Unbounded analysis operators*

H. Hosseini Giv M. Radjabalipour A. Askari Hemmat

To the memory of the late Batool Bagheri (1951-2012), the founder of Kerman Mathematics House.

Abstract

The paper studies bounded or unbounded operators which can act as analysis operators or synthesis operators of various signal processing including generalized frames, semi-frames, discrete frames, Fourier transforms, etc. The paper is concluded by a short discussion of the controllability of the behavior of the processed signals.

1 Introduction

The theory of frames, generalized frames, semi-frames or other generalizations of such classes have byproducts in the form of linear operators which theoretically help the better understanding of the subject; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 19]. In signal processing, the analysis operator sends each signal h from a Hilbert or Banach space X to the processed function $\tilde{h} \in L^2(Z, \mathcal{M}, \mu)$ for which every point evaluation $h \mapsto \tilde{h}(z) : H \to \mathbb{C}$ is a bounded or unbounded linear functional ($z \in Z$). The main goal of the present paper is to study those bounded or unbounded operators from a subspace of a Hilbert space H to some $L^2(\mu)$ whose closures can be admitted as the analysis operators of some signal processing. As we can always work in the smallest Hilbert space containing the signals of interest, we may and shall assume without loss of generality that the

*The research is supported by the Iran National Science Foundation (INSF).
Received by the editors March 2012.
Communicated by F. Bastin.
2010 *Mathematics Subject Classification* : Primary 42C15; Secondary 43A25.
Key words and phrases : Generalized frame, Coherent state system, Algebraic frame, Analytic frame, Fourier transform, Unbounded operators.

operators are densely defined. Also, to avoid null processing, we assume the operators are injective. Furthermore, since the analysis operators happen to be unbounded, we restrict ourselves to closable operators for a minimal controllability on the process.

Analysis operators corresponding to generalized frames are everywhere defined bounded operators; however, Fourier transformation on $\mathbb{R}^n$ is a unitary operator $\mathcal{F}$ such that the linear functionals $h \mapsto \hat{h}(z) = (\mathcal{F}h)(z)$ do not define bounded linear functionals. This is why Fourier transforms begin with signals in $L^1(\mu) \cap L^2(\mu)$ which is a dense linear subspace of a Hilbert space. This, among others, is a justification for beginning with linear operators which are densely defined.

Recall that the domain $\mathcal{D}(T)$ of an unbounded linear operator T is a linear subspace included in (but not necessarily equal to) a Hilbert space H and its codomain is a Hilbert space K. For the purposes of the present paper, we further assume that $K = L^2(\mu)$ for some positive measure μ, $\overline{\mathcal{D}} = H$ and $\overline{T}$ has an injective closure. The range and the graph of T are denoted by $\mathcal{R}(T)$ and $\mathcal{G}(T)$, respectively. The operator T is said to be everywhere defined if $\mathcal{D}(T) = H$. Note that, if $\overline{T}$ exists, it is uniquely determined by $\mathcal{G}(\overline{T}) = \overline{\mathcal{G}(T)}$. The operator T is called lower bounded, upper bounded, or doubly bounded, if, respectively, $||Th||/||h|| \geq \alpha$, $||Th||/||h|| \leq \beta$, or $\alpha \leq ||Th||/||h|| \leq \beta$ for all unit vectors $h \in \mathcal{D}(T)$ and some positive numbers α, β independent of h. If $S : \mathcal{D}(S) \subset K \to L$ is another unbounded operator, then $ST : \mathcal{D}(ST) \subset H \to L$ is a new operator for which $\mathcal{D}(ST) = \{x \in \mathcal{D}(T) : \ Tx \in \mathcal{D}(S)\}$ and $(ST)x = S(Tx)$ for all $x \in \mathcal{D}(ST)$. Recall that the operators T^{-1} and T^*, if it exists, are uniquely determined by the equations $\mathcal{G}(T^{-1})^{\text{flip}} = \mathcal{G}(T)$ and $\mathcal{G}(-T^*)^{\text{flip}} = \mathcal{G}(T)^{\perp}$, where "flip" is a map sending $x \oplus y$ to $y \oplus x$. Also, if $\overline{T}$ is injective, its polar decomposition $\overline{T} = V|T|$ consists of an isometry V and an unbounded selfadjoint operator $|T| := (T^*\overline{T})^{1/2}$. (The properties of unbounded operators can be found here and there in classical textbooks; we may also suggest to see [17].)

