example, in the complex Euclidean space  $\mathbb{C}^n$  (or more generally in a *Calabi-Yau* manifold), a locally defined angle function, the *Lagrangian angle*, is attached to any Lagrangian submanifold, and minimal

---

\*Supported by FAPESP (2012/02724-3)

Received by the editors in October 2015.

Communicated by J. Fine.

2010 *Mathematics Subject Classification* : 53A10, 53D12.

*Key words and phrases* : para-Kähler geometry - Lagrangian submanifolds - Minimal submanifolds, Self-similar solutions to the Mean Curvature Flow.

ones are characterized by the constancy of their Lagrangian angle. Recently, the study of Lagrangian submanifolds in pseudo-Riemannian Kähler manifolds has been addressed (see [An2],[An3]).

Para-complex geometry, an alternative to complex geometry, is the study of manifolds  $\mathcal{M}$  endowed with a  $(1,1)$ -tensor  $J$  satisfying  $J^2 = Id$  (instead of the usual relation  $J^2 = -Id$  characterizing complex geometry), satisfying in addition the rank condition  $\dim(Ker(J - Id)) = \dim(Ker(J + Id)) = \frac{1}{2}\dim \mathcal{M}$ . The model space of para-complex geometry is the Cartesian product  $\mathbb{D}^n$ , where  $\mathbb{D}$  is the module of para-complex numbers (see next section for the precise definition). Using the identification  $\mathbb{D}^n \simeq \mathbb{R}^{2n} \simeq T^*\mathbb{R}$ , we may also consider a natural symplectic structure  $\omega$  on  $\mathbb{D}^n$ . The pair  $(J, \omega)$  then defines a pseudo-riemannian metric by the relation  $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ , which happens to have neutral signature and makes the para-complex structure  $J$  compatible in a suitable sense (see next section for more detail).

This paper is devoted to the study of two classes of submanifolds that appear naturally in para-Kähler geometry, namely para-complex and Lagrangian submanifolds. We first prove that para-complex submanifolds are minimal (like complex submanifolds in Kähler geometry), but unstable (unlike complex submanifolds). Next, we describe the extrinsic geometry of Lagrangian submanifolds and define their Lagrangian angle, whose constancy is equivalent to minimality (like in the Kähler case). This is related to the fact, observed in [Me] (see also [HL2]), that minimal Lagrangian submanifolds of  $\mathbb{D}^n$  enjoy a kind of “Lagrangian calibration” and are therefore extremizers of the volume in their Lagrangian isotopy class (but not in their whole isotopy class). Next, we discuss a number of examples of Lagrangian submanifolds, such as minimal Lagrangian graphs, minimal normal bundles. We also characterize Lagrangian surfaces of  $\mathbb{D}^2$  which are minimal and have indefinite metric. Finally we describe the Lagrangian self-similar solutions of the Mean Curvature Flow which are  $SO(n)$ -equivariant.

The authors wish to thank the referee for her/his numerous remarks which significantly improved the paper.

## 1 Preliminaries

### 1.1 The space $\mathbb{D}^n$

The set  $\mathbb{D}$  of *para-complex* (or *split-complex*, or *double*) numbers is the two-dimensional real vector space  $\mathbb{R}^2$  endowed with the commutative algebra structure whose product rule is given by

$$(x, y)(x', y') = (xx' + yy', xy' + x'y).$$

The number  $(0, 1)$ , whose square is  $(1, 0)$  and not  $(-1, 0)$ , will be denoted by  $\tau$ . It is convenient to use the following notation:  $(x, y) \simeq z = x + \tau y$ . In particular, one has the same conjugation operator than in  $\mathbb{C}$ :

$$\overline{x + \tau y} = x - \tau y.$$

Since  $z\bar{z} = x^2 - y^2$ , it is only natural to introduce the Minkowski metric  $\langle \cdot, \cdot \rangle = dx^2 - dy^2$ , so that the squared norm  $\langle z, z \rangle$  of  $z$  is  $z\bar{z}$ .

We also introduce the *polar coordinates* as follows: the *radius* of  $z \in \mathbb{D}$  is  $r := |\langle z, z \rangle|^{1/2}$ . If  $z$  is non-null, it belongs to one of the two connected components of one of the two hyperbolas  $x^2 - y^2 = \pm r^2$ . Its *argument*  $\arg z$  is defined to be the unique real number  $\theta$  such that

$$z = \pm \tau^q r e^{\tau \theta} = \pm \tau^q r (\cosh(\theta) + \tau \sinh(\theta)), \quad q \in \{0, 1\}.$$

Next we define the “para-Cauchy-Riemann” operators on  $\mathbb{D}$  by

$$\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial y} \right) \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} - \tau \frac{\partial}{\partial y} \right).$$

A map  $f$  defined on a domain of  $\mathbb{D}$  is said to be *para-holomorphic* if it satisfies  $\frac{\partial f}{\partial \bar{z}} = 0$ . Observe that this is a hyperbolic equation, so a para-holomorphic map need not to be analytic, and may be merely continuously differentiable.

On the Cartesian  $n$ -product  $\mathbb{D}^n$  with para-complex coordinates  $(z_1, \dots, z_n)$ , we define the canonical para-Kähler structure  $(J, \langle \cdot, \cdot \rangle)$  by

$$J(z_1, \dots, z_n) := (\tau z_1, \dots, \tau z_n)$$

and

$$\langle \cdot, \cdot \rangle := \sum_{j=1}^n dz_j d\bar{z}_j = \sum_{j=1}^n dx_j^2 - dy_j^2.$$

We also introduce the “para-Hermitian” form:

$$\langle \langle \cdot, \cdot \rangle \rangle := \sum_{j=1}^n dz_j \otimes d\bar{z}_j = \sum_{j=1}^n (dx_j^2 - dy_j^2) - \tau \sum_{j=1}^n dx_j \wedge dy_j.$$

In other words we recover  $\langle \cdot, \cdot \rangle$  by taking the real part of  $\langle \langle \cdot, \cdot \rangle \rangle$ . On the other hand, its imaginary part is (up to sign) the canonical symplectic form  $\omega$  of  $T^*\mathbb{R}^n$  under the natural identification  $\mathbb{D}^n \simeq T^*\mathbb{R}^n$ . Finally, the three structures (metric, para-complex and symplectic) are related by the formula:  $\omega := \langle \cdot, J \cdot \rangle$ .

### 1.2 Para-complex and para-Kähler manifolds

Let  $\mathcal{M}$  be a differentiable manifold. An *almost para-complex structure* on  $\mathcal{M}$  is a  $(1, 1)$ -tensor  $J$  satisfying  $J^2 = Id$  and such that the two eigendistributions  $\text{Ker}(J \mp Id)$  have the same rank. On the other hand, a *para-complex structure* is an atlas on  $\mathcal{M}$  whose transition maps are local bi-para-holomorphic diffeomorphisms of  $\mathbb{D}^n$ . Exactly as in the complex case, a para-complex atlas defines an almost-complex structure  $J$  by the formula  $JX = d\varphi^{-1}(\tilde{J}d\varphi(X))$ , where we denoted by  $\tilde{J}$  the para-complex structure of  $\mathbb{D}^n$  and by  $\varphi$  a local chart on  $\mathcal{M}$ . The question of under which condition the converse is true, hence the para-complex version of Newlander-Nirenberg theorem, is the content of the following

**Theorem 1.** *The almost para-complex structure  $J$  comes from a para-complex structure if and only if its para-Nijenhuis tensor  $N^J$ , defined by*

$$N^J(X, Y) := [X, Y] + [JX, JY] - J[JX, Y] - J[X, JY]$$

*vanishes.*

The proof of this theorem, a consequence of Frobenius theorem, is much simpler than in the complex case. For the sake of completeness we state it in the Appendix.