2 Algebraic Frames: Fourier transforms revisited

Fourier transformations on $\mathbb{R}^n$ are unitary operators which yield the following unbounded linear functionals on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$:

$$\phi \mapsto \hat{\phi}(z) := \int_{\mathbb{R}^n} \phi(x) e^{-iz\cdot x} dx, \ \phi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

The above observation motivates the following definition whose notation and symbols are maintained throughout the paper.

Definition 2.1. Let $\mathcal{D}$ be a dense linear subspace of H and let $(\theta_z)_{z\in Z}$ be a family of (not necessarily bounded) linear functionals defined on $\mathcal{D}$, where Z is a set equipped with a (positive) measure μ. The family $(\theta_z)_{z\in Z}$ is called an *algebraic frame* if the unbounded operator $T : \mathcal{D}(T) \subset H \to L^2(\mu)$ defined by $(Th)(z) = \theta_z(h)$ $(h \in \mathcal{D}, \ z \in Z)$ has an injective closure $\overline{T}$. The measure space Z is called the index measure space of the algebraic frame and the closure $\overline{T}$ of T is called the analysis operator of the algebraic frame.

Note. The traditional definition of a generalized frame is a family of vectors $(\theta_z)_{z\in Z} \subset H$ indexed by a measure space $(Z, \mathcal{M}, \mu)$ which satisfies

$$A||h||^2 \leq \int_Z |\langle h, \theta_z\rangle|^2 d\mu(z) \leq B||h||^2$$

for all $h \in H$ and some positive constants A, B independent of h. (Note that by the Riesz representation theorem for linear functionals on Hilbert spaces, every bounded linear functional can be identified by a vector in the Hilbert space and vice versa.) In this case, the analysis operator T sending h to the complex-valued function $z \mapsto \langle h, \theta_z\rangle$ is a doubly bounded operator which is also closed. If A is allowed to be 0, then the family is called an upper semi-frame [1]. However, if $A > 0$ and $B = \infty$, the family is called a lower semi-frame [1] and T cannot be closable; hence, the family is not an algebraic frame. (For more on lower semi-frames, see [1, 2, 10].)

The following example revisits the Fourier transform on $\mathbb{R}^n$ as an algebraic frame.

Example 2.2. Let $H = L^2(\mathbb{R}^n)$. Let $\mathcal{D}(T)$ be the (dense) linear subspace of H consisting of all step functions and define $T : \mathcal{D}(T) \to L^2(\mathbb{R}^n)$ by

$$(T\phi)(z) = \int_{\mathbb{R}^n} \phi(x)e^{-2\pi i x\cdot z}dx, \;\; \forall \phi \in \mathcal{D}(T). \tag{2.1}$$

Then the closure of T is a multiple of a unitary operator and the right-hand side of (2.1) defines a family of unbounded linear functionals yielding a doubly bounded algebraic frame.

To see the frame properties of the Fourier transform on $\mathbb{R}^n$, consider a step function $\phi = \sum_{j=1}^h s_j\chi_{I_j}$, where s_j is a constant real number and $I_j = I_{j1} \times \cdots \times I_{jn} \subset \mathbb{R}^n$ is a cartesian product of some (closed or nonclosed) intervals $(j = 1, 2, \cdots, h)$. Let ℓ_n denote the n-dimensional volume in $\mathbb{R}^n$. Then, assuming without loss of generality that the cells $I_1, \cdots, I_h$ are disjoint,

$$\begin{aligned}
||T\phi||_2^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(x)\overline{\phi(y)}e^{2\pi i z\cdot(y-x)}dxdydz \\
&= \sum_{j,k=1}^h s_j\bar{s}_k \int_{\mathbb{R}^n}\left(\prod_{m=1}^n \int_{I_{jm}}\int_{I_{km}} e^{2\pi i z_m(y_m-x_m)}dx_m dy_m\right)dz \\
&= \sum_{j,k=1}^h s_j\bar{s}_k \prod_{m=1}^n \int_{-\infty}^{\infty}\int_{I_{jm}}\int_{I_{km}} e^{2\pi i z_m(y_m-x_m)}dx_m dy_m dz_m \\
&= \pi^{-n}\sum_{j,k=1}^h s_j\bar{s}_k \prod_{m=1}^n \ell_1(I_{jm}\cap I_{km})\int_{-\infty}^{\infty}[1-\cos t]/t^2 dt \qquad (2.2) \\
&= c^n\pi^{-n}\sum_{j=1}^h |s_j|^2\ell_n(I_j) = c^n\pi^{-n}||\phi||^2,
\end{aligned}$$

where

$$c = \int_{-\infty}^{\infty}[1-\cos t]/t^2 dt < \infty.$$

(The well-known fact $c = \pi$ is not needed here.) Thus $\pi^{n/2}c^{-n/2}T = V$ for some isometry V. (If we use the value $c = \pi$, it follows that T is in fact an isometry.)

To show T is a multiple of unitary, we prove that

$$(T^*\psi)(x) = \int_{\mathbb{R}^n} \psi(z)e^{2\pi i z\cdot x}dz. \tag{2.3}$$

If $\phi = \chi_A$ and $\psi = \chi_B$ for some cells $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, then

$$\langle \phi, T^*\psi\rangle = \langle T\phi, \psi\rangle = \int_B \int_A e^{-2\pi i z\cdot x}dxdz = \langle \phi, \int \psi(z)e^{2\pi i z\cdot x}dz\rangle,$$

which proves (2.3). Repeating the above computations for T^*, it follows that $\pi^{n/2}c^{-n/2}T^*$ is also an isometry. Hence, V is a unitary.

The next example revisits the Zak transformation [12] as an algebraic frame. In the following, if $\alpha, x \in \mathbb{R}^n$, by $\alpha > 0$, we mean all the coordinates $\alpha_1, \cdots, \alpha_n$ of α are positive, by α^{-1} we mean a vector with coordinates $\alpha_1^{-1}, \cdots, \alpha_n^{-1}$, and by αx, we mean a vector with coordinates $\alpha_1 x_1, \cdots, \alpha_n x_n$. Also, $[0, \alpha]$ denotes the cell $[0, \alpha_1] \times \cdots \times [0, \alpha_n] \subset \mathbb{R}^n$.

Example 2.3. Let λ be a finite (positive) measure on some space Y, let γ be the counting measure on $\mathbb{Z}^n$ and let dx be the (n-dimensional) Lebesgue measure on $[0, \alpha^{-1}]$ for some $0 < \alpha \in \mathbb{R}^n$. Let $H = L^2(\lambda \times \gamma)$ and $K = L^2(\lambda \times dx)$. Let F be a linear space of bounded functions supported on $Y \times J$ for some finite subset J of $\mathbb{Z}^n$, and assume the class $\mathcal{D}$ consisting of all functions almost equal to functions in F with respect to the measure $\lambda \times \gamma$ is dense in $L^2(\lambda \times \gamma)$. (For example, if Y is a locally compact space, F can be taken to be $C_c(Y \times \mathbb{Z}^n)$.) Define $T : \mathcal{D} \to K$ by

$$(T[f])(y, x) = \sum_{k\in\mathbb{Z}^n} f(y, k)e^{2\pi i(\alpha k).x}. \tag{2.4}$$