Unlike the case of complex manifolds, a para-complex manifold is not necessarily orientable: for example, it is easy to equip the Klein bottle with a para-complex structure: let  $\mathcal{M} = \mathbb{R}^2 / \sim$  where  $\sim$  is the equivalence relation defined by  $(u, v) \sim (u + 1, v) \sim (1 - u, v + 1)$ . Then the para-complex structure defined on  $\mathbb{R}^2$  by  $J\partial_u = \partial_u$  and  $J\partial_v = -\partial_v$  descends to  $\mathcal{M}$ .

A pair  $(J, g)$ , where  $J$  is para-complex structure, and  $g$  a compatible pseudo-Riemannian metric, is said to be a *para-Kähler structure* on  $\mathcal{M}$  if the alternating 2-form  $\omega := g(J\cdot, \cdot)$  is symplectic, i.e. it is closed (the non-degeneracy of  $\omega$  follows directly from that of  $g$ ). The simplest, non-flat example of such a para-Kähler manifold may be constructed in an analogous way to the complex projective space:  $\mathbb{D}\mathbb{P}^n$  is the set of two-dimensional subspaces of  $\mathbb{D}^{n+1}$  which are  $J$ -stable and have non-degenerate (hence indefinite) metric. Alternatively,  $\mathbb{D}\mathbb{P}^n$  is the quotient of the quadric  $\mathbb{Q} := \{\langle \cdot, \cdot \rangle = 1\}$  of  $\mathbb{D}^{n+1}$  by the Hopf action  $z \mapsto z \cdot e^{it}$ . In a para-Kähler manifold, exactly as in the Kähler case, the para-complex structure  $J$  is parallel with respect to the Levi-Civita connection of the metric. This fact will be useful in the next section.

## 2 Distinguished submanifolds in para-complex geometry

### 2.1 Para-complex submanifolds

It is well known that complex submanifolds in  $\mathbb{C}^n$  (or more generally in Kähler manifolds) are examples of not only minimal, but even calibrated submanifolds. In the para-complex case, some care must be taken to the definition of a para-complex submanifold: given a submanifold  $\mathcal{S}$  of a para-complex manifold  $(\mathcal{M}, J)$ , the assumption that the tangent bundle of  $\mathcal{S}$  is stable for  $J$  is not sufficient, since the restriction of the eigenspaces  $\text{Ker}(J \mp \text{Id})$  may not have the same rank. Hence a submanifold  $\mathcal{S}$  of  $(\mathcal{M}, J)$  will be called *para-complex* if the restriction of  $J$  to  $T\mathcal{S}$  is still an almost para-complex structure, i.e.  $T\mathcal{S}$  is  $J$ -stable and  $\text{Ker}(J \mp \text{Id}) \cap T\mathcal{S}$  have the same rank. It turns out that if  $\mathcal{M}$  is in addition para-Kähler, its para-complex submanifolds are minimal, but fail to be calibrated, being always unstable with respect to the volume form.

**Theorem 2.** *Let  $(\mathcal{M}, J, g, \omega)$  a para-Kähler manifold and  $\mathcal{S}$  a non-degenerate submanifold of  $\mathcal{M}$  which is para-complex. Then  $\mathcal{S}$  is minimal but unstable.*

*Proof.* Given a tangent vector field  $X$ ,  $JX$  is also tangent to  $\mathcal{S}$ . Moreover, since  $J$  is parallel with respect to the Levi-Civita connection  $D$ , we have  $D_Y JX = JD_Y X$ . On

the other hand, since the tangent bundle is  $J$ -invariant, so are the normal bundle. It follows that

$$(D_Y JX)^\perp = (JD_Y X)^\perp = J(D_Y X)^\perp,$$

i.e.  $h(JX, Y) = Jh(X, Y)$ , where  $h$  denotes the second fundamental form of  $\mathcal{S}$ , i.e. the two-tensor, valued in the normal bundle, defined by  $h(X, Y) = (D_X Y)^\perp$ . Next, an easy modification of the famous Gram-Schmidt process shows that there exists an orthonormal frame  $(e_1, \dots, e_{2k})$  on  $\mathcal{S}$  such that  $e_{2i} = Je_{2i-1}$ ,  $\forall i, 1 \leq i \leq k$  (which proves in particular that the dimension of  $\mathcal{S}$  is even). It follows that  $g(e_{2i}, e_{2i}) = -g(e_{2i-1}, e_{2i-1})$ . Setting  $\epsilon_i := g(e_i, e_i)$ , we deduce:

$$\begin{aligned} 2k\vec{H} &= \sum_{i=1}^{2k} \epsilon_i h(e_i, e_i) = \sum_{i=1}^k \epsilon_{2i-1} h(e_{2i-1}, e_{2i-1}) + \epsilon_{2i} h(e_{2i}, e_{2i}) \\ &= \sum_{i=1}^k \epsilon_{2i-1} \left( h(e_{2i-1}, e_{2i-1}) + h(Je_{2i-1}, Je_{2i-1}) \right) \\ &= \sum_{i=1}^k \epsilon_{2i-1} \left( h(e_{2i-1}, e_{2i-1}) - J^2 h(e_{2i-1}, e_{2i-1}) \right) = 0. \end{aligned}$$

The unstability of  $\mathcal{S}$  comes from the fact that its induced metric is indefinite: if  $X$  is not a null tangent vector,  $JX$  is tangent as well and  $g(JX, JX) = -g(X, X)$ . It has been proved in [An2] that a minimal submanifold with indefinite induced metric is always unstable. ■

## 2.2 Lagrangian submanifolds in $\mathbb{D}^n$

**Lemma 1.** *Let  $\mathcal{L}$  be a Lagrangian submanifold of  $\mathbb{D}^n$  and  $(X_1, \dots, X_n)$  a local tangent frame along  $\mathcal{L}$ . Then the para-complex number  $\det_{\mathbb{D}}(X_1, \dots, X_n)$  is non null if and only if the induced metric on  $\mathcal{L}$  is non-degenerate. Moreover, if  $(X_1, \dots, X_n)$  is orthonormal (so in particular the induced metric on  $\mathcal{L}$  is non-degenerate), then  $\det_{\mathbb{D}}(X_1, \dots, X_n)$  is unit, i.e.*

$$\left| \langle \det_{\mathbb{D}}(X_1, \dots, X_n), \det_{\mathbb{D}}(X_1, \dots, X_n) \rangle \right| = 1.$$

Finally, if the metric on  $\mathcal{L}$  is non-degenerate, the argument of the para-complex number  $\det_{\mathbb{D}}(X_1, \dots, X_n)$  does not depend on the choice of the frame  $(X_1, \dots, X_n)$ , but only on the submanifold  $\mathcal{L}$ .