(Here, it is important to distinguish between a function $g \in F$ and its equivalence class $[g] \in \mathcal{D}$; for example, if $g \in C([0, 1])$, $g(x)$ is well-defined but, if $h = g$ a.e., the expression $h(x)$ is not well-defined.) Then the closure of T is a multiple of a unitary operator and the functionals defined by (2.4) are unbounded linear functionals generating an algebraic frame. To see this, apply Tonelli's theorem to $|T[f]|^2$ $(f \in F)$, to observe that

$$\begin{aligned}
\int_{Y\times\mathbb{Z}^n} |f|^2 d\mu &= \sum_{k\in\mathbb{Z}^n} \int_Y |f(y,k)|^2 d\lambda(y) = \int_Y \sum_{k,j\in\mathbb{Z}^n} f(y,k)\overline{f}(y,j)\delta_{jk} d\lambda(y) \\
&= \alpha_\times \int_Y \sum_{k,j\in\mathbb{Z}^n} f(y,k)\overline{f}(y,j) \int_{[0,\alpha^{-1}]} e^{2\pi i\alpha(j-k)\cdot x} dx d\lambda(y) \\
&= \alpha_\times \int_Y \int_{[0,\alpha^{-1}]} |(T[f])(y,x)|^2 dx d\lambda(y) = \alpha_\times ||T[f]||^2 < \infty.
\end{aligned}$$

Thus, the closure of $(\alpha_\times)^{1/2}T$ defines an isometry V and, hence, the required (densely defined closed) analysis operator T is equal to $(\alpha_\times)^{-1/2}V$.

Next, assume $\psi \in L^2(\lambda \times d\omega)$ and $[f] \in \mathcal{D}$. Then

$$\begin{aligned} \langle T^*\psi, [f]\rangle_{L^2(\mu)} &= \langle \psi, T[f]\rangle \\ &= \int_Y \int_{[0,\alpha^{-1}]} \psi(y,x) \sum_k \overline{f}(y,k) e^{-2\pi i \alpha k\cdot x} dx d\lambda(y) \\ &= \langle \int_{[0,\alpha^{-1}]} \psi(y,x) e^{-2\pi i \alpha k\cdot x} dx, [f]\rangle_{L^2(\mu)}. \end{aligned} \quad (2.5)$$

Now, assume $T^*\psi = 0$ and fix $h \in \mathbb{Z}^n$ and $\eta \in L^2(\lambda) \cap L^\infty(\lambda)$. The function $f(y,k) := \delta_{kh}\eta(y)$ defines an element $[f] \in \mathcal{D}(T)$ and

$$0 = \langle T^*\psi, [f]\rangle_{L^2(\mu)} = \langle (T^*\psi)(\cdot, h), \eta\rangle_{L^2(\lambda)}.$$

Since $L^2(\lambda) \cap L^\infty(\lambda)$ is dense in $L^2(\lambda)$, it follows that, for each $k \in \mathbb{Z}^n$, there exists $\Gamma_k \subset Y$ such that $\lambda(Y \backslash \Gamma_k) = 0$ and $\int_{[0,\alpha^{-1}]} \psi(y,x) e^{-2\pi i \alpha k\cdot x} dx = 0$ for all $y \in \cap_k \Gamma_k$. Now, applying the theory of Fourier series inductively to each coordinate of x, one obtains measurable subsets $\Omega_j \subset [0, \alpha_j^{-1}]$ $(j = 1, 2, \cdots, n)$ such that, $[0, \alpha_j] \backslash \Omega_j$ has Lebesgue measure 0 and that $\psi(y,x) = 0$ for all $(y,x) \in (\cap_{h\in\mathbb{Z}^n} Y_h) \times \Omega_1 \times \cdots \times \Omega_n$. Hence $\psi = 0$ and, thus, V is a unitary operator.

The actual Zak transform is included in the following corollary.