This lemma allows us to give a satisfactory definition to the concept of *Lagrangian angle*:

**Definition 1.** *Let  $\mathcal{L}$  be a Lagrangian submanifold of  $\mathbb{D}^n$  with non-degenerate induced metric and  $(e_1, \dots, e_n)$  a local orthonormal tangent frame along  $\mathcal{L}$ . Then the para-complex number  $\Omega := \det_{\mathbb{D}}(e_1, \dots, e_n)$  is called para-holomorphic volume of  $\mathcal{L}$ . The argument  $\beta$  of  $\Omega$  is called Lagrangian angle of  $\mathcal{L}$ .*

*Proof of Lemma 1.* We denote by  $(e_1, \dots, e_n)$  the canonical para-Hermitian basis of  $(\mathbb{D}^n, \langle \langle \cdot, \cdot \rangle \rangle)$ . Observe first that given any two vectors  $X$  and  $Y$  of  $\mathbb{D}^n$ , we have

$$X = \sum_{i=1}^n \langle \langle X, e_i \rangle \rangle e_i$$

and

$$\langle\langle X, Y \rangle\rangle = \sum_{k=1}^n \langle\langle X, e_k \rangle\rangle \langle\langle e_k, Y \rangle\rangle = \sum_{k=1}^n \langle\langle X, e_k \rangle\rangle \overline{\langle\langle Y, e_k \rangle\rangle}.$$

Hence, setting  $M := [\langle\langle X_i, e_j \rangle\rangle]_{1 \leq i, j \leq n}$ , the coefficients of the first fundamental form (induced metric) are

$$\begin{aligned} g_{ij} &:= \langle X_i, X_j \rangle \\ &= \langle\langle X_i, X_j \rangle\rangle \\ &= \sum_{k=1}^n \langle\langle X_i, e_k \rangle\rangle \overline{\langle\langle X_j, e_k \rangle\rangle} \end{aligned}$$

(we have used the Lagrangian assumption to get the second equality). It follows that  $[g_{ij}]_{1 \leq i, j \leq n} = MM^*$ , where  $M^*$  denotes the conjugate matrix of  $M$ . It follows that

$$\det_{\mathbb{R}} [g_{ij}]_{1 \leq i, j \leq n} = \det_{\mathbb{D}} M \det_{\mathbb{D}} M^* = \det_{\mathbb{D}} M \overline{\det_{\mathbb{D}} M} = \langle \det_{\mathbb{D}} M, \det_{\mathbb{D}} M \rangle,$$

hence the induced metric is degenerate if and only if the para-complex number  $\det_{\mathbb{D}} M$  is null. Moreover, if the frame  $(X_1, \dots, X_n)$  is orthonormal, the matrix  $[g_{ij}]_{1 \leq i, j \leq n}$  is orthogonal, so that

$$\langle \det_{\mathbb{D}} M, \det_{\mathbb{D}} M \rangle = \det_{\mathbb{R}} [g_{ij}]_{1 \leq i, j \leq n} = \pm 1.$$

In order to conclude the proof it suffices to observe that if  $(Y_1, \dots, Y_n)$  is another local frame along  $\mathcal{L}$ , with  $Y_i = \sum_{j=1}^n a_{ij} X_j$ , we have

$$\det_{\mathbb{D}} (Y_1, \dots, Y_n) = \det_{\mathbb{D}} [a_{ij}]_{1 \leq i, j \leq n} \det_{\mathbb{D}} (X_1, \dots, X_n).$$

Since the coefficients  $a_{ij}$  are real, the two para-complex determinants above have the same argument.  $\blacksquare$

The proof of the next lemma can be found in [An2]:

**Lemma 2.** *Let  $\mathcal{L}$  be a non-degenerate, Lagrangian submanifold of  $\mathbb{D}^n$ . Denote by  $D$  the Levi-Civita connection of the induced metric on  $\mathcal{L}$ . Then the*

$$T(X, Y, Z) := \langle D_X Y, JZ \rangle$$

*is tensorial and tri-symmetric, i.e.*

$$T(X, Y, Z) = T(Y, X, Z) = T(X, Z, Y).$$

**Theorem 3.** *Let  $\mathcal{L}$  be a Lagrangian submanifold of  $\mathbb{D}^n$  with non-degenerate induced metric. Then the Lagrangian angle  $\beta$  and the mean curvature vector  $\vec{H}$  of  $\mathcal{L}$  are related by the formula*

$$n\vec{H} = -J\nabla\beta,$$

*where  $\nabla$  denotes the gradient operator on  $\mathcal{L}$  with respect to the induced metric.*

*Proof.* Let  $(e_1, \dots, e_n)$  a local orthonormal frame of  $\mathcal{L}$ . The Lagrangian assumption implies that it is also a para-Hermitian frame, i.e.

$$[\langle\langle e_j, e_k \rangle\rangle]_{1 \leq j, k \leq n} = \text{diag}(\epsilon_1, \dots, \epsilon_n),$$

where  $\epsilon_j = \pm 1$ . In particular, given any vector  $X$  of  $\mathbb{D}^n$ , we have

$$X = \sum_{j=1}^n \epsilon_j \langle\langle X, e_j \rangle\rangle e_j.$$

It is sufficient to prove that  $\langle n\vec{H}, Je_j \rangle = \langle \nabla \beta, e_j \rangle$ , i.e.  $\langle n\vec{H}, Je_j \rangle = d\beta(e_j)$ . Differentiating the identity  $e^{\tau\beta} = \Omega(e_1, \dots, e_n)$  with respect to the vector  $e_j$ , we have

$$\begin{aligned} \tau d\beta(e_j) e^{\tau\beta} &= \sum_{k=1}^n \Omega(e_1, \dots, D_{e_j} e_k, \dots, e_n) \\ &= \sum_{k=1}^n \Omega \left( e_1, \dots, \sum_{l=1}^n \langle\langle e_l, D_{e_j} e_k \rangle\rangle e_l, \dots, e_n \right) \\ &= \sum_{k=1}^n \Omega(e_1, \dots, \epsilon_k \langle\langle e_k, D_{e_j} e_k \rangle\rangle e_k, \dots, e_n) \\ &= \sum_{k=1}^n \epsilon_k \langle\langle e_k, D_{e_j} e_k \rangle\rangle e^{\tau\beta}, \end{aligned}$$

hence

$$\tau d\beta(e_j) = \sum_{k=1}^n \epsilon_k \langle\langle e_k, D_{e_j} e_k \rangle\rangle.$$

Recalling that  $\langle\langle \cdot, \cdot \rangle\rangle = \langle \cdot, \cdot \rangle - \tau\omega = \langle \cdot, \cdot \rangle - \tau \langle \cdot, J \cdot \rangle$ , we have

$$\langle\langle e_k, D_{e_j} e_k \rangle\rangle = \langle e_k, D_{e_j} e_k \rangle - \tau \langle e_k, J D_{e_j} e_k \rangle.$$

Differentiating the relation  $\langle e_k, e_k \rangle = \epsilon_k$  in the direction  $e_j$  yields  $\langle e_k, D_{e_j} e_k \rangle = 0$ , so that, taking into account that  $Je_j$  is a normal vector to  $\mathcal{L}$ ,

$$\langle\langle e_k, D_{e_j} e_k \rangle\rangle = \tau \langle Je_k, D_{e_j} e_k \rangle = \tau T(e_j, e_k, e_k).$$

By Lemma 2, we deduce that

$$d\beta(e_j) = \sum_{k=1}^n \epsilon_k \langle Je_j, h(e_k, e_k) \rangle = \langle Je_j, n\vec{H} \rangle = -\langle e_j, nJ\vec{H} \rangle,$$

which, by the very definition of  $\nabla$ , is equivalent to the claimed formula.  $\blacksquare$

**Corollary 1.** *Let  $\mathcal{L}$  be a Lagrangian submanifold of  $\mathbb{D}^n$  with non-degenerate induced metric. Then it is minimal if and only if it has constant Lagrangian angle.*

**Remark 1.** *The isometry  $e^{\tau\varphi_0}$ ,  $\varphi_0 \in \mathbb{R}$ , transforms a Lagrangian submanifold with para-holomorphic volume  $\Omega$  and Lagrangian angle  $\beta$  into a Lagrangian submanifold with para-holomorphic volume  $\Omega e^{\tau\varphi_0}$  and Lagrangian angle  $\beta + n\varphi_0$ . Hence, in the study of minimal Lagrangian submanifolds, there is no loss of generality in studying Lagrangian submanifolds with vanishing Lagrangian angle, or equivalently, with para-holomorphic volume satisfying  $\text{Re } \Omega = 0$  or  $\text{Im } \Omega = 0$  (unlike the Kähler or pseudo-Kähler case, these two equations are not equivalent).*

**Remark 2.** *It was proved in [Me] (see also [HL2]) that a minimal Lagrangian submanifold with definite induced metric is a volume extremizer in its Lagrangian homology class.*

### 3 Minimal Lagrangian submanifolds in $\mathbb{D}^n$

#### 3.1 A local characterization of minimal Lagrangian surfaces with indefinite metric

**Theorem 4.** *Let  $\mathcal{L}$  be a minimal Lagrangian surface of  $\mathbb{D}^2$  with indefinite induced metric. Then  $\mathcal{L}$  is the product  $\gamma_1 \times J\gamma_2 \subset P \oplus JP$ , where  $\gamma_1$  and  $\gamma_2$  are two planar curves contained in a non-Lagrangian (and therefore non-complex) null plane  $P$ .*

*Proof.* Since the induced metric is assumed to be indefinite, there exist local null coordinates  $(u, v)$  on  $\mathcal{L}$ , i.e. such that the induced metric takes the form  $F(u, v)dudv$  (see [We], [An2]). It follows that, given a local parametrization of  $\mathcal{L}$ , the mean curvature vector  $\vec{H}$  is given by the formula  $\partial_{uv}f$  (see [Ch], [An2]). Hence the minimality assumption amounts to the partial differential equation  $\partial_{uv}f = 0$ . Therefore, the local parametrization  $f$  takes the form  $f(u, v) = \gamma_1(u) + \tilde{\gamma}_2(v)$ , where  $\gamma_1, \tilde{\gamma}_2$  are two null curves of  $\mathbb{D}^2$  (i.e.,  $\langle \gamma'_1, \gamma'_1 \rangle$  and  $\langle \tilde{\gamma}'_2, \tilde{\gamma}'_2 \rangle$  vanish), and

$$\langle \gamma'_1(u), \tilde{\gamma}'_2(v) \rangle \neq 0, \quad \forall (u, v) \in I_1 \times I_2.$$

On the other hand the Lagrangian assumption is:

$$\omega(\gamma'_1(u), \tilde{\gamma}'_2(v)) = 0, \quad \forall (u, v) \in I_1 \times I_2.$$

The remainder of proof relies on the analysis of the dimension of the two linear spaces  $P_1 := \text{Span}\{\gamma'_1(u), u \in I_1\}$  and  $P_2 := \text{Span}\{\tilde{\gamma}'_2(v), v \in I_2\}$ . We first observe that  $\dim P_1, \dim P_2 \geq 1$  and that the case  $\dim P_1 = \dim P_2 = 1$  corresponds to the trivial case of  $\mathcal{L}$  being planar. Since the rôles of  $\gamma_1$  and  $\tilde{\gamma}_2$  are symmetric, we may assume without loss of generality, and we do so, that  $\dim P_1 > 1$ .

Next, denoting by  $P_i^\omega, i = 1, 2$ , the symplectic orthogonal of  $P_i$ , the Lagrangian assumption is equivalent to  $P_2 \subset P_1^\omega$  and  $P_1 \subset P_2^\omega$ , so  $\dim P_2 \leq \dim P_1^\omega$  and  $\dim P_1 \leq \dim P_2^\omega$ . By the non-degeneracy of  $\omega$ , it follows that  $\dim P_1 \leq \dim P_2^\omega = 4 - \dim P_2 \leq 3$ . We claim that in fact  $\dim P_1 = 2$ . To see this, assume by contradiction that  $\dim P_1 = 3$ . It follows that  $\dim P_2 \leq \dim P_1^\omega = 1$ , so the curve  $\tilde{\gamma}_2$  is a straight line, which may be parametrized as follows:  $\tilde{\gamma}_2(v) = e_0v$ , where  $e_0$  is a null vector of  $\mathbb{D}^2$ . Then  $\gamma'_1$  is contained in the intersection of the light cone  $\mathbb{Q}_{2,0}^4$  with the hyperplane  $\{e_0\}^\omega$ . Moreover,  $e_0 \notin \text{Ker}(J \mp \text{Id})$ . Hence, after a possible linear, para-complex, isometry of  $\mathbb{D}^2$  we may assume that  $e_0 = (1, 0, 0, \pm 1)$ . Then an easy computation shows that  $\mathbb{Q}_{2,0}^4 \cap \{e_0\}^\omega = \Pi_1 \cup \Pi_2$ , where  $\Pi_1$  and  $\Pi_2$  are two null planes. Moreover, one of these planes, say  $\Pi_2$ , is contained in the orthogonal  $\{e_0\}^\perp$  of  $e_0$  with respect to the metric of  $\mathbb{D}^2$ . By the non-degeneracy assumption,

$$\langle \gamma'_1(u), \tilde{\gamma}'_2(v) \rangle = v \langle \gamma'_1(u), e_0 \rangle \neq 0,$$

we deduce that  $\gamma'_1 \in \Pi_1$ , which implies that  $\dim P_1 \leq 2$ , a contradiction.

To conclude, observe that,  $\tilde{\gamma}_2 \in P_2 \subset P_1^\omega = JP_1$ . Hence we just need to set  $P := P_1$  and  $\gamma_2 := J\tilde{\gamma}_2$ , to get that  $\gamma_1, \gamma_2 \subset P$ , so that  $\mathcal{L}$  takes the required expression. ■

### 3.2 Minimal graphs

In this section, we look at the minimal Lagrangian graph equation. It is well known that the graph of a closed one-form of a manifold  $\mathcal{M}$  is a Lagrangian submanifold of  $T^*\mathcal{M}$ . In the case of  $\mathcal{M}$  being an open subset  $U$  of  $\mathbb{R}^n$ , it is equivalent, locally (or globally if  $U$  is simply connected), to look at the graph of the gradient of a map  $u \in C^2(U)$ , a Lagrangian in  $T^*U \subset T^*\mathbb{R}^n \simeq \mathbb{D}^n$ . In order to write the minimality condition with respect to the para-Kähler metric, we consider the immersion

$$F : U \longrightarrow \mathbb{D}^n$$

$$(x_1, \dots, x_n) \longmapsto \left( x_1 + \tau \frac{\partial u}{\partial x_1}, \dots, x_n + \tau \frac{\partial u}{\partial x_n} \right).$$

Since we have

$$\frac{\partial F}{\partial x_i} = \left( \tau \frac{\partial^2 u}{\partial x_i \partial x_1}, \dots, 1 + \tau \frac{\partial^2 u}{\partial x_i^2}, \dots, \tau \frac{\partial^2 u}{\partial x_i \partial x_n} \right),$$