Corollary 2.4. *Let $0 < \alpha \in \mathbb{R}^n$ and $0 < \beta \in \mathbb{R}^m$ for some positive integers m, n and assume $h : \mathbb{Z}^n \to \mathbb{Z}^m$ is a bijection. Let $X = C_c(\mathbb{R}^m)$ and let $Z := [0, \beta] \times [0, \alpha^{-1}]$ equipped with the Lebesgue measure $dy \times d\omega$. Then, for $z := (y, \omega) \in Z$, define*

$$\theta_z(\phi) = \sum_{k\in\mathbb{Z}^n} \phi(y + \beta h(k)) e^{2\pi i \alpha k\cdot\omega}.$$

Then $\Theta = (\theta_z)_{z\in Z}$ is a nonanalytic algebraic frame whose analyzing operator is a multiple of unitary. The Zak transformation is obtained if $\beta = \alpha = (\alpha_1, \alpha_1, \cdots, \alpha_1)$, $m = n$ and $h(k) \equiv k$.

In all the above examples, the analysis operator turned out to be a multiple of a unitary and thus the inversion formula could be written in the form of a multiple of the adjoint of the analysis operator. The following example is an unbounded analysis operator whose inverse is bounded and has nothing to do with the adjoint of the analysis operator.

Example 2.5. If $T = d/dx$ acts on $\mathcal{D}(T) = \{f \in C^{(1)}([0,1]) : f(0) = 0\}$, then the unbounded operator $T : \mathcal{D}(T) \subset L^2([0,1]) \to L^2([0,1])$ is a densely defined operator the inverse of whose closure is the bounded operator $(Sg)(x) = \int_0^x g(t)dt$. Thus, the functionals $f \mapsto f'(x)$ are unbounded linear functionals on $C^{(1)}([0,1])$ and define a lower bounded algebraic frame. On the other hand, the functionals $f \mapsto \int_0^x f(t)dt$ are bounded linear functionals on $L^2([0,1])$ and define an upper semi-frame.

3 Inversion formula

In this section, we establish an inversion formula for every algebraic frame. Recall that the closed injective operator $\overline{T}$ has the polar decomposition $\overline{T} = V|T|$, where $V : H \to L^2(\mu)$ is an isometry, $\mathcal{D}(\overline{T}) = \mathcal{D}(|T|)$ and $\mathcal{R}(V) = \overline{\mathcal{R}(T)}$. It is not difficult to show that, in our special case,

$$T^* = |T|V^*. \tag{3.1}$$

The synthesis operator of an algebraic frame generated by the unbounded operator $T : \mathcal{D}(T) \subset H \to L^2(\mu)$, must be a densely defined closed operator $S : \mathcal{D}(S) \subset L^2(\mu) \to H$ acting as a left inverse to T. To make it densely defined, we require S to satisfy the following conditions:

$$ST = I_{\mathcal{D}(T)} \text{ and } S(\mathcal{R}(T)^{\perp}) = \{0\}. \tag{3.2}$$

If T is an everywhere defined bounded operator, which is the case for generalized frames and upper semi-frames, then the inverse mapping theorem implies that $S = T^{-1}$ on $\mathcal{R}(T)$ and $S = 0$ on $\mathcal{R}(T)^{\perp}$; since S is closed, it is uniquely defined. The following theorem and the example following the theorem reveal that, to have a uniquely defined synthesis operator, we need to impose extra conditions on S. In what follows, the symbol $\dot{+}$ stands for decomposition of a general vector space as the direct sum of some linear subspaces, while $\oplus$ is used for the decomposition of a Banach space as the direct sum of closed subspaces; in Hilbert spaces we further assume the summands of $\oplus$ are mutually orthogonal unless otherwise specified.

Theorem 3.1. *Let $T : \mathcal{D}(T) \subset H \to L^2(\mu)$ and $S : \mathcal{D}(S) \subset L^2(\mu) \to H$ be unbounded operators such that $\overline{T}$ is the analysis operator of some algebraic frame and S is closed. Then the following assertions are equivalent.* (*Note that $\mathcal{G}(S) \subset L^2(\mu) \oplus H$.*)

1. $\mathcal{G}(\overline{T})^{\text{flip}} + [\mathcal{R}(T)^{\perp} \oplus \{0\}] \subset \mathcal{G}(S)$.