Hence, using Theorem 1, we obtain  $\beta = \arg \det_{\mathbb{D}}(Id + Hess(u))$ , where  $Hess$  denotes the matrix made of the second derivatives of  $u$ . We note that the PDE  $\arg \det_{\mathbb{D}}(Id + Hess(u)) = const$  may be elliptic or hyperbolic, depending on the causal character of the para-complex number  $\det_{\mathbb{D}}(Id + Hess(u))$ .<sup>1</sup>

In the case  $n = 2$ , this equation takes a remarkable form:

$$\arg(1 + \det_{\mathbb{R}}(Hess(u) + \tau \Delta u)) = const.$$

If the para-complex number  $1 + \det_{\mathbb{R}}(Hess(u) + \tau \Delta u)$  has positive squared norm, then the induced metric on the Lagrangian  $F(U)$  is definite. In this case, the equation  $\arg(1 + \det_{\mathbb{R}}(Hess(u) + \tau \Delta u)) = 0$  is equivalent, according to Remark 1, to the Laplace equation  $\Delta u = 0$ . If  $1 + \det_{\mathbb{R}}(Hess(u) + \tau \Delta u)$  has negative squared norm, the induced metric on  $F(U)$  is indefinite, and according to Remark 1, the equation amounts to the hyperbolic Monge-Ampère equation  $\det_{\mathbb{R}}(Hess(u)) = -1$ .

### 3.3 Minimal Equivariant Lagrangian submanifolds in $\mathbb{D}^n$

In this section we describe explicitly some solutions of the minimal equation which are equivariant. We define the following immersions

$$F : I \times \mathbb{S}^{n-1} \longrightarrow \mathbb{D}^n$$

$$(s, \sigma) \longmapsto \gamma(s)\iota(\sigma),$$

---

<sup>1</sup>The corresponding equation is elliptic in the Kähler case ([HL1]) and hyperbolic in the pseudo-Kähler case ([An2, An3]).

where  $\gamma \subset \mathbb{D}^2$  is a regular, planar curve with  $\gamma \neq 0$ , and  $\iota : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n$  denotes the canonical embedding of the sphere. In the following, we shall make the following abuse of notation:  $\iota(\sigma) = \sigma$ . The image of such an immersion is a Lagrangian submanifold of  $\mathbb{D}^n$  which is invariant under the action of  $SO(n)$  given by  $x + \tau y \mapsto Ax + \tau Ay$ , where  $x, y \in \mathbb{R}^n$  and  $A \in SO(n)$ . It is proved in [Sa] that if  $n \geq 3$ , any Lagrangian submanifold which is invariant by this action may be locally parametrized by such an immersion. This is, however, not true if  $n = 2$  (cf [Sa]).

We now proceed to calculate the para-holomorphic volume of  $X$ . Given  $\sigma \in \mathbb{S}^{n-1}$ , we introduce a local orthonormal frame  $(e_1, \dots, e_{n-1})$  in a neighbourhood of  $\sigma$ , which is parallel at  $\sigma$ . It follows that the vectors

$$dF(\partial_s) = F_s = \dot{\gamma}(s)\sigma$$

and

$$dF(e_i) = \gamma(s)d\iota(e_i) = \gamma(s)e_i, \quad 1 \leq i \leq n-1$$

span the tangent plane (here and in the following, the dot  $\dot{\cdot}$  denotes derivative with respect to the variable  $s$ ). So it is easily seen that the induced metric is degenerate if and only if  $\gamma$  is null or takes value in the light cone  $\langle z, z \rangle = 0$  and that  $\Omega$  is, up to a real constant,  $\dot{\gamma}(s)\gamma^{n-1}(s)$ . In particular, the induced metric is definite or Lorentzian, depending on whether  $\gamma$  and  $\dot{\gamma}$  have the same causal character, and the immersion  $F$  is minimal if and only if  $\arg(\dot{\gamma}\gamma^{n-1})$  is constant. According to Remark 1, it is sufficient to look at the two cases  $\operatorname{Re} \dot{\gamma}\gamma^{n-1} = 0$  and  $\operatorname{Im} \dot{\gamma}\gamma^{n-1} = 0$ . Integrating these equations, we obtain the algebraic equations  $\operatorname{Re} \gamma^n = C$  and  $\operatorname{Im} \gamma^n = C$ , where  $C$  is a real constant. Since the transformation  $z \mapsto Jz$  is an anti-isometry of  $\mathbb{D}^n$ , it preserves minimality. Accordingly, if  $\gamma$  is a solution of one of these two equations, so is  $\tau\gamma$ . We observe that in the even case, the solutions are *invariant* by the axial symmetry  $z \mapsto \tau z$ , while in the odd case, this symmetry *exchanges* solutions of  $\operatorname{Re} \gamma^n = C$  and  $\operatorname{Im} \gamma^n = C$  respectively. We observe that the minimality condition makes sense only if the curvature tensor is defined, i.e. if the induced metric is non-degenerate. Hence, in the next subsections, we take a closer look to the solutions and point out the existence of "singular points", i.e. where the induced metric becomes degenerate.

### The case $n = 2$

Here the solutions of the first equation corresponds to round circles  $x^2 + y^2 = C$ . These circles cross four times the light cone  $x^2 = y^2$  (i.e. the symmetry lines of  $z \mapsto \pm \tau z$ ). The resulting surface has equation

$$(x_1 - y_2)^2 + (y_1 + x_2)^2 = (x_1 + y_2)^2 + (x_2 - y_1)^2 = C.$$

It is a compact torus with four closed lines of null points and with indefinite metric elsewhere.

The solutions of the second equation are the hyperbolas<sup>2</sup>  $2xy = C$ . These curves cross once the light cone  $x^2 = y^2$ . The resulting surface is a cylinder with a circle of null points and with definite metric elsewhere.

<sup>2</sup>These curves do not have constant curvature with respect to the metric  $dx^2 - dy^2$ .

**The case  $n = 3$**

The first equation takes the form  $x^3 + 3xy^2 = C$ . The corresponding curve crosses twice the light cone  $x^2 = y^2$ . The resulting surface is a  $S^2 \times \mathbb{R}$  with two spheres  $S^2$  of null points bounding a cylinder with indefinite metric and two unbounded annuli with definite metric. The second equation  $3x^2y + y^3 = C$  is equivalent to the first one.

**The general case**

Using polar coordinates  $\gamma = \pm \tau^\alpha r e^{\tau\varphi}$  (with  $\alpha \in \{0, 1\}, r > 0$  and  $\varphi \in \mathbb{R}$ ), we get

$$\gamma^n = \pm \tau^{n\alpha} r^n e^{\tau n\varphi} = \pm \tau^{n\alpha} (r^n \cosh(n\varphi) + \tau r^n \sinh(n\varphi)).$$

We therefore obtain the general solutions

$$r = \left( \frac{C}{\cosh(n\varphi)} \right)^{1/n} \quad \text{or} \quad r = \left( \frac{C}{\sinh(n\varphi)} \right)^{1/n}.$$

In the first case, as  $\varphi \rightarrow \pm\infty$ , the curve  $\gamma$  tends to the two points  $\frac{(2C)^{1/n}}{2}(1, \pm 1)$  of the light cone, that it touches orthogonally.

In the second case, as  $\varphi \rightarrow +\infty$ , the curve  $\gamma$  tends again to a point of the light cone, that it touches orthogonally, while, when  $\varphi \rightarrow 0$ , the curve is asymptotic to a coordinate axis  $x = 0$  or  $y = 0$ .

**3.4 Normal bundles**

The *normal bundle*  $\overline{\mathcal{S}}$  of a submanifold  $\mathcal{S}$  of  $\mathbb{R}^n$  is the set

$$\overline{\mathcal{S}} := \{(x, v) \in \mathbb{D}^n = \mathbb{R}^n \oplus \mathbb{R}^n \mid x \in \mathcal{S}, v \in N_x \mathcal{S}\}.$$

We recall that a submanifold  $\sigma$  is said to be *austere* if, for any normal vector field  $v$ , the corresponding curvatures, i.e. the eigenvalues of the quadratic form  $A_v := \langle h(\cdot, \cdot), v \rangle$ , are symmetrically arranged around zero on the real line.

In the complex case  $\mathbb{C}^n$ , it has been proved in [HL1] that  $\overline{\mathcal{S}}$  is a Lagrangian submanifold, and that it is minimal if and only if  $\mathcal{S}$  is austere. Since the symplectic structure of  $\mathbb{D}^n$  is the same as that of  $\mathbb{C}^n$ , this fact that  $\overline{\mathcal{S}}$  is Lagrangian still holds true here. It turns out that the minimality criterion for  $\overline{\mathcal{S}}$  is the same in the para-complex case:

**Theorem 5.** *The Lagrangian submanifold  $\overline{\mathcal{S}}$  is minimal with respect to the neutral metric  $\langle \cdot, \cdot \rangle$  if and only if  $\mathcal{S}$  is austere.*

*Proof.* Let  $x$  be a point of  $\mathcal{S}$  and  $(v_1, \dots, v_{n-p})$  a local orthonormal frame of the normal space  $N\mathcal{S}$ , defined in a neighbourhood of  $x$ . There exists a local orthonormal frame  $(e_1, \dots, e_p)$  in a neighbourhood of  $x$  which is principal with respect to  $v_1$ , i.e. it diagonalizes  $A_{v_1}$ . We denote by  $\kappa_1, \dots, \kappa_p$  the corresponding principal curvatures, i.e.  $A_{v_1} e_i = \kappa_i e_i, \forall i, 1 \leq i \leq p$ . Observe that if  $t \in \mathbb{R}, A_{tv_1} = t A_{v_1}$ .

We are going to calculate the Lagrangian angle of  $\overline{\mathcal{S}}$  at a point  $(x, \nu)$ . Without loss of generality, we may assume that  $\nu = t\nu_1, t \in \mathbb{R}$ .

We set

$$E_i(x, \nu) := (e_i, -t\kappa_i e_i) \quad \forall i, 1 \leq i \leq p$$

and

$$E_{p+j}(x, \nu) := (0, \nu_j) \quad \forall j, 1 \leq j \leq n - p.$$

We therefore get a local tangent frame  $(E_1, \dots, E_n)$  along  $\overline{\mathcal{S}}$  in a neighbourhood of  $(x, \nu)$ . Next, we calculate

$$\det_{\mathbb{D}}(E_1, \dots, E_n) = \tau^{n-p} \prod_{i=1}^p (1 - \tau t \kappa_i).$$

Using Theorem 1, we get

$$\beta(x, \nu) = \arg \left( \tau^{n-p} \prod_{i=1}^p (1 - \tau t \kappa_i) \right).$$

Proceeding like in [HL1], it is easily seen that this number does not depend on  $t$  if and only if the principal curvatures  $\kappa_i$  are symmetrically arranged around zero on the real line, i.e.  $\mathcal{S}$  is austere. ■

#### 4 Equivariant self-similar Lagrangian submanifolds in $\mathbb{D}^n$

In this section, we describe some self-similar Lagrangian solutions of the Mean Curvature Flow (MCF). The simplest and most important example of a self-similar flow is when the evolution is a homothety. Such a self-similar submanifold  $F$  with mean curvature vector  $\vec{H}$  satisfies the following non-linear, elliptic system:

$$\vec{H} + \lambda F^\perp = 0, \tag{1}$$

where  $F^\perp$  denotes the projection of the position vector  $F$  onto the normal space of the submanifold, and  $\lambda$  is a real constant. If  $\lambda$  is strictly positive constant, the submanifold shrinks in finite time to a single point under the action of the MCF, its shape remaining unchanged. If  $\lambda$  is negative, the submanifold will expand, its shape again remaining the same. In the case of vanishing  $\lambda$ , we recover the case of minimal submanifolds, which of course are the stationary points of the MCF. We point out that in the para-Kähler setting, unlike the Kähler case, the shrinking and expanding case are roughly equivalent, since a change of sign in the metric  $\langle \cdot, \cdot \rangle \mapsto -\langle \cdot, \cdot \rangle$  (which can be explicitly performed by applying the point transformation  $z \mapsto Jz$ ) yields a change of sign in the mean curvature vector  $\vec{H} \mapsto -\vec{H}$  and leaves unchanged the term  $F^\perp$ . Hence, without loss of generality, we may assume that  $\lambda \geq 0$ .

As in Section 3.3, we consider the immersions

$$\begin{aligned} F : I \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{D}^n \\ (s, \sigma) &\longmapsto \gamma(s)\iota(\sigma), \end{aligned}$$

where  $\gamma \subset \mathbb{D}^2$  is a regular, planar curve with  $\dot{\gamma} \neq 0$ . We recall that  $(e_1, \dots, e_{n-1})$  is a local orthonormal frame. In particular, the vectors

$$JF_s = \tau\dot{\gamma}\sigma \text{ and } JdF(e_i) = \tau\gamma e_i, \quad 1 \leq i \leq n-1$$

span the normal space of  $F$ . A straightforward computation shows that  $\langle \vec{H}, JF(e_i) \rangle$  and  $\langle F, JF(e_i) \rangle$  vanish  $\forall i, 1 \leq i \leq n-1$ . Hence the self-similar equation (1) is equivalent to the scalar equation

$$\langle \vec{H}, JF_s \rangle + \lambda \langle F^\perp, JF_s \rangle = 0. \tag{2}$$

Next, we calculate

$$\langle \vec{H}, JF_s \rangle = \frac{\langle \ddot{\gamma}, J\dot{\gamma} \rangle}{\langle \dot{\gamma}, \dot{\gamma} \rangle} - (n-1) \frac{\langle \gamma, J\dot{\gamma} \rangle}{\langle \gamma, \gamma \rangle}.$$

We also have, recalling that  $JF_s$  is a normal vector,

$$\langle F^\perp, JF_s \rangle = \langle F, JF_s \rangle = \langle \gamma\sigma, J\dot{\gamma}\sigma \rangle = \langle \gamma, J\dot{\gamma} \rangle,$$

hence Equation (2) becomes

$$\frac{\langle \ddot{\gamma}, J\dot{\gamma} \rangle}{\langle \dot{\gamma}, \dot{\gamma} \rangle} + \left( -\frac{n-1}{\langle \gamma, \gamma \rangle} + \lambda \right) \langle \gamma, J\dot{\gamma} \rangle = 0. \tag{3}$$

We now assume that  $\gamma$  is parametrized by ‘‘arclength’’, i.e.  $\langle \dot{\gamma}, \dot{\gamma} \rangle = \pm 1 := \epsilon'$ . It follows that  $\nu := J\dot{\gamma}$  is a unit normal vector and that the acceleration vector  $\ddot{\gamma}$  is normal to the curve, hence collinear to  $\nu$ .