2. *The operator S satisfies* (3.2).

3. **(The inversion formula)** $\mathcal{D}(S) \supset \mathcal{R}(T)^{\perp} \dot{+} \mathcal{R}(T)$; *moreover, for all $g \in \mathcal{R}(T)^{\perp} \dot{+} \mathcal{R}(T)$ and for all $h \in \mathcal{R}(|T|)$,*

$$\langle Sg, h\rangle = \int_Z g(z)\overline{(V|T|^{-1}h)(z)}d\mu(z). \tag{3.3}$$

Moreover, under any one of the equivalent conditions $(1) - (3)$, *the operator T is an isometry if and only if $S = T^*$; in either case, $\mathcal{D}(\overline{T}) = H$.*

Proof. Assume S satisfies (1). Clearly, $S(\mathcal{R}(T)^{\perp}) = \{0\}$ and it remains to show that $ST|_{\mathcal{D}(T)} = I_{\mathcal{D}(T)}$. Let $x \in \mathcal{D}(T)$ be arbitrary. Then $Tx \oplus x \in \mathcal{G}(T)^{\text{flip}} \subset \mathcal{G}(\overline{T})^{\text{flip}} \subset \mathcal{G}(S)$ which proves that $x = STx$; hence, (1) $\Rightarrow$ (2).

Next, assume S satisfies (2) and let $h \in \mathcal{R}(|T|)$. If $g \perp \mathcal{R}(T)$, then $g \perp \mathcal{R}(V)$, $\langle Sg, h\rangle = 0$ and

$$\int g(z)\overline{(V|T|^{-1}h)(z)}d\mu(z) = \langle g, V|T|^{-1}h\rangle_{L^2(\mu)} = 0.$$

If $g \in \mathcal{R}(T)$, then $g \in \mathcal{D}(S)$ and

$$\begin{aligned}\langle Sg, h\rangle &= \langle Sg, |T||T|^{-1}h\rangle = \langle Sg, T^*V|T|^{-1}h\rangle \\ &= \langle TSg, V|T|^{-1}h\rangle = \langle g, V|T|^{-1}h\rangle \\ &= \int g(z)\overline{(V|T|^{-1}h)(z)}d\mu(z).\end{aligned}$$

The complete proof of (2) $\Rightarrow$ (3) follows from the linearity of S.

Now, assume (3) holds. If $g \perp \mathcal{R}(T)$, it follows from (3.3) that $\langle Sg, h\rangle = 0$ for all h in the dense subspace $\mathcal{R}(|T|)$ of H. Thus $Sg = 0$ and, hence, $\mathcal{R}(T)^{\perp} \oplus \{0\} \subset \mathcal{G}(S)$. If $k \in \mathcal{D}(\overline{T})$, then $k \oplus \overline{T}k = \lim_{n\to\infty} k_n \oplus Tk_n$ for some sequence $(k_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$. Fix $n \in \mathbb{N}$. By (3.3),

$$\begin{aligned}\langle STk_n, h\rangle &= \int_Z (Tk_n)(z)\overline{(V|T|^{-1}h)(z)}d\mu(z) \\ &= \langle Tk_n, V|T|^{-1}h\rangle = \langle V|T|k_n, V|T|^{-1}h\rangle \\ &= \langle |T|k_n, |T|^{-1}h\rangle = \langle k_n, h\rangle,\end{aligned}$$

for all $h \in \mathcal{R}(|T|)$. Since $\mathcal{R}(|T|)$ is dense in H, it follows that $STk_n = k_n$. Thus, $k \oplus \overline{T}k = \lim_{n\to\infty} k_n \oplus Tk_n = \lim_{n\to\infty} STk_n \oplus Tk_n \in \mathcal{G}(S)^{\text{flip}}$. Hence, $\mathcal{G}(\overline{T})^{\text{flip}} \subset \mathcal{G}(S)$ and the implication (3) $\Rightarrow$ (1) is proven.