It follows that there exists  $(p', q') \in \{0, 1\}^2$  and  $\theta : I \rightarrow \mathbb{R}$  such that

$$\dot{\gamma}(s) = (-1)^{p'} \tau^{q'} (\cosh(\theta(s)) + \tau \sinh(\theta(s))).$$

Differentiating, we get

$$\langle \ddot{\gamma}, J\dot{\gamma} \rangle = (-1)^{q'+1} \dot{\theta} = -\epsilon' \dot{\theta},$$

so that

$$\frac{\langle \ddot{\gamma}, J\dot{\gamma} \rangle}{\langle \dot{\gamma}, \dot{\gamma} \rangle} = \frac{-\epsilon' \dot{\theta}}{\epsilon'} = -\dot{\theta}.$$

In order to deal with the other terms of Equation (3), we use ‘‘polar’’ coordinates for  $\gamma$ :

$$\gamma(s) = (-1)^p \tau^q r(s) (\cosh(\varphi(s)) + \tau \sinh(\varphi(s)))$$

where  $(p, q) \in \{0, 1\}^2$ ,  $r(s) : I \rightarrow (0, \infty)$  and  $\varphi(s) : I \rightarrow \mathbb{R}$ .

Observe that  $q = \frac{1-\epsilon}{2}$ , where  $\epsilon \in \{-1, 1\}$  is defined by  $\langle \gamma, \gamma \rangle = \epsilon r^2$ . The induced metric is definite if  $q = q'$  and Lorentzian if  $q \neq q'$ . Hence Equation (3) becomes

$$-\dot{\theta} + \left( -\epsilon \frac{n-1}{r^2} + \lambda \right) \langle \gamma, \nu \rangle = 0. \tag{4}$$

For the remainder of the analysis of the equation it is convenient to deal separately with the definite and Lorentzian cases:

### The definite case

Differentiating  $\gamma$  and setting  $\alpha := \theta - \varphi$ , we have the following “compatibility equation”:

$$(\dot{r}, \dot{\varphi}) = \eta \left( \cosh \alpha, \frac{1}{r} \sinh \alpha \right)$$

where we have set for brevity  $\eta := (-1)^{p+p'}$ . On the other hand, a calculation gives

$$\langle \gamma, \nu \rangle = \eta \epsilon r \sinh \alpha.$$

So finally, putting together the compatibility equations and Equation (3), we get a  $3 \times 3$  system

$$\begin{cases} \dot{r} &= \eta \cosh \alpha \\ \dot{\varphi} &= \eta \frac{1}{r} \sinh \alpha \\ \dot{\theta} &= \eta \left( -\frac{n-1}{r} + \lambda' r \right) \sinh \alpha, \end{cases}$$

where we have set for convenience  $\lambda' := \epsilon \lambda$ . Observe that the integral curves of the system do not depend on  $\eta$ , so without loss of generality we may assume that  $\eta = 1$ . Moreover, recalling that  $\alpha = \theta - \varphi$ , we are left with the  $2 \times 2$  system:

$$\begin{cases} \dot{r} &= \cosh \alpha \\ \dot{\alpha} &= \left( -\frac{n}{r} + \lambda' r \right) \sinh \alpha \end{cases}$$

The system enjoys a first integral:  $E(r, \alpha) = r^n \exp(-\lambda' r^2/2) \sinh \alpha$ . It follows that the projection on the plane  $\{(r, \alpha)\}$  of the integral curves  $(r, \alpha, \varphi)$  have two ends, one with  $(r, \alpha) \rightarrow (0, \pm\infty)$  and the other with  $(r, \alpha) \rightarrow (+\infty, \pm\infty)$  if  $\lambda' > 0$  or  $(r, \alpha) \rightarrow (0, \pm\infty)$  if  $\lambda' \leq 0$ . In order to understand the asymptotic behaviour of  $\gamma$ , we need to study the variable  $\varphi$ . Using the fact that

$$\frac{d\varphi}{dr} = \frac{\sinh \alpha}{r \cosh \alpha} = \frac{1}{r} \left( \frac{r^{2n} \exp(-\lambda' r^2)}{E^2} + 1 \right)^{-1/2},$$

we obtain that

$$\varphi(s) = \varphi(0) + \int_0^s \dot{\varphi}(\sigma) d\sigma = \varphi(0) + \int_{r_0}^r \frac{1}{\rho} \left( \frac{\rho^{2n} \exp(-\lambda' \rho^2)}{E^2} + 1 \right)^{-1/2} d\rho.$$

We first see that  $\varphi \sim_{r \sim 0} \log(r) + C$ , which implies that the curve touches orthogonally the light cone as  $r \rightarrow 0$ . On the other hand, as  $r \rightarrow +\infty$ , the situation depends on  $\lambda'$ : in the case  $\lambda' > 0$ , we still have  $\varphi \sim_{r \sim +\infty} \log(r) + C$ , which implies that as  $r \rightarrow +\infty$ , the curve  $\gamma$  becomes asymptotically parallel to a branch of the light cone, while if  $\lambda' \leq 0$ ,  $\varphi$  converges to a constant, which means that the curve  $\gamma$  is asymptotic to a straight line.

### The Lorentzian case

We proceed exactly as in the previous case. Setting  $\alpha := \theta - \varphi$  and  $\lambda' := \epsilon \lambda$ , we are left with the  $2 \times 2$  system:

$$\begin{cases} \dot{r} &= \sinh \alpha \\ \dot{\alpha} &= \left( -\frac{n}{r} + \lambda' r \right) \cosh \alpha, \end{cases}$$

which enjoys a first integral:  $E(r, \alpha) = r^n \exp(-\lambda' r^2/2) \cosh \alpha$ . In order to describe the global behaviour of  $\gamma$ , we come back to the study of  $\varphi$ . Since we have

$$\frac{d\varphi}{dr} = \frac{1}{r} \left( 1 - \frac{r^{2n} \exp(-\lambda' r^2)}{E^2} \right)^{-1/2}$$

we obtain

$$\varphi(s) = \varphi(0) + \int_{r_0}^r \frac{1}{\rho} \left( 1 - \frac{\rho^{2n} \exp(-\lambda' \rho^2)}{E^2} \right)^{-1/2} d\rho.$$

In the case  $\lambda' \leq 0$ , the variable  $r$  is bounded on the integral curves  $(r, \alpha, \varphi)$ , which have two ends with  $(r, \alpha) \rightarrow (0, \pm\infty)$ . We still have  $\varphi \sim_{r \sim 0} \log(r) + C$  so again the curve  $\gamma$  touches orthogonally the light cone as  $r \rightarrow 0$  (in the case  $\lambda' = 0$  and  $n = 2$ , we recover the example of the torus found in Section 3.3).

If  $\lambda' > 0$ , the phase diagram is more complicated: it has a critical point  $(r_0, \alpha_0) = (\sqrt{\frac{n}{\lambda'}}, 0)$ , which corresponds to  $\gamma$  being a hyperbola (constant curvature curve): if  $\epsilon = -1$ , we have  $\gamma(s) = \pm r_0(\cosh(s/r_0), \sinh(s/r_0))$  which is “self-expanding” since here  $\lambda < 0$ , and if  $\epsilon = 1$ , we have  $\gamma(s) = \pm r_0(\sinh(s/r_0), \cosh(s/r_0))$ , (which is “self-shrinking” since  $\lambda > 0$ ).