Finally, assume T is an isometry. Since it is a closed, densely defined, bounded operator, it follows that $\mathcal{D}(T) = H$, $|T| = I$ and $T = V$. Since $SV = ST = I$, the operators S and V^* coincide on $\mathcal{R}(V)$ and since they both vanish on $\mathcal{R}(V)^{\perp}$, $S = V^* = T^* \in B(L^2(Z,\mu), H)$. Conversely, if $T^* = S$, it follows that $T^*T = ST = I_{\mathcal{D}(T)}$ and, hence, T is an isometry. ∎

The following example shows that the operator S satisfying any of the equivalent conclusions $(1) - (3)$ is not unique.

Example 3.2. Let $T = id/dx$ be the differential operator acting on the dense linear subspace $\mathcal{D}(T) := \{f \in C^{(1)}([0,1]) : f(0) = f(1) = 0\}$ of $L^2([0,1])$. Choosing $\widetilde{\mathcal{R}}(T) \subset C([0,1])$, we obtain an algebraic frame generated by T. Now, let $T_1 = id/dx$ be the differential operator on the dense linear subspace $\{f \in C^{(1)}([0,1]) : f(0) = f(1)\}$. It is known that $\overline{T}$ is an injective closed symmetric operator, $\overline{T}_1$ is a non injective selfadjoint operator, and $\overline{T} \subset \overline{T}_1 \subset T^*$ (see pp. $257 - 259$ of [18]). Define $S_1 : \ker(\overline{T}_1) \oplus \mathcal{R}(T_1) \to L^2([0,1])$ by $S_1 f = 0$ for $f \in \ker(\overline{T}_1)$ and $S_1(\overline{T}_1 f) = f$ for $f \in \overline{R(T_1)} \cap \mathcal{D}(\overline{T}_1)$. Similarly, define $S_2 : \ker(T^*) \oplus \mathcal{R}(T^*) \to L^2([0,1])$ by $S_2 f = 0$ for $f \in \ker(T^*)$ and $S_2(T^* f) = f$ for $f \in \overline{R(T)} \cap \mathcal{D}(T^*)$. The operators S_1 and S_2 are distinct closed operators satisfying the inversion formula (3.3) of Theorem 3.1.

The above observation suggests the following definition of the synthesis operator of an algebraic frame which coincides with the one defined for generalized frames.

Definition 3.3. The synthesis operator S of an algebraic frame having the analysis operator $\overline{T} : \mathcal{D}(\overline{T}) \subset H \to L^2(\mu)$ is defined to be the closed operator $S : \mathcal{D}(S) \subset L^2(\mu) \to H$ determined by

$$\mathcal{G}(S) = \mathcal{G}(\overline{T})^{\text{flip}} \oplus (\mathcal{R}(T)^{\perp} \oplus \{0\}).$$

4 Discussion

In this section, we discuss the effect of unboundedness of the analysis operators and/or the unboundedness of the linear functionals on the processed signals. The algebraic frame generated by the Fourier transformation consists of unbounded linear functionals; however, although the frequency of the signal is uncontrollable, the doubly boundedness of the analysis operator controls the energy of the processed signal. Also, there are other reasons to believe Fourier transforms have good behavior: if the signal itself is in $L^1(\mu)$, the processed signal is a $C^{(0)}$-function and the correspondence $\phi \mapsto \hat{\phi}$ is a continuous function from L^1-norm to L^∞-norm. For an analytic frame $(\theta_z)_{z \in Z}$, one can discard the zero functionals and normalize it to $\theta_z/||\theta_z||$ and modify the measure $d\mu(z)$ by $||\theta_z||^2 d\mu(z)$ to obtain a new analytic frame which is isometric to the original one. The analysis operator of the modified frame is continuous from the L^2-norm to L^∞-norm. (See [10].) In the worst case of the algebraic frames in which the linear functionals are not bounded, one can still have some kind of continuity: The analysis operator $\overline{T}$ has a polar decomposition $\overline{T} = V|T|$, where V is an isometry but $|T|$ is unbounded. The unbounded selfadjoint operator $|T|$ can be further decomposed as $|T| = (I - W)^{1/2} W^{-1/2}$, where $0 < W := (I + T^* \overline{T})^{-1} < I$. If $\epsilon > 0$ is small enough, the restriction $W|_{E([\epsilon, 1-\epsilon])H}$ provides a truncation T_ϵ which generates a doubly bounded analytic frame. Besides this, one has also to note that our unbounded operators have injective closures and, hence, the partial isometry V in the polar decomposition $V|T|$ of T is a well-behaved isometry.

References

[1] J-P. Antoine, P. Balazs, Frames and semi-frames. J. Physics A: Math. Theor., **44**, (2011), 205201.

[2] J-P. Antoine, P. Balazs, Frames and semi-frames. J. Physics A: Math. Theor., **44**, (2011), 479501.

[3] A. Askari-Hemmat, M.A. Dehghan, and M. Radjabalipour, Generalized frames and their redundancy. Proc. Amer. Math. Soc. **129**(2000), no. 4, 1143–1147.

[4] P. Casazza, O. Christensen, and D. T. Stoeva, Frame Expansions in Separable Banach Spaces. J. Math. Anal. Appl. **307**(2005), no. 2, 710–723.

[5] P. Casazza, D. Han, and D. R. Larson, *Frames for Banach spaces.* In: The functional and harmonic analysis of wavelets and frames, pp. 149-182, Contemp. Math., 247, Amer. Math. Soc., Providence, RI, 1999.

[6] I. Daubechies, The Wavelet Transform, Time-Frequency Localization and Signal Analysis. IEEE Trans. Inform. Theory **36**(1990), no. 5, 961–1005 .

[7] I. Daubechies, A. Grossmann and Y. Meyer, Painless Nonorthogonal Expansions. J. Math. Phys. **27**(1986), 1271–1283.

[8] R. J. Duffin and A. C. Schaeffer, A class of Nonharmonic Fourier Series. Trans. Amer. Math. Soc. **72** (1952), 341–366 .

[9] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.

[10] H.H. Giv and M. Radjabalipour, *On the structure and properties of lower semi-frames*, Iran. J. Sci. Tech. (IJST) (to appear).

[11] K. Gröchenig, Describing Functions: Atomic Decompositions versus Frames., Monatshefte für Mathematik, **112**(1991), 1–41.

[12] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser Boston, 2001.

[13] A. Grossmann, G. Morlet, and T. Paul, Transforms Associated to Square-Integrable Group Representations I: General Results. J. Math. Phys. **26** (1985), 2473–2479.

[14] A. Grossmann, G. Morlet, and T. Paul, Transforms Associated to Square-Integrable Group Representations II: Examples. Annales de l'institute Henri Poincare, Section A **45** (1986), 293–309.

[15] G. Kaiser, *A Friendly Guide to Wavelets.* Second Printing, Birkhäuser Boston, 1995.

[16] G. Kaiser, *Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis*, North-Holland, Amsterdam, 1990.

[17] M. Radjabalipour, *A survey of unbounded operators.* (to appear).

[18] M. Reed, and B. Simons, *Methods of Mathematical Physics. I: Functional Analysis, Revised and Enlarged Edition,* Academic Press, San Diego 1980.

[19] H. Zhang, and J. Zhang, Frames, Riesz Bases, and sampling expansions in Banach spaces via semi-inner products. Appl. Comput. Harmon. Anal. **31**(2011), no. 1, 1–25.

Center for Linear Algebra and Optimization,
Shahid Bahonar University of Kerman, Kerman, Iran
email: hossein.giv@gmail.com

Iranian Academy of Sciences, Tehran, Iran
email: radjabalipour@ias.ac.ir, mradjabalipour@gmail.com

Department of mathematics,
Kerman Graduate University of Technology, Kerman, Iran;
and Department of Mathematics,
Shahid Bahonar University of Kerman, Kerman, Iran.
email: askari@mail.uk.ac.ir