The properties of the other solutions depend on the energy level  $E_0 := (\frac{n}{\lambda'})^{n/2} \exp(-n^2/2)$ :

- Curves with  $E < E_0$  and  $r < r_0$ . They are symmetric with respect to the  $r$  axis and enjoy two ends with  $r \rightarrow 0$ . They are similar to the Lorentzian case with  $\lambda' > 0$ ;
- Curves with  $E < E_0$  and  $r > r_0$ . They are also symmetric with respect to the  $r$  axis and enjoy two ends with  $r \rightarrow +\infty$  which are asymptotically parallel to the light cone;
- Curves with  $E \geq E_0$ ; These curves enjoys one end with  $r \rightarrow 0$ , with the curve touching orthogonally the light cone, and another one with  $r \rightarrow +\infty$ , asymptotically parallel to the light cone.

## Appendix: para-complex and almost-para-complex manifolds

The purpose of this Appendix is to prove the following statement:

*An almost para-complex structure  $J$  comes from a para-complex structure if and only if its para-Nijenhuis tensor  $N^J$  vanishes.*

*Proof.* Given an almost para-complex structure  $J$ , we denote by  $\mathcal{D}^+$  and  $\mathcal{D}^-$  the two corresponding eigendistributions, i.e.  $\mathcal{D}^\pm := \text{Ker}(J \mp \text{Id})$ . The following lemma, whose proof we leave to the reader, expresses the para-Nijenhuis tensor in terms of the decomposition:

**Lemma 3.** *Take two vector fields  $X_1$  and  $X_2$  on  $\mathcal{M}$  and write the decomposition  $X_1 = U_1 + V_1$  and  $X_2 = U_2 + V_2$  in terms of  $T\mathcal{M} = \mathcal{D}^+ \oplus \mathcal{D}^-$ . Then we have*

$$N^J(X_1, X_2) = 2\left([U_1, U_2] - J[U_1, U_2] + [V_1, V_2] + J[V_1, V_2]\right).$$

We assume first that  $N^J$  vanishes. By Lemma 3,  $X_1 \in \mathcal{D}^+$  (hence  $V_1 = 0$ ) implies that  $[U_1, U_2] = J[U_1, U_2]$ , i.e.  $[U_1, U_2] \in \mathcal{D}^+$ , i.e.  $\mathcal{D}^+$  is integrable. Analogously, if  $X_2 \in \mathcal{D}^-$ , we get that  $[V_1, V_2] \in \mathcal{D}^-$ , i.e.  $\mathcal{D}^-$  is integrable.

Hence, in the neighbourhood of any point we can find local coordinates  $(u_1, \dots, u_n, v_1, \dots, v_n)$  on  $\mathcal{M}$  such that the integral submanifolds of  $\mathcal{D}^+$  (resp.  $\mathcal{D}^-$ ) are  $\{v_i = \text{const}, 1 \leq i \leq n\}$  (resp.  $\{u_i = \text{const}, 1 \leq i \leq n\}$ ). In particular, we have  $J\partial u_i = \partial u_i$  and  $J\partial v_i = -\partial v_i$ . We define another local system of coordinate setting  $x_i := u_i + v_i$ ,  $y_i := u_i - v_i$ . We claim that the collection of all such system of coordinates define an atlas of para-complex coordinates. To see this, we need to prove that the transition functions satisfy the para-Cauchy-Riemann equations. Let  $(x_1, \dots, x_n, y_1, \dots, y_n)$  and  $(x'_1, \dots, x'_n, y'_1, \dots, y'_n)$  be two local system of coordinates constructed as before. Let  $(u_1, \dots, u_n, v_1, \dots, v_n)$  and  $(u'_1, \dots, u'_n, v'_1, \dots, v'_n)$  be the associated “null” coordinates (between quotes because here there is no metric). By assumption, the integral submanifolds of  $\mathcal{D}^+$  and  $\mathcal{D}^-$  may be expressed either in terms of  $(u_1, \dots, u_n, v_1, \dots, v_n)$  or  $(u'_1, \dots, u'_n, v'_1, \dots, v'_n)$ : in particular,

$$\{u_i = \text{const.}, 1 \leq i \leq n\} = \{u'_i = \text{const.}, 1 \leq i \leq n\}$$

and

$$\{v_i = \text{const.}, 1 \leq i \leq n\} = \{v'_i = \text{const.}, 1 \leq i \leq n\},$$

therefore we have

$$\frac{\partial u'_i}{\partial v_j} = \frac{\partial v'_i}{\partial u_j} = 0, \quad \forall i, j, \quad 1 \leq i, j \leq n,$$

which is equivalent to the para-Cauchy-Riemann equations

$$\frac{\partial x'_i}{\partial x_j} = \frac{\partial y'_i}{\partial y_j} \quad \text{and} \quad \frac{\partial x'_i}{\partial y_j} = \frac{\partial y'_i}{\partial x_j}, \quad \forall i, j, \quad 1 \leq i, j \leq n.$$

Conversely, let  $\mathcal{M}$  be a manifold equipped with a para-complex structure, i.e. an atlas whose transition functions satisfy the para-Cauchy-Riemann equations. This implies the global existence of two foliations: locally, they are defined by  $\{u_i = \text{const}, 1 \leq i \leq n\}$  and  $\{v_i = \text{const}, 1 \leq i \leq n\}$  and the para-Cauchy-Riemann equations precisely say that the definition is independent of a particular choice of coordinates, and may therefore be extended globally. Denote by  $\mathcal{D}^+$  and  $\mathcal{D}^-$  the two distributions tangent to these foliations, which are the eigenspaces of  $J$ . By Frobenius theorem,  $[U_1, U_2] \in \mathcal{D}^+$  and  $[V_1, V_2] \in \mathcal{D}^-$ . By Lemma 3,  $N^J(X_1, X_2)$  vanishes. ■

## References

- [An1] H. Anciaux, *Construction of equivariant self-similar solutions to the mean curvature flow in  $\mathbb{C}^n$* , *Geom. Dedicata* **120** (2006), no. 1, 37–48

- [An2] H. Anciaux, *Minimal submanifolds in Pseudo-Riemannian geometry*, World Scientific, 2010
- [An3] H. Anciaux, *Minimal Lagrangian submanifolds in indefinite complex space*, Illinois J. of Math. **56** (2012), no. 4, 1331–1343
- [An4] H. Anciaux, *Spaces of geodesics of pseudo-Riemannian space forms and normal congruences of hypersurfaces*, Transactions of the AMS, **366** (2014), 2699–2718
- [Ch] B.-Y. Chen, *Pseudo-Riemannian Geometry,  $\delta$ -Invariants and Applications*, World Scientific, 2011
- [HL1] R. Harvey, H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982) 47–157
- [HL2] R. Harvey, H. B. Lawson, *Split Special Lagrangian Geometry*, Metric and Differential Geometry Progress in Mathematics **297** (2012) 43–89
- [Me] J. Mealy, *Volume maximization in Semi-Riemannian Manifolds*, Indiana Univ. Math. J. **40** (1991) 793–814
- [Sa] M. Samuays, *Subvariedades Lagrangeanas Mínicas e Autossimilares no Espaço Paracomplexo*, PhD thesis, Universidade de São Paulo
- [We] T. Weinstein, *An introduction to Lorentz surfaces*, de Gruyter Expositions in Mathematics, 22 Walter de Gruyter & Co., Berlin, 1996

Université Libre de Bruxelles,  
Boulevard du Triomphe,  
1050 Brussels, Belgium;  
email: henri.anciaux@gmail.com

Departamento de Matemática, IME,  
Universidade Federal da Bahia (UFBA)  
Av. Adhemar de Barros, S/N Ondina CEP:40170-110,  
Salvador, BA, Brazil.  
email: maikel.mat@gmail.com;